North University of Baia Mare
Faculty of Science

Resume of Ph.D. Thesis

FIXED POINTS AND BEST PROXIMITY POINTS THEOREMS FOR CYCLICAL CONTRACTIVE OPERATORS

Scientific Advisor:
Prof. Univ. Dr. Vasile BERINDE

Ph.D. Student:
Mihaela Ancuţa PETRIC

Baia Mare
2011

Contents of Thesis

Keywords and phrases: fixed point, periodic point, best proximity point, metric space, uniformly convex Banach space, cyclic operator, Banach operator, Kannan operator, Chatterjea operator, Zamfirescu operator, Reich-Rus operator, Ćirić operator, Hardy-Rogers operator, Bianchini operator, nonexpansive operator, almost contraction, almost φ-contraction, strict almost contraction, strict almost φ-contraction, fixed point structure, common fixed point, Maia type result, multivalued operator.

Introduction

The main topic of this thesis is concerned with cyclic type operators, both single-valued and multivalued, by considering several classical metrical contraction conditions. From the multitude of such conditions existing in the literature dedicated to fixed point theory, we will present some of them:

(i) (Banach, 1922) there exists a number $a \in [0,1)$, such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq ad(x, y);$$

(ii) (Kannan, 1968) there exists a number $a \in [0, 1/2)$, such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)];$$

(iii) (Chatterjea, 1972) there exist a number $b \in [0, 1/2)$, such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)];$$

(iv) (Zamfirescu, 1972) there exist the real numbers a, b and c satisfying

$$0 \leq a < 1, 0 \leq b < \frac{1}{2} \text{ and } 0 \leq c < \frac{1}{2};$$

such that, for each $x, y \in X$ at least one of the following is true:

(z1) $d(Tx, Ty) \leq ad(x, y)$;

(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;

(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

(v) (Bianchini, 1972) there exists a number $h \in [0, 1)$, such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\};$$

for all $x, y \in X$.

(vi) (Reich, 1971; Rus, 1971) there exist some real constants $a, b \in \mathbb{R}_+$ with $a + 2b < 1$ such that:

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$.

(vii) (Hardy-Rogers, 1973) there exist some real constants $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$, such that:

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y).$$

for all $x, y \in X$.

(viii) (Berinde, 2004) there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$, where T is an operator defined on a metric space (X, d) into itself.

Let us consider two nonempty subsets A and B of X. An operator $T : A \cup B \to A \cup B$ is called a *2-cyclic contraction*(see [**70**]) if

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;
(ii) $d(Tx, Ty) \leq kd(x, y)$, for all $x \in A$ and $y \in B$, where $k \in [0, 1)$.

The concept of 2-cyclic contraction has been introduced by W.A. Kirk, P.S. Srinivasan and P. Veeramani [**70**] in 2003, and afterwards studied by some authors [**1, 7, 47, 48, 49, 65, 67, 82, 83, 109, 111, 51, 66, 8, 113**]. Our material study centers around these contractive-type conditions when the contractive assumptions are restricted to pairs $x \in A$, $y \in B$, and has been organized in four chapters.

In [**47**] a more abstract formulation of 2-cyclic contraction was considered by replacing (ii) with the following contractive condition:

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)D(A, B),$$

where $D(A, B) = \inf\{d(x, y) | x \in A, y \in B\}$, and in this case the problem is not to seek for the existence of a fixed point, but for the existence of a point called *best proximity point* $x \in A$, that is a point satisfying:

$$d(x, Tx) = D(A, B).$$

If $D(A, B) = 0$ then from the above contraction condition we obtain (2). Moreover, $d(x, Tx) = 0$, i.e., x is a fixed point of T, situated in the intersection set, $A \cap B$. These results are useful when the fixed point equation

$$Tx = x,$$

does not necessarily possess a solution.

I am conscious off all the imperfections of this work and shall welcome any comments, remarks, suggestions and additional bibliographical references coming from the readers.

Acknowledgements

First of all, I am deeply thankful to my Ph.D. adviser, Professor Doctor Vasile Berinde, who has always been helpful and has offered invaluable suggestions, advice, support and guidance throughout this work, but also for teaching me how to attack problems.

My thanks also go to all the members of the Department of Mathematics and Computer Science of North University of Baia Mare for all their support and wonderful environment that they have created during the preparation of this work.

My love and thanks go to all the teachers and tutors that have helped me become what I am today.

I am also very grateful to Prof. Dr. Snezhana Gocheva-Ilieva and Asoc. Prof. B.G. Zlatanov for their help and support during my stay at Plovdiv University "Paisi Hilendarski" in the framework of a CEEPUS network mobility Grant. I take this opportunity to thank to Asoc. Prof. B.G. Zlatanov for his collaboration in our joint article, and for his advice.

I also want to thank to Prof. Dr. Vasile Câmpian, from Technical University of Cluj-Napoca, for opening my eyes in the field of research.

Thanks to all my friends for providing support and friendship that I need. I truly thank to Cristian Coteţ for sticking by my side, even when I was irritable and depressed. He has been a true and great supporter and has unconditionally loved me during my good and bad times.

Especially, I wish to express my loving thanks to my parents and my brother. Their love, encouragement, support and understanding have helped me immeasurably. There are no words to convey my gratitude for their sacrifice.

CHAPTER 1

Preliminaries

In this chapter we present some basic notions, like the concepts of metric and normed spaces, Banach space, concepts regarding comparison functions, the notion of fixed point structure accompanied by appropriate examples. The fourth section is dedicated to some concepts regarding functionals defined on $P(X)$, like the gap function between two sets, the excess functional of a set over another set, the generalized Pompeiu-Hausdorff functional with some properties.

The section *Operators and basic fixed point theorems* presents some classical contractive-type operators. Among them, one can find Banach contraction, Kannan operators, Chatterjea operators, Zamfirescu operators, Bianchini operators, Reich-Rus operators, Hardy-Rogers operators, Ćirić, φ-contraction, almost-contraction, with their corresponding fixed point theorems.

The basic notion used in the construction of this thesis, namely *cyclic operator*, is presented accompanied by some fixed point theorems given by W.A. Kirk, P.S. Srinivasan and P. Veeramani in [**70**].

Author's original contributions in this chapter are: Examples 1.5.3, 1.5.5 and 1.6.3.

Among the notions and results included in the first chapter we mention the basic notion used in the construction of this thesis is that of *cyclic operator*.

Let $p \in \mathbb{N}$, $p \geq 2$.

Definition 1.6.1. Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a metric space (X, d). An operator $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ is called a *cyclic operator* if the following condition is satisfied:

$$T(A_i) \subseteq A_{i+1}, \text{ for all } i \in \{1, 2, \ldots, p\} (\text{where} A_{p+1} = A_1). \tag{1.1}$$

In particular, for $p = 2$ we have the following condition for a cyclic operator:

$$T(A) \subseteq B \text{ and } T(B) \subseteq A. \tag{1.2}$$

Because the role played by the class of cyclic operators in the architecture of the present thesis is an important one, this section will present a brief survey of those fixed point results regarding them.

Fixed point theorems for cyclical operators

The first moment in the evolution of fixed point theory for cyclic operators is marked by the paper [**70**] of W.A. Kirk, P.S. Srinivasan and P. Veeramani, which deals with this type of cyclic operators by considering contractive assumptions restricted to pairs $(x, y) \in A_i \times A_{i+1}$. Their theorems illustrate the methodology to obtain other extension of Banach's Theorem, which is the main aim of this thesis. This is the reason of the presence of the proofs for some of the following results.

First W.A. Kirk, P.S. Srinivasan and P. Veeramani [**70**] considered the following extension of Banach's theorem, for two sets:

Theorem 1.6.1.([**70**]) *Let A and B be two non-empty closed subsets of a complete metric space X, and suppose $T : X \to X$ satisfies (1.2) and*

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x \in A, y \in B, \tag{1.3}$$

where $k \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

An interesting feature about the above observation is that continuity of T is no longer needed. Also it is possible to reformulate this result as a common fixed point theorem for two mappings.

Corollary 1.6.1.([**70**]) *Let A and B be two non-empty closed subsets of a complete metric space (X, d). Let $f : A \to B$ and $g : B \to A$ be two functions such that*

$$d(f(x), g(y)) \leq kd(x, y), \text{ for all } x \in A \text{ and } y \in B, \tag{1.4}$$

where $k \in (0, 1)$. Then there exists a unique $x_0 \in A \cap B$ such that

$$f(x_0) = g(x_0) = x_0.$$

Obviously the reasoning in Theorem 1.6.1 can be extended to a collection of finite sets.

Theorem 1.6.2.([**70**]) *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space, and suppose $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ satisfies (1.1) and the following one (where $A_{p+1} = A_p$):*

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{1.5}$$

where $k \in (0, 1)$. Then T has a unique fixed point.

In [**70**] Edelstein's result in [**46**], Geraghty's result in [**52**], the well known Boyd-Wong result in [**35**] and Caristi's theorem in [**60**] were similarly extended. Suggested by these considerations, in [**109**] the following concept was introduced.

Definition 1.6.2.([**109**]) Let X be a nonempty set, $T : X \to X$ an operator. By definition, $X = \bigcup_{i=1}^{p} A_i$ is a *cyclic representation* of X with respect to T if

(i) $A_i, i = \overline{1,p}$, are nonempty sets;

(ii) $T(A_1) \subset A_2, \ldots, T(A_{p-1}) \subset A_p, T(A_p) \subset A_1$.

Example 1.6.3. Let $A_1 := \{2n : n \in \mathbb{N}\}$ and $A_2 := \{2n-1 : n \in \mathbb{N}^*\}$ and an operator defined by

$$Tx = 3x + 1, \text{ for all } x \in A_1 \cup A_2.$$

Then $\mathbb{N} = A_1 \cup A_2$ is a cyclic representation of $\mathbb{N}$ with respect to T.

Regarding the concept of cyclic representations I.A. Rus in [**109**] gave two lemmas: Lemma 2.1.2 and 2.1.3.

Among the results obtained for cyclic operators we mention the following one:

Theorem 1.6.7.([**91**]) *Let (X, d) be a complete metric space, $A_1, A_2, \ldots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a (c)-comparison function and $T : Y \to Y$ an operator. Assume that $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T and T is a cyclic φ-contraction, i.e., it satisfies*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \text{ for all } x \in A_i, y \in A_{i+1}.$$

Then

(i) T has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and the Picard sequence $\{T^n x\}$ converges to x^* for any starting point $x \in Y$;

(ii) the following estimates hold:

$$d(T^n x, x^*) \leq s(\varphi^n(d(x, Tx))), n \geq 1;$$

$$d(T^n x, x^*) \leq s(\varphi^n(d(T^n x, T^{n+1} x))), n \geq 1;$$

(iii) for any $x \in Y$:

$$d(x, x^*) \leq s(d(x, Tx))$$

where $s : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t).$$

Best proximity point theorems for cyclical operators

As one can notice, in the previous theorems, for a cyclic operator, if a fixed point exists then it is located in the intersection of the sets, therefore the intersection set is non-empty. A. Anthony Eldred and P. Veeramani, in [**47**], considered the case when the intersection of the sets is empty. In this case they seek not for the existence of a fixed point but for the existence of a *best proximity point*, as is defined below.

Definition 1.6.3. *Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ an operator. A point $x \in A \cup B$ is called a* best proximity point *if*

$$d(x, Tx) = D(A, B),$$

where

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

A. Anthony Eldred and P. Veeramani in [**47**], considered the contractive condition from Theorem 1.6.1, and with respect to this condition it was defined the following operator:

Definition 1.6.4. [**47**] *Let A and B be two nonempty subsets of a metric space (X, d). An operator $T : A \cup B \to A \cup B$ is called a* cyclic contraction operator *if it is cyclic operator and for some $k \in (0, 1)$ T satisfies*

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)D(A, B), \text{ for all } x \in A, y \in B.$$

For this operator, in [**47**], was given some existence and convergence result, which form an algorithm to find a best proximity point for an operator in the settings of a uniformly convex Banach space. This algorithm will be used in Chapter 4 to obtain similar results for other cyclic operators. For this reason we will present here the algorithm from [**47**], used for cyclic contraction operators.

Proposition 1.6.1. ([**47**])*Let A and B be nonempty subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction operator. Then starting with any $x_0 \in A \cup B$ we have*

$$d(T^n x, T^{n+1} x) \to D(A, B), n = 0, 1, 2, \ldots.$$

Proposition 1.6.2. ([**47**]) *Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction operator, let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = D(A, B)$.*

Lemma 1.6.1. ([**47**]) *Let A be a nonempty closed subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $\|z_n - y_n\| \to D(A, B)$;

(ii) for every $\epsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,

$$\|x_m - y_n\| \leq D(A, B) + \epsilon.$$

Then, for every $\epsilon > 0$ there exists N_1 such that for all $m > n \geq N_1$,

$$\|x_m - z_n\| \leq \epsilon.$$

Lemma 1.6.2. ([**47**]) *Let A be a nonempty closed and convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $\|x_n - y_n\| \to D(A, B)$,

(ii) $\|z_n - y_n\| \to D(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

Theorem 1.6.8. ([**47**])*Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map, then there exists a unique best proximity point $x \in A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n, n \geq 0$, then $\{x_{2n}\}$ converges to the best proximity point.*

This chapter is developed on the following books/ papers: [**4, 5, 6, 16, 22, 24, 25, 26, 30, 31, 32, 33, 34, 35, 36, 37, 40, 42, 43, 45, 46, 47, 50, 52, 53, 54, 55, 56, 57, 58, 60, 63, 64, 68, 69, 70, 74, 76, 79, 80, 84, 85, 86, 88, 91, 92, 93, 95, 97, 98, 101, 102, 104, 105, 106, 108, 109, 110, 125, 129, 130, 131, 132**].

CHAPTER 2

Fixed Point Theorems for Cyclical Contractive Operators

In this chapter we will consider extensions of some of classical metrical fixed point theorems due to Banach, Kannan, Chatterjea, Zamfirescu, Bianchini, Hardy-Rogers, Reich-Rus, Ćirić and many others by considering cyclical operators and by restricting the contractive conditions. For some of this results both a priori and a posteriori error estimates are available. It is possible for part of the proofs to use convergence theorems given by P.D. Proinov in [**84, 85, 86**], but for the sake of simplicity we prefer to use the technique of the classical proof of the Banach Contraction Mapping Principle. We also provide very interesting examples in support of our results.

This chapter starts with the proof of a technical lemma, which will be constant used in the proof of the new results presented in this chapter. In each section first we shall introduce the corresponding cyclic operator, and for some of this operators we will present fixed point theorems with a priori and a posteriori error estimates and the rate of convergence. Very interesting examples usually accompany the new results.

A fixed point of an operator usually corresponds to a solution of a certain equation. Some equations have more than one solution. This reason suggested to consider operators with a non-unique fixed point. Ćirić considered several contractive conditions which assures the existence of the fixed point. We present some of this conditions in the eighth section of this chapter, by considering cyclic operators.

We also present in this chapter some applications of fixed point structures in the study of cyclic operators. Using a fixed point lemma, given by I.A. Rus, in [**109**], we can prove some fixed point results from the previous sections.

It is very natural to use some of the cyclic fixed point theorems to obtain common fixed point results. But, this is possible only if in the right side of the contractive condition we have a function which depends only on the distance $d(x, y)$. Two applications of common fixed point theorems close the section *Common Fixed Point of Contractive Type Operators*.

Author's original contributions in this chapter are: Definitions 2.1.1, 2.3.1-2.3.4, 2.4.1, 2.5.1, 2.6.1, 2.6.2, 2.7.1, 2.7.3, 2.7.4, Examples 2.2.1, 2.2.2, 2.3.1, 2.3.2, 2.3.3, 2.9.4-2.9.8, Lemma 2.1.1, Corollaries 2.10.1, 2.10.2, Theorems 2.2.1, 2.3.1, 2.3.2, 2.3.3, 2.4.1, 2.5.1-2.5.4, 2.6.1, 2.6.2, 2.7.1, 2.7.3, 2.7.4, 2.8.1-2.8.4, 2.11.2.

1. Fixed Point Lemmas

Definition 2.1.1. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space (X,d). An operator $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called *cyclic graphic contraction* if it is a cyclic operator, i.e., satisfies (1.1), and there exists $k \in [0,1)$ such that

$$d(Tx, T^2x) \le kd(x, Tx), \tag{2.6}$$

for all $x \in \bigcup_{i=1}^{p} A_i$.

The following lemma is very simple but useful in the proof of the results presented in this chapter.

Lemma 2.1.1. (Petric [**125, 124**]) *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d). If $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a cyclic graphic contraction, with the constant $k \in [0,1)$, then:*

(i) $d(T^n x, T^{n+1}x) \le k^n d(x, Tx)$, *for all* $n = 0, 1, 2, \ldots$.

(ii) For each $x \in \bigcup_{i=1}^{p} A_i$ *the Picard iteration defined by*

$$x_{n+1} = Tx_n, n = 0, 1, 2 \ldots \tag{2.7}$$

converges to a point $x^* \in \bigcap_{i=1}^{p} A_i$.

(iii) $\bigcap_{i=1}^{p} A_i \ne \emptyset$.

Lemma 2.1.2. ([**109**]) *Let $(X, S(X), M)$ be a fixed point structureon an L-space $(X, \to)$. Let $A_1, A_2, \ldots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. We suppose that*

(i) Y is a cyclic representation of Y with respect to T;

(ii) there exists $x \in Y$ such that $\{T^n x\}$ converges;

(iii) if $A = \bigcap_{i=1}^{p} A_i$, then $A \in S(X)$ and $T|_A \in M(A)$.

Then T has at least one fixed point.

Lemma 2.1.3. ([**109**]) *Let $(X, S(X), M)$ be a fixed point structureon an L-space $(X, \to)$. Let $A_1, A_2, \ldots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. We suppose that*

(i) Y is a cyclic representation of Y with respect to T;

(ii) at least $A_i \in S(X)$;

(iii) $g, h \in \Rightarrow g \circ h \in M$.

Then T^m has at least one fixed point.

2. Banach Contraction Principle for Cyclical Operators

First, we take up the question, if we can complete Theorem 1.6.2 with some error estimates, to obtain a similar formulation as Banach's Contraction Mapping Principle. The answer at this question can be found in the following theorem.

Theorem 2.2.1. (Petric-Zlatanov [**128**]) *Let all the assumptions in Theorem* 1 *be satisfied, with the constant* $a \in [0,1)$*. Then*

(i) T *has a unique fixed point* x^* *in* $\bigcap_{i=1}^{p} A_i$*;*

(ii) The Picard iteration $\{x_n\}_{n\geq 0}$ *defined by* (2.7) *converges to* x^**, for any* $x_0 \in \bigcup_{i=1}^{p} A_i$*;*

(iii) The following a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), n = 0, 1, 2, \dots \tag{2.8}$$

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x_n), n = 1, 2, \dots \tag{2.9}$$

hold;

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq a d(x_{n-1}, x^*), n = 1, 2, \dots \tag{2.10}$$

Example 2.2.1. (Petric-Zlatanov [**128**]) Consider the space $C[0,1]$ endowed with the metric

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

Define the subsets of $C[0,1]$:

$$A_1 = \overline{\left\{ \sum_{k=1}^{n} \alpha_k x^{2k}, \alpha_k \geq 0, \sum_{k=1}^{n} \alpha_k \leq 1, n \in \mathbb{N} \right\}}$$

$$A_2 = \overline{\left\{ \sum_{k=1}^{n} \beta_k x^{2k-1}, \beta_k \geq 0, \sum_{k=1}^{n} \beta_k \leq 1, n \in \mathbb{N} \right\}}.$$

By definition the sets A_1 and A_2 are closed and bounded subsets in $C[0,1]$. Define the map $T : C[0,1] \to C[0,1]$ by

$$Tf = \frac{1}{2}\int_0^x f(t)dt.$$

We will show that the map T satisfies the conditions of Theorem 2.2.1. Let $f \in A_1$. We need to show that Tf is in A_2.

Case I. Let $f(x) = \sum_{k=1}^{n} \alpha_k x^{2k}$, $\alpha_k \geq 0$, $\sum_{k=1}^{n} \alpha_k \leq 1$. Then

$$Tf = \frac{1}{2}\int_0^x f(t)dt = \frac{1}{2}\int_0^x \sum_{k=1}^{n} \alpha_k t^{2k} = \sum_{k=1}^{n} \frac{\alpha_k}{2(2k+1)} x^{2k+1} \in A_2,$$

because $\sum_{k=1}^{n} \frac{\alpha_k}{2(2k+1)} \leq 1$.

Case II. There exists a sequence $f_n(x) = \sum_{k=1}^{p_n} \alpha_k^{(n)} x^{2k}$, $\alpha_k^{(n)} \geq 0$, $\sum_{k=1}^{n} \alpha_k^{(n)} \leq 1$, which is uniformly convergent to f. By the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ we have that

$$\lim_{n\to\infty} \frac{1}{2}\int_0^x f_n(t)dt = \frac{1}{2}\int_0^x f(t)dt.$$

By Case I we have

$$\frac{1}{2}\int_0^x f_n(t)dt = \sum_{k=1}^{p_n} \beta_k^{(n)} x^{2k+1} = g_n \in A_2,$$

where $\beta_k = \dfrac{\alpha_k}{2(2k+1)}$. By the uniform convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ we have that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$, such that for every $n, m \geq N$ holds

$$\max_{x\in[0,1]} |f_n(x) - f_m(x)| < \epsilon.$$

Then it is easy to see that

$$\begin{aligned}
\max_{x\in[0,1]} |g_n(x) - g_m(x)| &= \max_{x\in[0,1]} \left| \sum_{k=1}^{p_n} \beta_k^{(n)} x^{2k+1} - \sum_{k=1}^{p_m} k^{(m)} x^{2k+1} \right| \\
&\leq \frac{1}{2} \max_{x\in[0,1]} \left| \sum_{k=1}^{p_n} \alpha_k^{(n)} x^{2k} - \sum_{k=1}^{p_m} k^{(m)} x^{2k} \right| \\
&= \frac{1}{2} \max_{x\in[0,1]} |f_n(x) - f_m(x)| < \epsilon,
\end{aligned}$$

which ensures that $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A_2. Therefore there exists $g \in A_2$, such that $\lim_{n\to\infty} g_n = g$ and thus $\frac{1}{2}\int_0^x f(t)dt = g \in A_2$.

The proof that $T : A_2 \subseteq A_1$ is similar. It is well known that T is a contraction. The constant zero is a fixed point of the map T and $0 \in A_1 \cap A_2$.

Let us mention that the sets A_1 and A_2 consists not only of polynomial functions. For example $\dfrac{1}{4}e^{x^2} \in A_1$. Indeed

$$\frac{1}{4}e^{x^2} = \lim_{n\to\infty} \left(\frac{1}{4} + \sum_{k=1}^{n} \frac{x^{2k}}{4.k!} \right).$$

Thus the function $\dfrac{1}{8}\displaystyle\int_0^x e^{x^2}$ is in A_2.

Example 2.2.2. (Petric-Zlatanov [**128**]) Consider the sets:

$$A = [0,1] \bigcup \left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right), \quad B = [-1,0] \bigcup \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right).$$

Define the map $Tx = \frac{x}{4}$. Then T satisfies the conditions of Theorem 2.2.1. It is easy to see that

$$T\left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right) \subseteq \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right)$$

and

$$T\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right) \subseteq \left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right)$$

and 0 is the fixed point of T. The interior of the intersection $A \cap B$ is not an empty set, but the fixed point of T is not in the interior of $A \cap B$.

For any $x_0 \in A \cap B$, $x_n = Tx_{n-1} \in A \cap B$ and for any $x_0 \notin A \cap B$ $x_n = Tx_{n-1} \notin A \cap B$.

3. Cyclical Quasi-Nonexpansive Operators

A generalization of a nonexpansive operator, with at least one fixed point, is that of the quasi nonexpansive operators. This class of quasi-nonexpansive operators is strongly connected to the Newton's iterative method, see [**10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 27, 28, 29, 89, 90, 43**].

Obviously, for a cyclic quasi-nonexpansive operator, we have the following definition:

Definition 2.3.1. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space X. An operator $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is said to be *cyclical qausi-nonexpansive* if

(i) T is a cyclic operator, i.e., satisfies (1.1);

(ii) T has at least one fixed point in $\bigcap\limits_{i=1}^p A_i$;

(iii) for each fixed point $x^* \in \bigcap\limits_{i=1}^p A_i$, we have

$$d(Tx, x^*) \leq d(x, x^*), \text{ for all } x \in \bigcup_{i=1}^p A_i. \tag{2.11}$$

In this section we will present three examples of cyclic quasi-nonexpansive operators.

3.1. Cyclical Kannan's operators

In [**26**] was proved that the class of Kannan operators is included in the class of quasi-nonexpansive operators. Let us define a cyclic Kannan operator:

Definition 2.3.2. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space X. An cyclic operator $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is said to be *cyclical Kannan operator* if there exists $a \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq a\Big[d(x, Tx) + d(y, Ty)\Big], \text{for all } x \in A_i, y \in A_{i+1}, \text{ for } 1 \leq i \leq p. \tag{2.12}$$

Regarding this definition M.A. Petric and B.G. Zlatanov, in [**128**], proved the following theorem.

Theorem 2.3.1. *Let* $\{A_i\}_{i=1}^p$ *be nonempty closed subsets of a complete metric space* X *and suppose* $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$*, is a cyclical Kannan operator with* $a \in [0, 1/2)$*. Then*

(i) T *has a unique fixed point* x^* *in* $\bigcap\limits_{i=1}^p A_i$*.*

(ii) The Picard iteration $\{x_n\}_{n\geq 0}$ *given by (2.7) converges to* x^* *for any starting point* $x_0 \in \bigcup\limits_{i=1}^p A_i$*;*

(iii) The following estimates hold

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1), n = 0, 1, 2 \ldots \tag{2.13}$$

$$d(x_n, x^*) \leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n), n = 1, 2 \ldots \tag{2.14}$$

where $\lambda = \dfrac{a}{1-a}$.

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq \lambda d(x_{n-1}, x^*), n = 1, 2, \ldots \tag{2.15}$$

where $\lambda = \dfrac{a}{1-a}$.

It is a simple exercise to show that any cyclic Kannan operator with constant $a \in \left[0, \dfrac{1}{2}\right)$ is a cyclic quasi-nonexpansive operator.

Example 2.3.1. (Petric-Zlatanov [**128**]) Consider the function:

$$f(x) = \begin{cases} -1/3|x \sin \frac{1}{x}|, & x \in (0, \pi], \\ 1/3|x \sin \frac{1}{x}|, & x \in [-\pi, 0), \\ 0, & x = 0. \end{cases}$$

Obviously $f : [0, \pi] \to [-\pi, 0]$ and $f : [-\pi, 0] \to [0, \pi]$.

It is easy to see that for $x \in [0, \pi]$ and $y \in [-\pi, 0]$ holds

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{3}|x \sin \frac{1}{x}| + \frac{1}{3}|y \sin \frac{1}{y}| \right| \leq \frac{1}{3}(|x| + |y|) \\ &\leq \frac{1}{3}\left(\left| -\frac{1}{3}|x \sin \frac{1}{x}| - x \right| + \left| \frac{1}{3}|y \sin \frac{1}{y}| - y \right| \right) \\ &= \frac{1}{3}(|f(x) - x| + |f(y) - y|). \end{aligned}$$

If $y \in [0, \pi]$ and $x \in [-\pi, 0]$ it is proved similarly that

$$|f(x) - f(y)| \leq \frac{1}{3}(|f(x) - x| + |f(y) - y|).$$

So f satisfies all the conditions of Theorem 2.3.1 and thus it has a fixed point which is the intersection of the sets $[0, \pi]$ and $[-\pi, 0]$.

It is interesting in this example that there is no a constant $a > 0$ such that

$$|f(x) - f(y)| \leq a|x - y|.$$

Indeed if we take

$$x_n = \frac{1}{2n\pi + \frac{1}{n}}$$

and

$$y_n = \frac{1}{2n\pi}$$

then we have

$$\lim_{n\to\infty}\frac{|f(x_n)-f(y_n)|}{|x_n-y_n|}=\lim_{n\to\infty}\frac{2n\pi\sin(1/n)}{3/n}=+\infty$$

and therefore there is no $a>0$ so that $|f(x_n)-f(y_n)|\le a|x_n-y_n|$.

Example 2.3.2. (Petric-Zlatanov [**128**]) Consider the map $T:\ell_2\to\ell_2$ defined by

$$T(\{x_k\})=\begin{cases}\{-1/3|x_k\sin(1/x_k)|\}_{k=1}^{\infty}, & x_k\ge 0, k\in\mathbb{N}\\ \{1/3|x_k\sin(1/x_k)|\}_{k=1}^{\infty}, & x_k\le 0, k\in\mathbb{N}\\ \{0\}_{k=1}^{\infty}, & x_k=0, k\in\mathbb{N}.\end{cases}$$

Consider the sets

$$A_1=\{\{x_k\}_{k=1}:\sum_{k=1}^{\infty}x_k^2\le 1, x_k\ge 0\}$$

and

$$A_2=\{\{x_k\}_{k=1}:\sum_{k=1}^{\infty}x_k^2\le 1, x_k\le 0\}.$$

Obviously $T:A_1\to A_2$ and $T:A_2\to A_1$. It is easy to show, similarly to Example 2.3.1, that for $x=\{x_k\}_{k=1}^{\infty}$ and $y=\{y_k\}_{k=1}^{\infty}\in A_2$ holds

$$\|Tx-Ty\|\le\frac{1}{3}(\|Tx-x\|+\|Ty-y\|).$$

So T satisfies all the conditions of Theorem 2.3.1 and thus it has a fixed point which is the intersection of the sets A_1 and A_2.

Example 2.3.3. (Petric-Zlatanov [**128**]) Consider the sets:

$$A_1=\{0\}\cup\{1/n\}_{n=1}^{\infty}\cup\left\{\frac{-1}{2n}\right\}_{n=1}^{\infty}$$

and

$$A_2=\{0\}\cup\{-1/n\}_{n=1}^{\infty}\cup\left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}.$$

Define the map

$$Tx=\begin{cases}-\dfrac{x}{x+4}, & x\in A_1\\ -\dfrac{x}{4}, & x\in A_2\end{cases}$$

It is easily to be checked that $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$.

For any $x\in A_1$ and any $y\in A_2$ we have the chain of inequalities

$$\begin{aligned}|Tx-Ty| &= \left|\frac{x}{x+4}-\frac{y}{4}\right|\le\frac{1}{4}(|x|+|y|)\le\frac{1}{4}\left(|x|\left|1+\frac{1}{x+4}\right|+|y|\left|1+\frac{1}{4}\right|\right)\\ &= \frac{1}{4}\left(\left|\frac{-x}{x+4}-x\right|+\left|\frac{-y}{4}-y\right|\right)=\frac{1}{4}(|Tx-x|+|Ty-y|).\end{aligned}$$

So T satisfies all the conditions of Theorem 2.3.1 and thus it has a fixed point $x^*=0\in A_1\cap A_2$. It is interesting in this example that the intersection of A_1 and A_2 is with empty interior and that if we start from an arbitrary point $x_0\notin A_1\cap A_2$ then for any $n\in\mathbb{N}$ the Picard iteration $x_n=Tx_{n-1}\notin A_1\cap A_2$.

3.2. Cyclical Chatterjea operators

Another example of cyclic quasi-nonexpansive operators consists in the cyclic Chatterjea operators.

Definition 2.3.3. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space X. An cyclic operator $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is said to be *cyclical Chatterjea operator* if there exists $b \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq b\Big[d(x, Ty) + d(y, Tx)\Big], \text{for all } x \in A_i, y \in A_{i+1}, \text{ for } 1 \leq i \leq p. \tag{2.16}$$

In the following we will prove the existence of a unique fixed point, located in the intersection of the sets, for cyclic Chatterjea operators. The next result is included in the paper [**125**], by M.A. Petric.

Theorem 2.3.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space, and suppose $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is a cyclical Chatterjea operator. Then T has a unique fixed point in $\bigcap\limits_{i=1}^p A_i$.*

If T is a cyclic Chatterjea operator, then taking in (2.16) $y := p$ we obtain that T is a cyclic quasi-nonexpansive operator.

The fixed point result due to Chatterjea [**36**] can be obtained by taking $A_i = A_j$ for all i, j in the above theorem.

Example 2.3.4. ([**88**]) Let $X = [0, 1]$ with the usual norm and $A = B = [0, 1]$ be two closed sets. Define $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} \dfrac{1}{5}, & x \in \left[0, \frac{8}{15}\right) \\ \dfrac{1}{3}, & x \in \left[\frac{8}{15}, 1\right] \end{cases}$$

Then T is a Chatterjea operator with constant $c = \dfrac{2}{5}$, and the fixed point $\dfrac{1}{5} \in A \cap B$.

3.3. Cyclical Zamfirescu 0perators

Now, as we have done before, we will begin this section with the definition of a cyclic Zamfirescu operator.

Definition 2.3.4. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space X. An cyclic operator $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is said to be *cyclical Zamfirescu operator* if there exists the real numbers $a \in [0, 1), b \in [0, \frac{1}{2})$ and $c \in [0, \frac{1}{2})$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $1 \leq i \leq p$, at least one of the following is true:

(z1) $d(Tx, Ty) \leq a d(x, y)$;
(z2) $d(Tx, Ty) \leq b\Big[d(x, Tx) + d(y, Ty)\Big]$;
(z3) $d(Tx, Ty) \leq c\Big[d(x, Ty) + d(y, Tx)\Big]$.

Using the same technique, we will prove the following result, included in the original paper Petric-Zlatanov [**128**].

Theorem 2.3.3. *Let* $A_1, A_2, \dots, A_p, A_{p+1} = A_1$ *be nonempty closed subsets of a complete metric space* X *and suppose* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *is a cyclical Zamfirescu operator. Then*

(i) T *has a unique fixed point* x^* *in* $\bigcap_{i=1}^{p} A_i$.

(ii) The Picard iteration $\{x_n\}_{n\geq 0}$ *given by (2.7) converges to* x^* *for any starting point* $x_0 \in \bigcup_{i=1}^{p} A_i$;

(iii) The following estimates hold

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1), n = 0, 1, 2 \dots \tag{2.17}$$

$$d(x_n, x^*) \leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n), n = 0, 1, 2 \dots \tag{2.18}$$

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq \lambda d(x_{n-1}, x^*), n = 1, 2, \dots \tag{2.19}$$

where $\lambda = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

It is easy to check that if T is cyclic Zamfirescu operator then T is also o cyclic quasi-nonexpansive operator.

4. Cyclical Bianchini Operators

The preceding ideas lead us also to an analogous extension of Bianchini's fixed point theorem [**33**], which is included in Petric [**125**]. It is natural to define the cyclic Bianchini operator.

Definition 2.4.1. Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a metric space X. An cyclic operator $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is said to be *cyclical Bianchini operator* if there exists $h \in [0, 1)$ such that

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\}, \text{for all } x \in A_i, y \in A_{i+1}, \text{ for } 1 \leq i \leq p. \tag{2.20}$$

For this type of operators we have the following existence and uniqueness result of a fixed point.

Theorem 2.4.1. *Let* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of a complete metric space, and suppose* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *is a cyclical Bianchini operator. Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

5. Cyclical Reich-Rus Type Operators

In a similar manner to the previous results in this chapter the theorems established in the following are included in our paper Petric [**124**]. First we considered the cyclic version of Reich-Rus operators and secondly, we study the existence and uniqueness for some generalizations of this contractive type operator.

5.1. Cyclic Reich-Rus operators

Let us consider the following class of cyclic contractive-type operators.

Definition 2.5.1. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete space X. An cyclic operator $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is said to be *cyclical Reich-Rus operator* if there there exist the constants $a, b \in \mathbb{R}_+$ with $a + 2b < 1$ such that

$$d(Tx, Ty) \le ad(x, y) + b\Big[d(x, Tx) + d(y, Ty)\Big], \tag{2.21}$$

for all $x \in A_i, y \in A_{i+1}, 1 \le i \le p$.

We extend the fixed point theorems of S.Reich [**93**] and I.A. Rus [**102**] by imposing the cyclic condition (1.1) to the operator.

Theorem 2.5.1. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X, d), and suppose $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ is a cyclic Reich-Rus operator. Then T has a unique fixed point in $\bigcap\limits_{i=1}^p A_i$.*

Let us mention that in the case (2.21) is replaced by the condition

$$d(Tx, Ty) \le ad(x, y) + bd(x, Tx) + cd(y, Ty),$$

with $a, b, c \in \mathbb{R}_+$, $a + b + c < 1$, the corresponding fixed point result can by found in the original paper Petric [**125**].

As condition (2.21) is obtained from (1.5) by adding the extra term from (2.12), it is a very natural question of whether another contractive conditions of the same type can be considered.

5.2. Generalized Reich-Rus operators

In this section we shall study some contractive conditions of Reich-Rus type.

First time we consider a contractive condition formed by adding to (1.5) an extra term from (2.16). We have the following result.

Theorem 2.5.2. *Let (X, d) be a complete metric space and $\{A_i\}_{i=1}^p$ be nonempty closed subsets of X. Suppose that a mapping $T : \bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ satisfies the conditions (1.1) and there exist two constants $a, b \in \mathbb{R}_+$ with $a + 2b < 1$ such that*

$$d(Tx, Ty) \le ad(x, y) + b\Big[d(x, Ty) + d(y, Tx)\Big], \tag{2.22}$$

for all $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$. *Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

In the following we shall follow the mentioned steps by considering a contractive condition formed by adding to (1.5) an extra term from (2.20), establishing the fixed point theorem:

Theorem 2.5.3. *Let* (X, d) *be a complete metric space and* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of* X. *Suppose that an operator* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *satisfies the conditions (1.1) and there exist two constants* $a, b \in \mathbb{R}_+$ *with* $a + b < 1$ *such that*

$$d(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\}, \tag{2.23}$$

for each $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$. *Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

In a similar way we can build a contractive condition using one similar with Bianchini but of Chatterjea's type:

Theorem 2.5.4. *Let* (X, d) *be a complete metric space and* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of* X. *Suppose that a mapping* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *satisfies the conditions (1.1) and there exist two constants* $a, b \in \mathbb{R}_+$ *with* $a + 2b < 1$ *such that*

$$d(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Ty), d(y, Tx)\}, \tag{2.24}$$

for each $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$. *Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

The condition (2.24) appears in Rhoades's classification [**97**].

6. Cyclical Hardy-Rogers Operators

We proceed to obtain a more general fixed point result for the class of cyclical operators.

Definition 2.6.1. Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete space X. An cyclic operator $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is said to be *cyclical Hardy-Rogers operator* if there there exist the constants $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$ such that

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y), \tag{2.25}$$

for all $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$.

Theorem 2.6.1. *Let* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of a complete metric space* (X, d), *and suppose* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *is a cyclic Hardy-Rogers operator. Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

Now we will investigate a class of generalized contractions introduced by Lj. B Ćirić in [**40**].

Definition 2.6.2. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete space X. An cyclic operator $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ is said to be *cyclic Ćirić generalized contraction* if for every $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$ there exist nonnegative numbers q, r, s and t which may depend on both x and y, such that

$$\sup\{q + r + s + 2t : x, y \in X\} < 1$$

and

$$d(Tx, Ty) \leq qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, Tx)]. \tag{2.26}$$

Now, in the same manner, we are in position to prove the next extension of the above result, from the original paper [**125**].

Theorem 2.6.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X, d), and suppose $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ is a cyclic Ćirić generalized contraction. Then T has a unique fixed point in $\bigcap\limits_{i=1}^{p} A_i$.*

7. Cyclical Almost φ-Contractions

The present section contains a similar approach of a very general class of contractions, called *almost φ-contractions*, introduced by V. Berinde in [**23**].

7.1. Cyclical almost contractions

In [**87, 88**] the following definition was introduced.

Definition 2.7.1. Let (X, d) be a metric space, $A_1, \ldots, A_p \in P_{cl}(X)$, $Y = \bigcup\limits_{i=1}^{p} A_i$ and $T : Y \to Y$ an operator. If

(i) $\bigcup\limits_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(ii) there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \tag{2.27}$$

for all $x \in A_i, y \in A_{i+1}$, where $A_{p+1} = A_1$,

then T is a *cyclic almost contraction.*

Taking into account this definition we will prove the following result which ensures the convergence of the fixed point, which always exists, but is not necessary unique.

Theorem 2.7.1. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X, d) and $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ be an cyclic almost contraction. Then*

(i) T has at least one fixed point in $\bigcap\limits_{i=1}^{p} A_i$;

(ii) The Picard iteration $\{x_n\}_{n\geq 0}$ *defined by (2.7) converges to some* $x^* \in F_T$*, for any starting point* $x_0 \in \bigcup_{i=1}^{p} A_i$.

(iii) The error estimates are given by

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), n \geq 0, \tag{2.28}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), n \geq 1, \tag{2.29}$$

where δ *is the constant appearing in (2.27).*

As the previous theorem does not posses sufficient conditions to assure the uniqueness of a fixed point, the following definition, gives am additional condition.

Definition 2.7.2. ([**88**]) Let (X, d) be a metric space, $A_1, \dots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(ii) T is a cyclic almost contraction with constants $\delta \in [0, 1)$ and $L \geq 0$;

(iii) there exist $\delta_u \in [0, 1)$ and $L_u \geq 0$ such that

$$d(Tx, Ty) \leq \delta_u d(x, y) + L_u d(x, Tx), \tag{2.30}$$

for all $x \in A_i, y \in A_{i+1}$, where $A_{p+1} = A_1$,

then T is a *cyclic strict almost contraction.*

In view of these definition, we will present the main result of the paper [**87**].

Theorem 2.7.2. ([**87, 88**]) *Let* (X, d) *be a complete metric space,* $A_1, A_2, \dots, A_p \in P_{cl}(X)$*,* $Y = \bigcup_{i=1}^{p} A_i$*, and* $T : T \to Y$ *an operator. Assume that* $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of* Y *with respect to* T *and* T *is a cyclic strict almost contraction with constants* $\delta \in [0, 1), L \geq 0$ *and* $\delta_u \in [0, 1), L_u \geq 0$*, respectively. Then*

(i) T *has a unique fixed point* $x^* \in \bigcap_{i=1}^{p} A_i$ *and the Picard iteration* $\{x_n\}$ *given by (2.7) converges to* x^* *for any starting point* $x_0 \in Y$*;*

(ii) the following estimates hold:

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), n \geq 1,$$

$$d(x_n, x^*) \leq \frac{\delta_u}{1-\delta_u} d(x_{n-1}, x_n), n \geq 1,$$

(iii) for any $x \in Y$

$$d(x, x^*) \leq \frac{1}{1-\delta_u} d(x, Tx).$$

7.2. Cyclical almost φ-contractions

Using the comparison functions and their properties, presented in Section 3 of chapter 1, we can extend the results from the previous section, in the following way. The concepts and results in this subsection were introduced in our paper [**126**].

Definition 2.7.3. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space (X, d) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be an cyclic operator, i.e., it satisfies (1.1). If there exists φ a (c)-comparison function and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(y, Tx), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{2.31}$$

then T is a *cyclic φ-almost contraction*.

Clearly, any cyclic almost contraction is cyclic almost φ-contraction, with $\varphi(t) = \delta t$, $t \in \mathbb{R}_+$ and $0 < \delta < 1$. Theorem 2.7.1 could now be easily extended to cyclic almost φ-contractions.

Theorem 2.7.3. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X, d) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be an cyclic φ-almost contraction. Then*

(i) T has at least one fixed point in $\bigcap_{i=1}^p A_i$

(ii) the Picard iteration $\{x_n\}$ given by (2.7) converges to some x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$;

(iii) the following estimates hold:

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), n \geq 0, \tag{2.32}$$

$$d(x_n, x^*) \leq s(\varphi(d(x_n, x_{n-1}))), n \geq 1, \tag{2.33}$$

where $s(t) = \sum_{n=0}^{\infty} \varphi^n(t)$, for all $t \in \mathbb{R}_+$.

It is possible to force the uniqueness of the fixed point of a cyclic almost φ-contraction, like in the case of simple almost contraction, by imposing an additional contractive condition, quite similar to (2.30), as shows the next definition

Definition 2.7.4. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space (X, d) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be an cyclic operator, i.e., it satisfies (1.1). If T is a cyclic almost φ-contraction with φ a (c)-comparison function and there exists ψ a comparison function and some $L_1 \geq 0$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)) + L_1 d(x, Tx), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p. \tag{2.34}$$

where $A_{p+1} = A_1$, then T is a *cyclic strict almost φ-contraction*.

Regarding this definition we have the following new result.

Theorem 2.7.4. *Let* $\{A_i\}_{i=1}^p$ *be nonempty closed subsets of a complete metric space* (X,d) *and* $T:\bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ *be an cyclic strict almost* φ*-contraction. Then*

(i) T *has at least one fixed point in* $\bigcap\limits_{i=1}^p A_i$*;*

(ii) The Picard iteration $\{x_n\}_{n\geq 0}$ *defined by (2.7) converges to some* $x^* \in F_T$*, for any starting point* $x_o \in \bigcup\limits_{i=1}^p A_i$*.*

(iii) The error estimates are given by (2.32) and (2.33).

(iv) The rate of the convergence is given by

$$d(x_n,x^*) \leq \psi(d(x_{n-1},x^*)), n \geq 1.$$

8. Cyclical Operators of Ćirić Type with a Non-unique Fixed Point

A fixed point of a mapping usually corresponds to a solution of certain equation. Some equations have more than one solution. For solving an equation, one care only the existence of a solution and how to approximate it. The concepts and results in this subsection were introduced in our paper [**121**].

In 1974 Ćirić introduced and considered the class of mappings which satisfy the following contractive-type condition:

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\} - \min\{d(x,Ty),d(y,Tx)\} \leq \lambda d(x,y) \tag{2.35}$$

where $\lambda \in [0,1)$.

Remark 2.8.1. The class of almost contractions is independent of the class of Ćirić operators satisfying (2.35). Indeed, see Example 4.3, pp 80 in [**37**]:

Let $X=[0,1]$ with the usual norm and $T:[0,1]\to[0,1]$ be defined by

$$Tx = \begin{cases} \dfrac{x}{2}, & x \in (0,1] \\ 1, & x = 0 \end{cases}$$

T does satisfy (2.35) but does not satisfy (2.7.3), since T is fixed point free and any almost contraction has a fixed point. This this motivates the study of Ćirić operators satisfying (2.35).

We can also prove a corresponding results for cyclic operators which satisfy this contractive type condition.

Theorem 2.8.1. *Let* (X,d) *be a complete metric space and* $\{A_i\}_{i=1}^p$ *be nonempty, closed subsets of* X*. Let* $T:\bigcup\limits_{i=1}^p A_i \to \bigcup\limits_{i=1}^p A_i$ *be an cyclical orbitally continuous mapping. Suppose that* T *satisfies (2.35) for each* $x\in A_i, y \in A_{i+1}, 1\leq i\leq p$*. Then*

(i) T *has at least one fixed point in* $\bigcap\limits_{i=1}^p A_i$*;*

(ii) The Picard sequence $\{T^n x\}$ *is convergent for each starting point* $x \in \bigcup\limits_{i=1}^p A_i$ *and its limit is a fixed point of* T*;*

(iii) The following error estimates hold

$$d(T^n x, x^*) \le \frac{\lambda^n}{1-\lambda} d(x, Tx), n \ge 0.$$

$$d(T^n x, x^*) \le \frac{\lambda}{1-\lambda} d(T^{n-1}x, T^n x), n > 0.$$

Now we will consider a contractive condition stronger then (2.35), which guarantees the existence of a fixed point, without added condition T to be (orbitally) continuous (see [**37**]).

Theorem 2.8.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a cyclical mapping satisfying:*

$$\min\{d(Tx,Ty), \max\{d(x,Tx), d(y,Ty)\}\} - \min\{d(x,Ty), d(y,Tx)\} \le \lambda d(x,y) \tag{2.36}$$

for all $x \in A_i, y \in A_{i+1}, 1 \le i \le p$, where $\lambda \in [0,1)$. Then

(i) T has at least one fixed point in $\bigcap_{i=1}^p A_i$.

(ii) The Picard sequence $\{T^n x\}$ is convergent for each starting point $x \in \bigcup_{i=1}^p A_i$ and its limit is a fixed point of T.

(iii) The following error estimates hold

$$d(T^n x, x^*) \le \frac{\lambda^n}{1-\lambda} d(x, Tx), \forall n \ge 0.$$

$$d(T^n x, x^*) \le \frac{1}{1-\lambda} d(T^{n-1}x, T^n x), \forall n > 0.$$

Ćirić and Jotić [**41**] considered a wider class of maps which still have a non-unique fixed point. We will prove the cyclic version of this theorem.

Theorem 2.8.3. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a cyclical orbitally continuous mapping satisfying:*

$$\begin{aligned} \min & \Big\{ d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \\ & \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty)\}}{d(x,y)} \Big\} \\ - & \, a \min\{d(x,Ty), d(y,Tx)\} \le \lambda \max\{d(x,y), d(x,Tx)\} \end{aligned} \tag{2.37}$$

for all $x \in A_i, y \in A_{i+1}, 1 \le i \le p$, $x \ne y$, where $a \ge 0$ and $\lambda \in [0,1)$. Then

(i) T has at least one fixed point in $\bigcap_{i=1}^p A_i$.

(ii) The Picard sequence $\{T^n x\}$ is convergent for each starting point $x \in \bigcup_{i=1}^p A_i$ and its limit is a fixed point of T.

(iii) The following error estimates hold

$$d(T^n x, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x, Tx), n \geq 0.$$

$$d(T^n x, x^*) \leq \frac{1}{1-\lambda} d(T^{n-1}x, T^n x), n > 0.$$

In [**41**] is considered another contractive condition, quite similar to (2.37). But in this case the space must be T-orbitally complete, and also it is needed that T to be orbitally continuous. We will state now the cyclic version for this result.

Theorem 2.8.4. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a T-orbitally complete metric space (X, d) and $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ be a cyclical orbitally continuous mapping satisfying:*

$$\min\left\{d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Ty)[1 + d(x, y)]}{1 + d(x, Tx)}\right\}$$

$$- a \min\{d(x, Ty), d(y, Tx)\} \leq \lambda \max\{d(x, y), d(x, Tx)\}. \tag{2.38}$$

for all $x \in A_i, y \in A_{i+1}, 1 \leq i \leq p$, where $a \geq 0$ and $\lambda \in [0, 1)$. Then

(i) T has at least one fixed point in $\bigcap\limits_{i=1}^{p} A_i$.

(ii) The Picard sequence $\{T^n x\}$ is convergent for each starting point $x \in \bigcup\limits_{i=1}^{p} A_i$ and its limit is a fixed point of T.

(iii) The following error estimates hold

$$d(T^n x, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x, Tx), n \geq 0.$$

$$d(T^n x, x^*) \leq \frac{1}{1-\lambda} d(T^{n-1}x, T^n x), n > 0.$$

Open Problem Almost contractions and Ćirić non-unique type conditions are related? Find examples.

9. Applications of Fixed Point Structure Theory

We remark that using Lemma 2.1.1 and the definition of a fixed point structure we also can prove some of the results from this chapter. Therefore those theorems can be considered as applications of the fixed point structure theory.

10. Common Fixed Points of Contractive Type Operators

In this section we present some applications of fixed point theorems for cyclic operators to common fixed point results. We will consider the case of $p \geq 2$ sets.

Corollary 2.10.1. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and suppose $f_i : A_i \to A_{i+1}$, for $1 \le i \le p$, with $A_{p+1} = A_1$, be p functions such that:*

$$d(f_i(x), f_{i+1}(y)) \le kd(x,y), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \le i \le p \tag{2.39}$$

where $k \in [0,1)$. Then there exists a unique $x_0 \in \bigcap_{i=1}^{p} A_i$ such that

$$f_i(x_0) = x_0, \text{ for all } i \in \{1, 2, \dots, p\}.$$

Corollary 2.10.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and suppose $f_i : A_i \to A_{i+1}$, for $1 \le i \le p$, with $A_{p+1} = A_1$, be p functions such that:*

$$d(f_i(x), f_{i+1}(y)) \le \varphi(d(x,y)), \text{ for all } x \in A_i, y \in A_{i+1}, 1 \le i \le p \tag{2.40}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function. Then there exists a unique $x_0 \in \bigcap_{i=1}^{p} A_i$ such that

$$f_i(x_0) = x_0, \text{ for all } i \in \{1, 2, \dots, p\}.$$

11. Maia's Fixed Point Theorem for Cyclical Operators

We end this chapter by stating fixed point theorems of Maia's type for cyclic operators. A Maia type result regarding cyclic φ-contractions with φ a (c)-comparison function was given in [**91**], in the following form:

Theorem 2.11.1. ([**91**]) *Let X be a nonempty set, d and ρ two metrics on X, p a positive integer, $A_1, A_2, \dots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$ an operator. We assume that:*

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of Y with respect to T;*
(ii) $d(x,y) \le \rho(x,y)$, *for any $x, y \in Y$;*
(iii) (X,d) *is a complete metric space;*
(iv) $T : (Y,d) \to (Y,d)$ *is continuous;*
(v) $T : (Y,\rho) \to (Y,\rho)$ *is a cyclic φ-contraction with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a (c)-comparison function.*

Then $F_T = \{x^*\}$ and the sequence $\{T^n x_0\}$ converges to x^* in (X,d), for any $x_0 \in X$.

Notice that condition (v) in Theorem 2.11.1 may be replaced by one of the following conditions:

(i) $T : (Y,\rho) \to (Y,\rho)$ is a cyclic contraction, i.e., satisfies (1.5);
(ii) $T : (Y,\rho) \to (Y,\rho)$ is a cyclic Kannan operator, i.e., satisfies (2.12);
(iii) $T : (Y,\rho) \to (Y,\rho)$ is a cyclic Chatterjea operator, i.e., satisfies (2.16);
(iv) $T : (Y,\rho) \to (Y,\rho)$ is a cyclic Zamfirescu operator;

(v) $T : (Y, \rho) \to (Y, \rho)$ is a cyclic Bianchini operator, i.e., satisfies (2.20);

(vi) $T : (Y, \rho) \to (Y, \rho)$ is a cyclic Reich-rus type operator;

(vii) $T : (Y, \rho) \to (Y, \rho)$ is a cyclic Hardy-Rogers operator, i.e., satisfies (2.25);

(viii) $T : (Y, \rho) \to (Y, \rho)$ is a cyclic strict almost φ-contractions.

As it is an easy task to construct the Maia type results for this cyclic contractive-type conditions, we will present only the one corresponding to cyclic strict almost φ-contractions.

Theorem 2.11.2. *Let X be a nonempty set, d and ρ two metrics on X, p a positive integer, $A_1, A_2, \ldots, A_p \in P_{cl}(X)$, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$ an operator. We assume that:*

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(ii) $d(x, y) \le \rho(x, y)$, for any $x, y \in Y$;

(iii) (X, d) is a complete metric space;

(iv) $T : (Y, d) \to (Y, d)$ is continuous;

(v) $T : (Y, \rho) \to (Y, \rho)$ is a cyclic strict almost φ-contraction with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a (c)-comparison function, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function.

Then $F_T = \{x^*\}$ and the sequence $\{T^n x_0\}$ converges to x^* in (X, d), for any $x_0 \in X$.

For other Maia type results see for example [**77, 78**].

CHAPTER 3

Fixed Foint and Strict Fixed Point Principles for Multivalued Cyclic Contraction Operators

A very large and a very important branch of fixed point theory is devoted to the fixed point theory for multivaled operators. The study of fixed point theorems for multivalued operators has been initiated by Markin [**73**] and Nadler [**79**]. The fundamental result in this theory is Nadler's theorem, publised in 1969 in [**79**]. Since the pioneering works of Markin [**73**] and Nadled [**79**], an extensive literature has been developed, consisting in many theorems which deal with fixed point for multivalued operators, see for example [**2, 3, 37, 38, 39, 44, 59, 61, 62, 71, 72, 75, 81, 94, 96, 99, 97, 103, 107, 112, 100, 114, 115**].

This chapter is a natural continuation of Chapter 2, and presents fixed point theorems for cyclic multivalued operators. All the new results extend and generalize the fundamental results for multivalued mappings.

The first section of this chapter presents the following abstract results: intersection, fixed points and periodic points theorems for multivalued cyclical operators on a metric space with cyclic representation, obtained by I.A. Rus, A. Petruşel and G. Petruşel in [**111**].

The second section of this chapter is concerned with obtaining fixed point theorems for multivaled cyclic contractions, Kannan, Chatterjea, Zamfirescu, Bianchini, Reich-Rus operators and cyclic almost contractions. Further, we give a version of a generalized multivalued almost contraction considered in [**111**]

The last section of this chapter discusses strict fixed point results for multivaled cyclic Kannan, Bianchini and Reich-Rus operators.

Author's original contributions in this chapter are: Theorems 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5, 3.2.6, 3.2.7, 3.2.8, 3.3.1, , 3.3.2, 3.3.3.

1. Intersection, Fixed Points and Periodic Points Theorems for Multivalued Cyclical Operators

In this section we consider the following abstract results for a multivalued operator on a metric space with cyclic representations, included in [**111**].

Theorem 3.1.1. ([**111**]) *Let $(X, \to)$ be an L-space, $T : X \to P(X)$ a multi-valued operator and $X = \bigcup_{i=1}^{p} A_i$ be a cyclic representation of X with respect to T. Suppose that:*

(i) $A_i \in P_{cl}(X)$, *for all* $i \in \{1, 2, \ldots, p\}$;

(ii) there exists a convergent sequence $\{x_n\}$, *where* $x_n \in X$, $x_{n+1} \in T(x_n)$, *for each* $n \in \mathbb{N}$.

Then $\bigcap_{i=1}^{p} A_i \neq \emptyset$.

As consequence of the above result we have the following fixed point theorem.

Theorem 3.1.2. ([**111**]) *Let $(X, S(X), \mathbb{M})$ be a fixed point structure, where $(X, \to)$ is an L-space. Let $A_i \in P_{cl}(X)$, for each $i \in \{1, 2, \ldots, p\}$. Denote $Y = \bigcup_{i=1}^{p} A_i$ and consider $T : Y \to P(Y)$ a multi-valued operator. Suppose that:*

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of* Y *with respect to* T;

(ii) there exists a convergent sequence $\{x_n\}$, *where* $x_n \in X$, $x_{n+1} \in T(x_n)$, *for each* $n \in \mathbb{N}$;

(iii) if $A = \bigcap_{i=1}^{p} A_i \neq \emptyset$ *then* $A \in S(X)$ *and* $T|_A \in M(A)$.

Then $F_T \neq \emptyset$.

Some applications of the above results can be obtained by taking in Theorem 3.1.2 some particular fixed point structures.

Theorem 3.1.3. ([**111**]) *Let $(X, S(X), \mathbb{M})$ be a fixed point structure, where X is a nonempty set. Let $A_i \in P(X)$, for each $i \in \{1, 2, \ldots, p\}$. Denote $Y = \bigcup_{i=1}^{p} A_i$ and consider $T : Y \to P(Y)$ a multi-valued operator. Suppose that:*

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of* Y *with respect to* T;

(ii) $A_i \in S(X)$ *for some* $i \in \{1, 2, \ldots, p\}$;

(iii) $G_1, G_2 \in \mathbb{M}$ *implies* $G_1 \circ G_2 \in \mathbb{M}$.

Then $F_{T^p} \neq \emptyset$.

2. Fixed Point Principles for Multivalued Cyclical Operators

Starting from the background of the previous chapters, the main aim of this section is to extend the classical concepts of contractive type operators from the single-valued case to multivalued case. In this way we shall obtain general fixed point theorems that extend, improve and unify a multitude of corresponding results in literature dedicated to fixed point theory, for both single-valued and multivalued operators. To prove the new results in this section we shall need the properties of functional on $P(X)$ presented in the first chapter of this thesis.

All the results from this section will be considered in our paper Petric [**122**].

2.1. A fixed point theorem for multivalued cyclical contraction operators

We consider the extension of Nadler's theorem to multivalued cyclic operators.

Theorem 3.2.1. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let*

$$T:\bigcup_{i=1}^p A_i \to P\Big(\bigcup_{i=1}^p A_i\Big)$$

be a multivalued cyclic contraction, i.e., it satisfies

$$T(A_i)\subseteq A_{i+1},$$

for all $i\in\{1,2,\dots,p\}$, where $A_{p+1}=A_1$, and

$$H(Tx,Ty)\le kd(x,y), \text{for all } x\in A_i, y\in A_{i+1}, 1\le i\le p, \tag{3.41}$$

where $k\in(0,1)$. Then there exists a sequence of successive approximations of T starting from $x_0\in\bigcup_{i=1}^p A_i$ which converges to a fixed point $x^*\in Tx^*\subseteq\bigcap_{i=1}^p A_i$.

2.2. A fixed point theorem for multivalued cyclical Kannan operators

Theorem 3.2.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T:\bigcup_{i=1}^p A_i \to P\Big(\bigcup_{i=1}^p A_i\Big)$ be a multivalued cyclic Kannan operator, i.e., it satisfies*

$$T(A_i)\subseteq A_{i+1},$$

for all $i\in\{1,2,\dots,p\}$, where $A_{p+1}=A_1$, and

$$H(Tx,Ty)\le a[D(x,Tx)+D(y,Ty)], \text{for all } x\in A_i, y\in A_{i+1}, 1\le i\le p, \tag{3.42}$$

where $a\in(0,\frac{1}{2})$. Then there exists a sequence of successive approximations of T starting from $x_0\in\bigcup_{i=1}^p A_i$ which converges to a fixed point $x^*\in Tx^*\subseteq\bigcap_{i=1}^p A_i$.

2.3. A fixed point theorem for multivalued cyclical Chatterjea operators

Theorem 3.2.3. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T:\bigcup_{i=1}^p A_i \to P\Big(\bigcup_{i=1}^p A_i\Big)$ be a multivalued cyclic Chatterjea operator, i.e., it satisfies*

$$T(A_i)\subseteq A_{i+1},$$

for all $i\in\{1,2,\dots,p\}$, where $A_{p+1}=A_1$, and

$$H(Tx,Ty)\le b[D(x,Ty)+D(y,Tx)], \text{for all } x\in A_i, y\in A_{i+1}, 1\le i\le p, \tag{3.43}$$

where $b\in(0,\frac{1}{2})$. Then there exists a sequence of successive approximations of T starting from $x_0\in\bigcup_{i=1}^p A_i$ which converges to a fixed point $x^*\in Tx^*\subseteq\bigcap_{i=1}^p A_i$.

2.4. A fixed point theorem for multivalued cyclical Zamfirescu operators

Theorem 3.2.4. *Let $A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space X and $T : \bigcup_{i=1}^{p} A_i \to P\Big(\bigcup_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Zamfirescu operator, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \ldots, p\}$, where $A_{p+1} = A_1$, and there exists exist the real numbers $a \in (0, 1), b \in (0, \frac{1}{2})$ and $c \in (0, \frac{1}{2})$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $1 \le i \le p$, at least one of the following is true:

(z1) $H(Tx, Ty) \le ad(x, y)$;
(z2) $H(Tx, Ty) \le b\Big[D(x, Tx) + D(y, Ty)\Big]$;
(z3) $H(Tx, Ty) \le c\Big[D(x, Ty) + D(y, Tx)\Big]$.

Then there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup_{i=1}^{p} A_i$ which converges to a fixed point $x^* \in Tx^* \subseteq \bigcap_{i=1}^{p} A_i$.

2.5. A fixed point theorem for multivalued cyclical Bianchini operators

Theorem 3.2.5. *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space (X, d), and let $T : \bigcup_{i=1}^{p} A_i \to P\Big(\bigcup_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Bianchini mappings, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \ldots, p\}$, where $A_{p+1} = A_1$, and

$$H(Tx, Ty) \le h \max\{D(x, Tx), D(y, Ty)\}, \text{for all } x \in A_i, y \in A_{i+1}, 1 \le i \le p, \tag{3.44}$$

where $h \in (0, 1)$. Then there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup_{i=1}^{p} A_i$ which converges to a fixed point $x^* \in Tx^* \subseteq \bigcap_{i=1}^{p} A_i$.

2.6. A fixed point theorem for multivalued cyclical Reich-Rus operators

Theorem 3.2.6. *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space (X, d), and let $T : \bigcup_{i=1}^{p} A_i \to P\Big(\bigcup_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Reich-Rus operators, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \ldots, p\}$, where $A_{p+1} = A_1$, and

(3.45)

$$H(Tx, Ty) \le ad(x, y) + b[D(x, Tx) + D(y, Ty)], \text{for all } x \in A_i, y \in A_{i+1}, 1 \le i \le p,$$

where $a, b \in \mathbb{R}_+$ with $a + 2b < 1$. Then there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup_{i=1}^{p} A_i$ which converges to a fixed point $x^* \in Tx^* \subseteq \bigcap_{i=1}^{p} A_i$.

2.7. A fixed point theorem for multivalued cyclical almost contractions

In this section the concept of almost contraction from the both single valued and multivalued case is extended to multivalued cyclic almost contraction and the corresponding convergence theorem for the Picard iteration associated to multivalued cyclic almost contraction are obtained.

Theorem 3.2.7. *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space (X, d), and let $T : \bigcup_{i=1}^{p} A_i \to P\Big(\bigcup_{i=1}^{p} A_i\Big)$ be a multivalued cyclic almost contraction or a multivalued (ϕ, L)-almost contraction, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \dots, p\}$, where $A_{p+1} = A_1$, and there exist two constants $\phi \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \phi d(x, y) + LD(y, Tx), \text{for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p. \tag{3.46}$$

Then there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup_{i=1}^{p} A_i$ that converges to a fixed point $x^* \in Tx^* \subseteq \bigcap_{i=1}^{p} A_i$, for which the following estimates hold:

$$d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x_0, x_1), n \geq 0, \tag{3.47}$$

$$d(x_n, x^*) \leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n), n \geq 1, \tag{3.48}$$

for a certain constant $\alpha < 1$.

Remark 3.2.1. Due to the symmetry of the metrics d and H, in order to check that T is a multivalued cyclic almost contraction we have also to check the dual of (3.46), that is to check that T verifies:

$$H(Tx, Ty) \leq \phi d(x, y) + LD(x, Ty).$$

Remark 3.2.2. By taking $A_i = A_j$, for all i, j in Theorem 3.2.7 we can obtain Theorem 3 in M Berinde and V. Berinde [**9**]. This contractive condition introduced by V. Berinde is really very general because it doesn't ask $\phi + L$ be less than 1, as happens in almost all fixed point theorems based on contractive conditions that involve one or more of the displacements:

$$d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx).$$

Example 3.2.1. ([**79**]) Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ be a multivalued cyclic α-contraction, with $\alpha \in (0,1)$. Then T is a multivalued cyclic almost contraction. Nadler's fixed point theorem can be obtained by taking .

Example 3.2.2. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Kannan operator. Then T is a multivalued cyclic almost contraction.

Example 3.2.3. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Chatterjea operator. Then T is a multivalued cyclic almost contraction.

Example 3.2.4. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Zamfirescu operator. Then T is a multivalued cyclic almost contraction.

Example 3.2.5. ([**94, 95, 96**]) Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ be a multivalued cyclic Reich operator for which there exist $a, b, c \in \mathbb{R}_+$, $a + b + c < 1$ such that

$$H(Tx, Ty) \le ad(x, y) + bD(x, Tx) + cD(y, Ty),$$

for all $x \in A_i, y \in A_{i+1}, 1 \le i \le p$. Then T is a multivalued cyclic almost contraction.

2.8. A fixed point theorem for generalized multivalued cyclical almost contractions

One can extend all results in this section by considering various generalizations of Nadled's fixed point theorem, see M Berinde and V. Berinde [**9**] and references therein, by considering by considering instead of the term $\phi d(x,y)$ in (3.46) the expression

$$k\Big(d(x,y)\Big)d(x,y), \text{ where } k : [0,\infty) \to [0,1)$$

is a function satisfying certain conditions.

We will present the cyclic version of Theorem 4 in [**9**].

Theorem 3.2.8. *Let* $\{A_i\}_{i=1}^p$ *be nonempty closed subsets of a complete metric space* (X,d)*, and let* $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ *be a generalize multivalued cyclic* (α, L)*-almost contraction, i.e.,*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \ldots, p\}$*, where* $A_{p+1} = A_1$*, and there exists a function* $\alpha : [0,\infty) \to [0,1)$ *satisfying*

$$\limsup_{r \to t^+} \alpha(r) < 1, \text{ for every } t \in [0,\infty),$$

such that

(3.49)
$$H(Tx,Ty) \leq \alpha\Big(d(x,y)\Big)d(x,y) + LD(y,Tx), \text{ for all } x \in A_i, y \in A_{i+1}, i \leq i \leq p,$$

Then T has at least one fixed point.

3. Strict Fixed Point Principles for Multivalued Cyclical Operators

The following results are included in the original paper [**123**].

3.1. A strict fixed point theorem for multivalued cyclical Kannan operators

Theorem 3.3.1. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^p A_i \to P\Big(\bigcup\limits_{i=1}^p A_i\Big)$ be a multivalued cyclic Kannan operator, i.e. it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1,2,\ldots,p\}$, where $A_{p+1} = A_1$, and

$$\delta(Tx,Ty) \leq a[\delta(x,Tx) + \delta(y,Ty)], \text{for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{3.50}$$

where $a \in (0,1/2)$. Then

(i) T has a unique strict fixed point in $\bigcap\limits_{i=1}^p A_i$;

(ii) there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup\limits_{i=1}^p A_i$ which converges to a fixed point $x^* \in Tx^* \subseteq \bigcap\limits_{i=1}^p A_i$;

(iii) for all $n \geq 0$ we have the following a priori estimate

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \tag{3.51}$$

where $\alpha := \dfrac{aq}{1-a}$, with arbitrary $q < \dfrac{1-a}{1}$.

3.2. A strict fixed point theorem for multivalued cyclical Bianchini operators

Theorem 3.3.2. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d), and let $T : \bigcup\limits_{i=1}^p A_i \to P\Big(\bigcup\limits_{i=1}^p A_i\Big)$ be a multivalued cyclic Bianchini operator, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1,2,\ldots,p\}$, where $A_{p+1} = A_1$, and

$$\delta(Tx,Ty) \leq h \max\{\delta(x,Tx), \delta(y,Ty)\}, \text{ for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{3.52}$$

where $h \in (0,1)$. Then

(i) T has a unique strict fixed point in $\bigcap\limits_{i=1}^{p} A_i$;

(ii) there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup\limits_{i=1}^{p} A_i$ *which converges to a fixed point* $x^* \in Tx^* \subseteq \bigcap\limits_{i=1}^{p} A_i$;

(iii) for all $n \geq 0$ *we have the following a priori estimate*

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \tag{3.53}$$

where $\alpha := \frac{h}{q}$, *with arbitrary* $h < q < 1$.

3.3. A strict fixed point theorem for multivalued cyclical Reich-Rus operators

Theorem 3.3.3. *Let* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of a complete metric space* (X, d), *and let* $T : \bigcup\limits_{i=1}^{p} A_i \to P\Big(\bigcup\limits_{i=1}^{p} A_i\Big)$ *be a multivalued Reich-Rus operator, i.e., it satisfies*

$$T(A_i) \subseteq A_{i+1},$$

for all $i \in \{1, 2, \dots, p\}$, *where* $A_{p+1} = A_1$, *and*

$$\delta(Tx, Ty) \leq ad(x, y) + b[\delta(x, Tx) + \delta(y, Ty)], \text{for all } x \in A_i, y \in A_{i+1}, 1 \leq i \leq p, \tag{3.54}$$

where $a, b, c \in \mathbb{R}_+$, *with* $a + b + c < 1$. *Then*

(i) T has a unique strict fixed point in $\bigcap\limits_{i=1}^{p} A_i$;

(ii) there exists a sequence of successive approximations of T starting from $x_0 \in \bigcup\limits_{i=1}^{p} A_i$ *which converges to a fixed point* $x^* \in Tx^* \subseteq \bigcap\limits_{i=1}^{p} A_i$;

(iii) for all $n \geq 0$ *we have the following a priori estimate*

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \tag{3.55}$$

where $\alpha := \frac{aq + b}{q(1-c)}$, *with arbitrary* $q < \frac{1-a-c}{1-b}$.

CHAPTER 4

Best Proximity Points for Cyclical Contractive Operators

The main aim of this chapter is to obtain existence and convergence results for best proximity points considering different contractive type conditions. Kirk, Srinivasan and Veeramani in [**70**], introduced the notion of contractions under cyclic conditions. Actually in this case the problem is solved under the hypothesis that the intersection of the sets involved in the cyclic contraction is nonempty, that is $A \cap B \neq \emptyset$. Moreover, the fixed points are situated in the intersection set. In the case $A \cap B = \emptyset$, it is natural to try to find an approximative solution for the fixed point problem, that is a *best proximity point*.

In this chapter we will extend further Theorem 1.6.1, using the technique in [**47**], by considering certain contractive conditions. The fixed point theorems are usually proven by using the fact that the geometric series $\sum\limits_{n=0}^{\infty} q^n$ is convergent for $q \in [0, 1)$. The A. A. Eldred and P. Veeramani technique does not use the geometric series.

The best proximity point theorems are obtained in the framework of a uniformly convex Banach space, and the contractive condition imposed to the operator is weakened. Basically, we are speaking about weak cyclic Kannan contractions, weak cyclic Chatterjea contractions, weak Bianchini contractions and weak cyclic Reich-Rus contractions. In notion of weak cyclic Kannan contraction was generalized to weak p-cyclic Kannan contraction. In this case it was proved that the distance between two adjacent sets are equal.

Let us mention that almost all the papers concerned with best proximity points for cyclic operators present results for operators with nonexpansivity property, see for example [**65, 67, 7, 1, 116, 48, 49**].

Author's original contributions in this chapter are: Definitions 4.2.1, 4.3.1, 4.4.1 and 4.5.1, Examples 4.1.1, 4.2.1 and 4.2.2, Lemmas 4.1.1 and 4.2.1, Remarks 4.1.1, 4.2.1, 4.3.1, 4.4.1 and 4.5.1, Theorems 4.2.1, 4.2.2, 4.2.3, 4.3.1, 4.3.2, 4.4.1, 4.4.2, 4.5.1 and 4.5.2.

1. Preliminaries

In [**47**] A. Antony Eldred and P.Veeramani extended Theorem 1.6.1 to the case when $A \cap B = \emptyset$, and in this case they didn't asked for the existence of a fixed point of

T but for the existence of a *best proximity point*, that is, a point x in $A \cup B$ such that

$$d(x, Tx) = D(A, B).$$

where

$$D(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}.$$

Let us mention that an operator is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \text{ for all } x, y \in X.$$

Example 4.1.1. Let $X = [0, 1]$ with the usual norm and $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \begin{cases} \frac{2}{5}, & x \in \left[0, \frac{2}{3}\right) \\ \frac{1}{5}, & \left[\frac{2}{3}, 1\right] \end{cases}$$

T is a Kannan operator (see [**88**]), but T is not a nonexpansive operator. Take, for example $x = \frac{16}{25}$ and $y = \frac{17}{25}$. In this particular the nonexpansive condition becomes

$$\frac{1}{5} \leq \frac{1}{25},$$

which is impossible.

This example lead us to investigate some classes of contractive type conditions which do not imply the nonexpansivity property of the operators.

In order to obtain the existence and uniqueness of a best proximity point we will use two important convergence lemma's given in [**47**] and reasserted in the first chapter of this thesis: Lemma 1.6.1 and Lemma 1.6.2. The following lemma is also an important tool used in the proofs the new results.

Lemma 4.1.1. (Petric [**119**]) *Let X be a uniformly convex Banach space, A and B be two nonempty, closed, convex subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic operator satisfying*

$$\|Tx - T^2x\| \leq \alpha\|x - Tx\| + (1 - \alpha)D(A, B), \text{ for all } x \in A \cup B, \tag{4.56}$$

where $\alpha \in [0, 1)$. Then

(i) $\|T^n x - T^{n+1}x\| \leq \alpha^n\|x - Tx\| + (1 - \alpha^n)D(A, B)$, for all $x \in A \cup B$ and $n \geq 0$.
(ii) $\|T^n x - T^{n+1}x\| \to D(A, B)$, as $n \to \infty$, for all $x \in A \cup B$.
(iii) $\|T^{2n}x - T^{2n+-2}x\| \to 0$, as $n \to \infty$, for all $x \in A \cup B$.
(iv) z is a best proximity point if and only if z is a fixed point of T^2

Remark 4.1.1. The uniform convexity of the space X needed to prove (iii) and (iv). Therefore (i) and (ii) can be used even if we have a metric space.

2. Best Proximity Point Theorems for Weak Cyclic Kannan Contractions

In this section we will introduce a new class of contraction, called *weak cyclic Kannan contraction*, and we will give convergence and existence results for best proximity points. The results in this section are included in the original paper [**119**].

Definition 4.2.1. Let (X, d) be a metric space and let A and B be nonempty subsets of X. Then an cyclic operator $T : A \cup B \to A \cup B$ is called *weak cyclic Kannan contraction* if it satisfies the following condition

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + (1 - 2a)D(A, B), \text{ for all } x \in A, y \in B, \tag{4.57}$$

where $a \in \left[0, \frac{1}{2}\right)$.

Example 4.2.1. Let $X = l^p$, $1 \leq p \leq \infty$ and $k \in (0, 1)$. Define two sets by

$$A = \{((1 + k^{2n})e_{2n}) : n \in \mathbb{N}\}, B = \{((1 + k^{2m-1})e_{2m-1}) : m \in \mathbb{N}\}.$$

Define an operator $T : A \cup B \to A \cup B$ by

$$T((1 + k^{2n})e_{2n}) = (1 + k^{2n+1})e_{2n+1},$$

and

$$T((1 + k^{2m-1})e_{2m-1}) = (1 + k^{2m})e_{2m}.$$

Then $D(A, B) = 2^{\frac{1}{p}}$ and T is a cyclic contraction, see [**47**]. Also T is a weak cyclic Kannan contraction. Indeed, we have

$$\begin{aligned}
\Big[(1 \; &+ \; k^{2n+1})^p + (1 + k^{2m})^p\Big]^{\frac{1}{p}} \\
&\leq \left[\left(1 - k + \frac{k^{2n+1}}{2} + \frac{k^{2n+1}}{2} + k\right)^p + \left(1 - k + \frac{k^{2m}}{2} + \frac{k^{2m}}{2} + k\right)^p\right]^{\frac{1}{p}} \\
&\leq \left[\left(1 - k + \frac{k^{2n+1}}{2} + \frac{k^{2n+2}}{2} + k\right)^p + \left(1 - k + \frac{k^{2m}}{2} + \frac{k^{2m+1}}{2} + k\right)^p\right]^{\frac{1}{p}} \\
&\leq \frac{k}{2}\left\{\Big[(1 + k^{2n})^p + (1 + k^{2n+1})^p\Big]^{\frac{1}{p}} + \Big[(1 + k^{2m-1})^p + (1 + k^{2m})^p\Big]^{\frac{1}{p}}\right\} \\
&+ \; (1 - k)2^{\frac{1}{p}}.
\end{aligned}$$

Consequently, T is weak cyclic Kannan contraction with $a := \frac{k}{2} \in \left(0, \frac{1}{2}\right)$.

Example 4.2.2. Let $X = \mathbb{R}$ with the usual norm, and

$$0 \leq b \leq \alpha \leq c \leq e \leq \beta \leq f,$$

be real numbers. Let's denote by $A := [b, c]$ and $B := [e, f]$ and define an operator $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \beta & , \text{if } b \leq x \leq c \\ \alpha & , \text{if } e \leq x \leq f \end{cases}.$$

Obviously T is a cyclical map. We want to find out in which conditions this is a weak cyclic Kannan operator.

Suppose $c < e$. Then D(A,B)=e-c. Let $x \in A$ and $y \in B$. In this particular situation condition (4.57) becomes:

$$\beta - \alpha \leq a(y - x + \beta - \alpha) + (1 - 2a)(e - c). \tag{4.58}$$

As $b \leq x \leq c$ and $e \leq y \leq f$ it follows that

$$(y - x + \beta - \alpha) \in [e - c + \beta - \alpha, f - b + \beta - \alpha].$$

Since the minimum value of the right side in (4.58) is $a(e-c+\beta-\alpha)+(1-2a)(e-c)$, then for (4.58) to hold it is necessary that

$$\beta - \alpha \leq a(e - c + \beta - \alpha) + (1 - 2a)(e - c),$$

which implies $\beta - \alpha \leq e - c$, that is true only for $\beta = e$ and $\alpha = c$.

If $c = e$ then $D(A, B) = 0$ and (4.57), for $x \in A, y \in B$, becomes:

$$\beta - \alpha \leq a(y - x + \beta - \alpha). \tag{4.59}$$

Since the minimum value of the right side in (4.59) is $a(\beta - \alpha)$, so for (4.59) to hold it is necessary that

$$\beta - \alpha \leq a(\beta - \alpha),$$

which is impossible since $a \in \left(0, \frac{1}{2}\right)$.

Analyzing these situations we conclude that in order for (4.57) to be fullfiled, T must be defined by

$$Tx = \begin{cases} e & , \text{if } b \leq x \leq c \\ c & , \text{if } e \leq x \leq f \end{cases},$$

where $0 \leq b \leq c < e \leq f$, are real numbers. It is easy to see that T is also a cyclic contraction.

Example 4.2.3. ([**97**]) Let $A = B = [0, 1]$ be two subsets of $X = \mathbb{R}$ with the usual norm, and a map defined by $Tx = \dfrac{x}{3}$, for all $x \in [0, 1]$.

Then $D(A, B) = 0$. T is cyclic contraction but it is not a weak cyclic Kannan contraction.

Example 4.2.4. ([**26**]) Let $A = B = \mathbb{R}$ be two subsets of $X = \mathbb{R}$ with the usual norm, and a map $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = \begin{cases} 0 & , \text{if } x \in (-\infty, 2] \\ -\frac{1}{2} & , \text{if } x \in (2, \infty) \end{cases},$$

Then $D(A, B) = 0$. T is not a cyclic contraction but T is a weak cyclic Kannan contraction.

Remark 4.2.1. If we take $y := Tx$ in (4.57) then we obtain (4.56) where $\alpha = \dfrac{a}{1-a}$. Hence we can apply Lemma 4.1.1. Since $a \in \left[0, \dfrac{1}{2}\right)$ it is easy to see that $\alpha \in [0, 1)$.

The following result gives a necessary condition for the existence of a best proximity point, for weak cyclic Kannan contractions.

Theorem 4.2.1. *Let (X, d) be a metric space, let A and B be nonempty subsets of X and let $T : A \cup B \to A \cup B$ be a weak cyclic Kannan contraction. Let $x \in A$ such that a subsequence $\{T^{2n_i}x\}$ of $\{T^{2n}x\}$ converges to $z \in A$. Then z is the unique best proximity point of T.*

We can now state the convergence and existence result for a weak cyclic Kannan contraction.

Theorem 4.2.2. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a weak cyclic Kannan contraction map. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

Best Proximity Point Theorems for Weak p-Cyclic Kannan Contractions

Consider $p \in \mathbb{N}, p > 1$.

Definition 4.2.2. Consider a self map T defined on the union of p nonempty subsets $A_1, A_2, \ldots, A_p$ of a metric space (X, d), satisfying the following *generalized cyclical conditions*

$$T(A_i) \subseteq A_{i+1},$$

for all $1 \le i \le p$, where $A_{p+1} = A_1$. In this case T is called a *weak p-cyclic Kannan contraction if it also satisfies the contractive condition*

$$d(Tx, Ty) \le a[d(x, Tx) + d(y, Ty)] + (1 - 2a)D(A_i, A_{i+1}), \tag{4.60}$$

for all $x \in A_i$ and $y \in A_{i+1}$, for each $1 \le i \le p$, where $a \in \left(0, \frac{1}{2}\right)$.

The following lemma shows that the distance between the adjacent sets are equal under weak p-cyclic Kannan contraction.

Lemma 4.2.1. *Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space X, $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup i = 1^p A_i$ be a weak p-cyclic Kannan contraction. Then $D(A_i, A_{i+1}) = D(A_{i+1}, A_{i+2})$, for all $1 \le i \le p$, where $A_{p+1} = A_1$.*

In a similar manner one can extend the main result of this section, namely Theorem 4.2.2, which is formulated for $p = 2$ to the case general $p \geq 2$, as follows

Theorem 4.2.3. *Let $\{A_i\}_{i=1}^{p}$ be nonempty, closed and convex subsets of a uniformly convex Banach space. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a weak p-cyclic Kannan contraction. Then*

(i) for each $i, 1 \leq i \leq p$, there exists a unique best proximity point x_i^*, of T in A_i;
(ii) the sequence $\{T^{pn}x\}$ converges to x_i^* for any starting point $x \in A_i$;
(iii) x_i^* is the unique fixed point of T^p;
(iv) $T^j x_i^* = x_{i+j}^*$ is a best proximity point of T in A_{i+j}, for $j = \overline{0, p-1}$.

The complete proof of this result we be included in our joint paper [**127**].

3. Best Proximity Point Theorems for Weak Cyclic Chatterjea Contractions

The aim of this section is to introduce the notion of *weak cyclic Chatterjea contraction* and to present existence and convergence results for this type of operators. The results in this section are included in the original paper [**118**].

Definition 4.3.1. Let (X, d) be a metric space and let A and B be nonempty subsets of X. Then an cyclic operator $T : A \cup B \to A \cup B$ is called *weak cyclic Chatterjea contraction* if it satisfies the following condition

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)] + (1 - 2b)D(A, B), \text{ for all } x \in A, y \in B, \quad (4.61)$$

where $b \in \left(0, \dfrac{1}{2}\right)$.

Remark 4.3.1. If we take $y := Tx$ in (4.61) then we obtain (4.56) where $\alpha = \frac{b}{1-b}$. Hence we can apply Lemma 4.1.1. Since $b \in \left(0, \dfrac{1}{2}\right)$ it is easy to see that $\alpha \in (0, 1)$.

In the following we shall prove the existence and uniqueness result of best proximity point for weak cyclic Chatterjea contractions.

Theorem 4.3.1. *Let (X, d) be a metric space, let A and B be nonempty subsets of X and let $T : A \cup B \to A \cup B$ be a weak cyclic Chatterjea contraction. Let $x \in A$ such that a subsequence $\{T^{2n_i}x\}$ of $\{T^{2n}x\}$ converges to $z \in A$. Then z is a best proximity point of T. Moreover, if*

$$diam(A) + diam(B) \leq 2D(A, B) \quad (4.62)$$

then z is the unique best proximity point.

The main result of this section is:

Theorem 4.3.2. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space, such that (4.62) is satisfied. Suppose $T : A \cup B \to A \cup B$ is a weak cyclic Chatterjea contraction map. Then*

(i) T has a unique best proximity point z in A.

(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.

(iii) z is the unique fixed point of T^2.

(iv) Tz is a best proximity point of T in B.

4. Best Proximity Point Theorems for Weak Cyclic Bianchini Contractions

Regarding Bianchini contractive condition we can define the following notion:

Definition 4.4.1. Let (X, d) be a metric space and let A and B be nonempty subsets of X. Then an cyclic operator $T : A \cup B \to A \cup B$ is called *weak cyclic Bianchini contraction* if it satisfies the following condition

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\} + (1 - h)D(A, B), \forall x \in A, y \in B, \tag{4.63}$$

where $h \in [0, 1)$.

The above definition, and the following results in this section are included in the original paper [**117**]. It ie easy to see that:

Remark 4.4.1. Let $x \in A \cup B$. Since T is cyclic we can take $y := Tx$ in (4.63). We obtain

$$d(Tx, T^2x) \leq h \max\{d(x, Tx), d(Tx, T^2x)\} + (1 - h)D(A, B).$$

If $d(Tx, T^2x) > d(x, Tx)$ then from the above inequality it follows that $d(Tx, T^2x) = D(A, B)$, and hence $d(x, Tx) = D(A, B) = d(Tx, T^2x)$, a contradiction. Therefore $d(Tx, T^2x) \leq d(x, Tx)$ and (4.63) implies (4.56) with $\alpha = h \in [0, 1)$. Hence we can apply Lemma 4.1.1.

Hawing in view this definition and remark we can prove now the following:

Theorem 4.4.1. *Let (X, d) be a metric space, let A and B be nonempty subsets of X and let $T : A \cup B \to A \cup B$ be a weak cyclic Bianchini contraction. Suppose that A is a convex set and let $x \in A$ such that a subsequence $\{T^{2n_k}x\}$ of $\{T^{2n}x\}$ converges to $z \in A$. Then z is the unique best proximity point of T.*

As in the previous sections, we will prove a convergence and existence result for weak cyclic Bianchini contractions.

Theorem 4.4.2. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a weak cyclic Bianchini contraction map. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

5. Best Proximity Point Theorems for Weak Cyclic Reich-Rus Contraction

In a similar manner to the previous sections, in this section we shall extend Reich-Rus operators to a cyclic contractive-type condition.

Definition 4.5.1. (Petric [**120**]) Let (X, d) be a metric space and let A and B be nonempty subsets of X. Then an cyclic operator $T : A \cup B \to A \cup B$ is called *weak cyclic Reich-Rus contraction* if it satisfies the following condition

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + (1 - a - 2b)D(A, B), \tag{4.64}$$

for all $x \in A, y \in B$, where a and b are nonnegative numbers such that $a + 2b < 1$.

In the following considerations we will use the below remark.

Remark 4.5.1. (Petric [**120**]) Let $x \in A \cup B$. Since T is cyclic we can take $y := Tx$ in (4.63). We obtain

$$d(Tx, T^2x) \leq \frac{a+b}{1-b} d(x, Tx) + \left(1 - \frac{a+b}{1-b}\right) D(A, B).$$

If we denote $\alpha = \dfrac{a+b}{1-b}$ then the above inequality is in fact the relation (4.56), and since $a + 2b < 1$ it follows that $\alpha \in [0, 1)$. Hence we can apply Lemma 4.1.1.

Considering this remark we can prove the following existence and uniqueness result:

Theorem 4.5.1. (Petric [**120**]) *Let (X, d) be a metric space, let A and B be nonempty subsets of X and let $T : A \cup B \to A \cup B$ be a weak cyclic Reich-Rus contraction. Suppose that A is a convex set and let $x \in A$ such that a subsequence $\{T^{2n_k}x\}$ of $\{T^{2n}x\}$ converges to $z \in A$. Then z is the unique best proximity point of T.*

We can also prove the following existence, uniqueness and convergence result for best proximity points in the case of a weak cyclic Reich-Rus operator.

Theorem 4.5.2. (Petric [**120**]) *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a weak cyclic Reich-Rus contraction. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

Bibliography

[1] M.A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal., **70(10)** (2009), 3665–3671.

[2] N.A. Assad, *Fixed point theorems for set-valued transformations on compact set*, Boll. Un. Mat. Ital., **4** (1973), 1–7.

[3] N.A. Assad and W.A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43** (1972), 553–562.

[4] G.V.R. Babu, M.L. Sandhya, and M.V.R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math., **24** (2008), no. 1, 8–12.

[5] J. S. Bae, *Fixed point theorems of generalized nonexpansive maps*, J. Korean Math. Soc., **21** (1984), no. 2, 233–248.

[6] S. Banach, *Sur les operatoions dans les ebsembles abstraits et leur applications aux èquations intègrales*, Fund. Math., **3** (1922), 133–181.

[7] C. Di Bari, T. Suzuki, and C. Vetro, *Best proximity points for cyclic Meir-Keeler contractions*, Nonlinear Anal., **69(11)** (2008), 3790–3794.

[8] S. Sadiq Basha, *Global optimal approximate solutions*, Optimization Letters, **Article in press**, 1–7.

[9] M. Berinde and V. Berinde, *On a class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl., **326** (2007), 772–782.

[10] V. Berinde, *A chaos phenomenon in applying the Newton method*, Proceed.Int.Conf. MICROCAD '95, Univ. Miskolc, 21-24 Februarie 1995, 48–51.

[11] V. Berinde, *On the extended Newton's method*, Advances in Difference Equations, **1997**, 81–88.

[12] V. Berinde, *On the rate of convergence of some Newton-type iterative methods*, Proceed.Int.Conf. MICROCAD '97, Univ. of Miskolc, febr. 1997, 39–44.

[13] V. Berinde, *Conditions for the convergence of the Newton method*, Analele Univ. "Ovidius" Constanþa, **3** (1995), 22–28.

[14] V. Berinde, *Remarks on the convergence of the Newton-Raphson method*, Rev. d'anal. numer. theor. approx., **Tome 24** (1995), no. 1-2, 15–21.

[15] V. Berinde, *Weak exit criteria for some Newton type methods*, Bulletins for Applied Mathematics, **77** (1995), 27–32.

[16] V. Berinde, *Contracţii generalizate şi aplicaţii*, Editura Cub Press 22, Baia Mare, 1997.

[17] V. Berinde, *On some exit criteria for the Newton method*, Novi Sad J. Math., **27** (1997), no. 1, 19–26.

[18] V. Berinde, *On the convergence of the Newton method*, Trans. of the Tech. Univ. of Kosice (1997), no. 1, 68–77.

[19] V. Berinde, *Weak smoothness conditions and the NewtonŠs method*, in Finite Difference Methods:Theory and Applications(A. Samarskii, P.N. Vabishchevich, L. Vulkov - eds), Nova Science Publishers, (1999), 1–8.

[20] V. Berinde, *A fixed point proof of the convergence of the Newton method*, Proceed. Int. Conf. MICROCAD '94, Univ. Miskolc (2-4 Martie 1994), 14–21.

[21] V. Berinde, *Error estimates for some Newton-type methods obtained by fixed point techniques*, Proceed. Int. Sc. Conf. on Math., Herlany, 21-23 oct 1999, Technical Univ. Kosice (2000), 19–22.

[22] V. Berinde, *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory, **4** (2003), no. 2, 131–142.

[23] V. Berinde, *On the approximation of fixed points of weak φ-contractive operators*, Fixed Point Theory, **4** (2003), no. 2, 131–142.

[24] V. Berinde, *On the approximations of fixed points of weak contractive mappings*, Carpathian J. Math„ **19** (2003), no. 1, 7–22.

[25] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9** (2004), 43–53.

[26] V. Berinde, *Iterative Approximation of Fixed Points*, Springer Verlag Berlin Heidelberg New York, 2007.

[27] V. Berinde, *On some smoothness conditions and NewtonŠs method*, Proceed. of the First Southern Symp. on Computation, Univ. of Southern Mississippi (4-5 dec 1998).

[28] V. Berinde, *An extension of the Newton-Raphson method for solving nonlinear equations*, Proceed.Int.Conf. MICROCAD '93, Tech. Univ.Kosice (9-10 nov 1993), 63–64.

[29] V. Berinde and M. Păcurar, *A fixed point proof of the convergence o a Newton-type method*, Fixed Point Theory, **7** (2006), no. 2, 235–244.

[30] V. Berinde and M. Păcurar, *Iterative Approximation of Fixed Points of Almost Contractions*, in Proceedings of Ninth International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC 2007) Timişoara, IEEE Computer Society, Los Alamitos, California, Washington, Tokyo (2007), 387–394.

[31] V. Berinde and M. Păcurar, *A note on the paper "Remarks on fixed point theorems of Berinde"*, Nonlinear Analysis Forum, **14** (2009), 119–124.

[32] R.M. BIANCHINI and M. GRANDOLFI, *Transformazioni di tipo contractivo generalizato in uno spazio metrico*, Ann. Acad. Naz. Lincei, **45** (1968), 212–216.

[33] R.M.T. BIANCHINI, *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Bolletino U.M.I., **(4)5** (1972), 103–106.

[34] J. BOGIN, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi*, Canad. Math. Bull., **19** (1976), 7–12.

[35] D.W. BOYD and J.S.W. WONG, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458–464.

[36] S.K. CHATTERJEA, *Fixed-point theorems*, C.R. Acad. Bulgare Sci., **25** (1972), 727–730.

[37] L. ĆIRIĆ, *Fixed Point Theory (Contraction Mapping Principle)*, Fakulty of Mechanical Engineering, Beograd, 2003.

[38] L.B. ĆIRIĆ and J.S. UME, *Common fixed point theorems for multi-valued nonself mappings*, Publ. Math. Debrecen, **60** (2002), no. 2-3, 359–371.

[39] L.B. ĆIRIĆ and J.S. UME, *On the convergence of the Ishikawa iterates to a common fixed point of multi-valued mappings*, Demonstration Math., **36** (2003), no. 4, 951–956.

[40] LJ. ĆIRIĆ, *Generalized contractions and fixed point theorems*, Publ. Inst. Math., **12(26)** (1971), 19–26.

[41] LJ.B. ĆIRIĆ and N. JOTIĆ, *A further extension of maps with non-unique fixed points*, Mat. Vesnik, **50** (1998), 1–4.

[42] P. COLLAÇO and J. CARVALHO E SILVA, *A complete comparison of 25 contraction conditions*, Nonlinear Anal., TMA, **30** (1997), 741–746.

[43] ŞT. MĂRUŞTER, *Metode numerice în rezolvarea ecuaţiilor neliniare [Numerical Methods in Solving Nonlinear Equations]*, Ed. Tehnică, Bucureşti, 1981.

[44] P.Z. DAFFER and H. KANEKO, *Fixed points of generalized contractive multivalued mappings*, J. Math. Anal. Appl., **192** (1995), 655–666.

[45] J. DUGUNDJI, *Topology*, Boston, 1966.

[46] M. EDELSTEIN, *On fixed and periodic point under contractive mappings*, J. London Math. Soc., **25** (1982), 139–144.

[47] A. A. ELDRED and P. VEERAMANI, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006.

[48] A.A. ELDRED, W.A. KIRK, and P. VEERAMANI, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math., **171** (2005), 283–293.

[49] R. ESPÍNOLA and A. FERNÁNDEZ-LEÓN, *On best proximity points in metric and Banach spaces*, Canadian J. Math., **(in press)**.

[50] M. FURI, *Un teorema di punto fisso per transformazioni di uno spazio metrico completo in se*, Rend. Acad. Naz. dei Licei„ **Serie VIII, Vol XLV** (1968), no. 5, 207–211.

[51] M. Gabeleh and A. Abkar, *Results on the existence and convergence of best proximity points*, Fixed Point Theory and Applications Volume 2010, 2010, Article number 386037, **2010** (2010), no. 386037.

[52] M.A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608.

[53] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press., Cambridge, 1990.

[54] K. Goebel, W.A. Kirk, and T.N. Chimi, *Fixed Point Theorem in Uniformly Convex Spaces*, Boll. Union. Math. Ital., **7** (1973), 67–75.

[55] M. Gregus, *A fixed point theorem in Banach spaces*, Bolletino. Unione Matematica Italiana A. Series V, **17** (1980), no. 1, 193–198.

[56] G.E. Hardy and T.D. Rogers, *A generalization of a ixed point theorem of Reich*, Canadian Mathematical Bulletin, **16** (1973), 201–206.

[57] T.L. Hicks and B.E. Rhoades, *A Banach type fixed point theorem*, Math. Jap., **24** (1979), 327–330.

[58] V.I. Istrăţescu, *Fixed Point Theory (An Introduction)*, D. Reidel Pulishing Company, Dordrecht: Holland/ Boston: U.S.A./ London: England, 1979.

[59] S. Itoh, *Multi-valued generalized contractions and fixed point theorems*, Comment. Math. Univ. Carolin., **18** (1977), 247–258.

[60] J.Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215** (1976), 241–251.

[61] H. Kaneko, *A general principle for fixed points of contractive multi-valued mappings*, Math. Japon., **31** (1986), 407–411.

[62] H. Kaneko, *Generalized contractive multi-valued mappings and their fixed points*, Math. Japon., **33** (1988), 57–64.

[63] R. Kannan, *Some results on fixed points*, Bull. Calcutta. Math. Soc., **10** (1968), 71–76.

[64] R. Kannan, *On certain sets and fixed point theorems*, Roum. Math. Pure. Appl., **14** (1969), 51–54.

[65] S. Karpagam and S. Agrawal, *Best proximity point for p-cyclic contractions.*

[66] S. Karpagam and S. Agrawal, *Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps*, Nonlinear Analysis, Theory, Methods and Applications, **Article in press**.

[67] S. Karpagam and S. Agrawal, *Best proximity point theorems for p-cyclic Meir-Keeler contractions*, Fixed Point Theory Appl., **Art ID. 197308** (2009), 9pp.

[68] S. Kasahara, *On some generalizations of the Banach contraction theorem*, Math. Sem. Notes, **3** (1975), 161–169.

[69] W.A. KIRK, *A fixed point theorem for mappings which do not increase distances*, The American MAthematical Monthly, **72** (1965), no. 9, 1004–1006.

[70] W.A. KIRK, P.S. SRINIVASAN, and P.VEERAMANI, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, **4** (2003), no. 1, 79–89.

[71] I. KUBIACZYK and N.M. ALI, *On the convergence of the Ishikawa iterates to a common fixed point for a pair of multi-valued mappings*, Acta Math. Hungar., **75** (1997), no. 3, 253–257.

[72] T.-C. LIM, *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., **110** (1985), no. 2, 436–441.

[73] J.T. MARKIN, *A fixed point theorem for set-valued mappings*, Bull. Amer. Math. Soc., **74** (1968), 639–640.

[74] J. MESZAROS, *A comparison of various definitions of contractive type mappings*, Bull. Calcutta Math., **84** (1992), no. 2, 167–194.

[75] N. MIZOGUCHI and W. TAKAHASHI, *Fixed point theorems for multi-valued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177–188.

[76] R. MORALES and E. ROJAS, *Some results for one class of discontinuous operators with fixed points*, Acta. Math. Univ. Comenianae, **LXXVI, 2** (2007), 149–160.

[77] A.S. MUREŞAN, *Some fixed point theorems of Maia type*, Sem. on Fixed Point Theory, Babeş-Bolyai Univ. Cluj-Napoca, **Preprint No. 3** (1988), 35–42.

[78] A.S. MUREŞAN, *From Maia fixed point theorem to fixed point theory in a set with two metrics*, Carpathian J. Math, **23** (2007), 133–140.

[79] S.B. NADLER, *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

[80] A. PETRUŞEL, *Multifunctions and Applications*, Cluj University Press, 2002.

[81] A. PETRUŞEL, *On Frigon-Granas-type multifunctions*, Nonlinear Anal. Forum, **7** (2002), 113–121.

[82] G. PETRUŞEL, *Cyclic representations and periodic points*, Studia Univ. Babeş-Bolyai Math., **50** (2005), no. 3, 107–112.

[83] B. PIĄTEK, *On cyclic MeirŰKeeler contractions in metric spaces*, Nonlinear Analysis: Theory, Methods & Applications, **74** (January 2011), no. 1, 35–40.

[84] P.D. PROINOV, *General local convergence theory for a class of iterative processes and its applications to Newton's method*, J. Complexity, **25** (2009), 38–62.

[85] P.D. PROINOV, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich theorems*, J. Complexity, **26** (2010), 3–42.

[86] P.D. Proniov, *A generalization of the Banach contraction principle with high order of convergence of succesive approximations*, Nonlinear Anal., **67** (2007), 2361–2369.

[87] M. Păcurar, *Fixed point theory for cylic strict almost contractions.*

[88] M. Păcurar, *Iterative methods for fixed point approximation*, Ph.D. thesis, "Babeş-Bolyai" University, Cluj-Napoca, Faculty of Mathematics and Computer Science, 2009.

[89] M. Păcurar and V. Berinde, *A convergence theorem for some Newton type methods under weak smoothness conditions*, Proc. Conf. Aplimat 2007, Bratislava, Slov. Rep. (2007), 265–270.

[90] M. Păcurar and V. Berinde, *A convergence theorem for some Newton type methods under weak smoothness conditions*, An. Univ. Vest Timiž. Ser. Mat.-Inform., **46** (2008), 115–122.

[91] M. Păcurar and I.A. Rus, *Fixed point theorems for cyclic φ-contractions*, Nonlinear Analysis: Theory, Methods and Applications, **72** (2010), no. Issues 3-4, 1181–1187.

[92] S. Reich, *Kannan's fixed point theorem*, Boll. Un. Mat. Ital., **4** (1971), 1–11.

[93] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14(1)** (1971), 121–124.

[94] S. Reich, *A fixed point theorem for locally contractive multi-valued functions*, Rev. Roum. Math. Pures Appl., **17** (1972), 569–572.

[95] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **5** (1972), 26–42.

[96] S. Reich, *Some problems and results in fixed point theory*, Contemp. Math., **21** (1983), 179–187.

[97] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc, **226** (1977), 257–290.

[98] B.E. Rhoades, *Contractive definitions revisited*, Contemporany Mathematics, **21** (1983), 189–205.

[99] B.E. Rhoades, *A fixed point theorem for a multi-valued non-self mapping*, Contemp. Math., **37** (1996), 401–404.

[100] B.E. Rhoades and B. Watson, *Fixed points for multi-valued mappings on metric spaces*, Math. Japon., **35** (1990), no. 4, 735–743.

[101] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.

[102] I.A. Rus, *Some fixed point theorems in metric spaces*, Rend. Ist. di Matem., Unic. di Trieste, **3** (1971), 1–4.

[103] I.A. Rus, *Fixed point theorems for multivalued mappings in complete metric spaces*, Math. Japon., **20** (1975), 21–24.

[104] I.A. Rus, *Principles and Applications of the Fixed Point Theory (in Romanian), Editura Dacia*, Editura Dacia, Cluj-Napoca, 1979.

[105] I.A. Rus, *Generalized contractions*, Seminar on Fixed Point Theory, **3** (1983), 1–130.

[106] I.A. Rus, *Further remarks on the fixed point structure*, Studia Univ. Babeş-Bolyai, **31** (1986), 41–43.

[107] I.A. Rus, *Basic problems of the metric fixed point theory revisited (II)*, Stud. Univ. Babeş-Bolyai, **36** (1991), 81–99.

[108] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[109] I.A. Rus, *Cyclic representatioms and fixed points*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, **3** (2005), 171–178.

[110] I.A. Rus, *Fixed Point Structure Theory*, Cluj University Press, Cluj-Napoca, 2006.

[111] I.A. Rus, A. Petruşel, and G. Petruşel, *Fixed point theorems for set-valued Y-contractions*, Fixed Point Theory and its Applications, Banach Center Publ., Polish Acad. Sci. Warsaw, **77** (2007), 227–237.

[112] I.A. Rus, A. Petruşel, and A. Sîntămărian, *Data dependence of the fixed point set of some multi-valued weakly Picard operators*, Nonlinear Anal. Forum, **52** (2003), 1947–1959.

[113] N. Shahzad, S. Sadiq Basha, and R. Jeyaraj, *Common Best Proximity Points: Global Optimal Solutions*, Journal of Optimization Theory and Applications (2010), 1–10.

[114] S.L. Singh, C. Bhatnagar, and A.M. Hashim, *Round-off stability of Picard iterative procedure for multi-valued operators*, Nonlinear Anal. Forum, **10** (2005), 13–19.

[115] A. Sîntămărian, *Some pairs of multi-valued operators*, Carpathian J. Math, **21** (2005), 115–125.

[116] T. Suzuki, M. Kikkawa, and C. Vetro, *The existence of the best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918–2926.

[117] **M.A. Petric**, *Best proximity point theorems for weak cyclic Bianchini contractions*, (submitted).

[118] **M.A. Petric**, *Best proximity point theorems for weak cyclic Chatterjea contractions*, (submitted).

[119] **M.A. Petric**, *Best proximity point theorems for weak cyclic Kannan contractions*, Filomat, **(accepted)**.

[120] **M.A. Petric**, *Best proximity point theorems for weak Reich-Rus contractions*, (submitted).

[121] **M.A. Petric**, *Cyclical operators of Ćirić type with a non-unique fixed point.*

[122] **M.A. Petric**, *Fixed point principles for multivalued cyclic operators*, (submitted).

[123] **M.A. Petric**, *Strict fixed point principle for multivalued cyclic operators*, (submitted).

[124] **M.A. Petric**, *Some remarks concerning Círíc-Reich-Rus operators*, Creative Math. Inform., **18** (2009), 188–193.

[125] **M.A. Petric**, *Some results concerning cyclical contractive conditions*, General Mathematics, **18** (2010), no. 4, 213–226.

[126] **M.A. Petric** and M. Păcurar, *Fixed point theorems for cyclic φ-almost contractions.*

[127] **M.A. Petric** and B.G. Zlatanov, *Best proximity point theorems for weak p-cyclic Kannan contractions.*

[128] **M.A. Petric** and B.G. Zlatanov, *Fixed point theorems of Kannan type for cyclical contractive conditions*, Proceedings of the Anniversary International Conference REMIA 2010, Plovdiv, Bulgaria, 187–194.

[129] C. Vladimirescu and C. Avramescu, *Applications of the Fixed Point Method to Ordinary Differential and Integral Equations on Noncompact Intervals*, Universitaria Press, Craiova, 2006.

[130] Ch. S. Wong, *On Kannan maps*, Proc. Amer. Math. Soc, **47** (1975), 105–111.

[131] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292–298.

[132] A. Zitarosa, *Una generalizatione del teorema di Banach sulle contrazioni*, Le Matematiche, **23** (1968), 417–424.

Addend: Published and Accepted Papers

This thesis is developed on the basis of the following published/accepted and communicated works:

List of Research Papers Published

1. *Fixed Point Theorems of Kannan Type for Cyclical Contractive Conditions*, Proceedings of the Anniversary International Conference REMIA 2010, Plovdiv, Bulgaria 187-194, (jointly with B.G. Zlatanov)
2. *Some remarks concerning Ćirić-Reich-Rus operators*, Creative Math. Inform., 18 (2009), 188-193.
3. *Some results concerning cyclical contractive conditions*, General Mathematics, Vol. 18, No. 4 (2010), 213-226.

List of Research Papers Accepted

1. *Best proximity point theorems for weak cyclic kannan contractions*, Filomat, (accepted).

List of Research Papers Communicated

1. *Best proximity point theorems for weak Reich-Rus contractions*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010.
2. *Fixed point theorems for cyclic φ-almost contractions*, (jointly with M. Păcurar).
3. *Best proximity point theorems for weak p-cyclic kannan contractions*, (jointly with B.G. Zlatanov).
4. *Best proximity point theorems for weak cyclic Chatterjea contractions.*
5. *Best proximity point theorems for weak cyclic Bianchini contractions.*
6. *Fixed point principles for multivalued cyclic operators.*
7. *Strict fixed point principle for multivalued cyclic operators.*
8. *Cyclical operators of Ćirić type with a non-unique fixed point.*