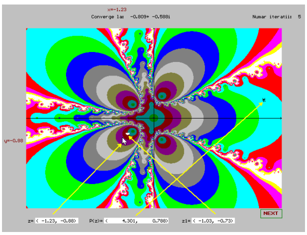
Technical University of Cluj-Napoca
North University Center at Baia Mare
Faculty of Sciences

Ph.D. Thesis

(ABSTRACT)

THE ATTRACTION BASINS OF ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS



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Contents

Preliminaries

Historical aspects

Solving nonlinear equations is an important part of numerical analysis.

First result in solving nonlinear equations is the well-known Newton's method (1643-1727), from *Method of Fluxions* [**100**], developed in 1671, but posthumously published only in 1736. In this work, Sir I. Newton describes his method in a more complicated way than we know it today. As an example, Sir I. Newton presented how to find the real root for the equation $x^3 - 2x - 5 = 0$ and, in addition, it was shown that the root is greater than 2 and less than 3.

In 1690, Raphson(1648-1715) published *Analysis aequationum universalis* [**123**], in which is presented Newton's method to approximate the roots of an equation in a simplified way. It seems that Raphson was one of the few people for who Newton allowed to see his mathematical works (see the Raphson's biography [**104**]).

Although known as Newton-Raphson method, according to N. Kollerstrom, *Thomas Simpson and 'Newton's method of approximation': an enduring myth* [**84**], the actual form of method belongs to T. Simpson and it was published in 1740, see *Essays ... on Mathematics* [**129**].

In his work, *Historical note on Newton-Raphson method of approximation*, published in 1911 [**31**], Florian Cajori presents some interesting considerations about Newton method. Among others, he said that the first printed form of Newton method is in 94th chapter of Wallis work, *Algebra*, published at London in 1685.

In 1948, Kantorovich proves the quadratic convergence on Banach spaces of Newton method. After that the method is known as Newton-Kantorovich method.

Regardless of any existing controversy, after pioneering of Newton method, there were established many others famous method, such as Euler and Chebyshev [**128**], Halley [**65, 66**], Kantorovich [**78**], Ostrowski [**105, 106**].

Chronologically, many papers was written in the literature. These works presented either new methods or some new improvements of old version of Newton method. In this way, it was started a contest to improve the []

We recall some of these works: S. Weerakcoon and T. G. I. Fernando [**140**], T. Yamamoto [**143**], J. M. Gutiérez and M. A. Hernández [**64**], S. Abbasbandy [**1**], S. Amat and all. [**5**], I. K. Argyros [**19, 20, 21**], A. Y. Ozban [**107**], H. H. H. Homeier [**71**], C. Chun [**39, 41, 40**], J. Kou şi and all [**85, 87, 88, 89, 86**], Y. Ham and all [**67**], Z. Xiaojian [**142**], M. A. Noor and all [**102, 101, 103**] or Ljiljana şi Miodrag Petković [**113, 111, 112, 57**] etc.

Now, if we refer to the important monograph in the literature, we can record the works of: J. F. Traub [**132**], A. M. Ostrowschi [**105, 106**], A. Householder [**72**], W. Gautschi [**62**], C. T. Kelley [**82, 81**], D. Kincaid şi W. Cheney [**83**], I. K. Argyros şi F. Szidarovszky [**22**], or the works of Romanian authors: G. Coman [**49**], Şt. Măruşter [**98, 99, 96, 97**], V. Berinde [**26, 27, 25, 28, 29**], E. Cătinaş [**36, 37, 35, 34, 33**], O. Cira [**43, 45, 44, 47, 46**], A. Diaconu [**54, 55**], O. Agratini and all [**2**], I. Păvăloiu [**119, 118, 120, 121**], A. Lupaş [**93**], etc.

The structure of PhD Thesis

This PhD thesis contains five chapters and one annex.

First Chapter contains some notions and preliminaries results. Newton method is presented here, along with some methods for numerical approximation of solutions of nonlinear equations. Also, some proofs of convergence are presented.

In the second chapter, some methods of Newton type used to solving nonlinear equations are presented. A new own method is proposed here. Also, two new methods of representation for the attraction basins of roots of nonlinear equations are presented. In the end of this chapter, a comparative study of some iterative methods is presented. The study is made with respect to attraction basins related to each method.

In Chapter 3, iterative methods and attraction basins of zeroes of polynomial equation with complex coefficients are presented. Also, a comparative study based on attraction basins is presented here.

The fourth chapter contains the results related to a new topological structure introduced by author.

A didactic application, developed by author, is presented in chapter 5. It is an application which highlights some elements related to attraction basins of zeroes of equations with complex coefficients.

The annex contains the Mathcad and Pascal programmes(sours code), which were used to generate the attraction basins.

Personal contributions

In that follows, there are the main results of author:

1. We introduced, for the first time, of two graphical representations of attraction area of roots of nonlinear equations with real coefficients, **G.**

Ardelean [**13**].

2. We developed a new iterative method of Newton type to solve the nonlinear equations. This new Newton method has a large convergence's area than other known methods, **G. Ardelean** [**14**].

3. We proved the convergence of some Newton methods using the symbolic computation by Mathematica and Maple, **G. Ardelean** [**18, 17, 15**].

4. We developed one soft to generate the attraction basins, **G. Ardelean** [**10**].

5. We made a comparative study of iterative methods for solving some nonlinear equations with real coefficients, using the attraction basins, **G. Ardelean** [**14**].

6. We made a comparative study of iterative methods for determining the zeros of polynomials equations with complex coefficients, using the attraction basins, **G. Ardelean** [**11**].

7. We have formulated and proved some properties of left topology related to a quasiorder relation, **G. Ardelean** [**8, 7**].

8. We introduced a new topological structure on attraction basins and proved some properties of this topology, **G. Ardelean** [**16**].

9. We developed a soft which is necessary to view the attraction basins and to represent, in a graphic and interactive way, some specific elements of these areas , **G. Ardelean** [**12**].

10. We made a DVD attached to this thesis. This DVD contains electronic form, the didactic soft and the necessary instructions for use.

Thanks

We thank you, Mr. Prof. Ion Păvăloiu, for your careful coordination that we received throughout the elaboration of this PhD thesis.

We thank you, Mr. Prof. Vasile Berinde, for your relevant remarks expressed relative to the way of presentation of some results.

Also, our thanks go to our colleague Andrei Horvat-Marc, for his support to typing this thesis.

CHAPTER 1

Preliminaries

This chapter includes some mathematical results and fundamental concepts that will be used in presentation of the results in the thesis. In this chapter we used the presentation from cite agra1, gaut1, kinc1, pava1, trau1.

1. Preliminary results, fundamental concepts and notations

2. Newton'smethod for solving nonlinear equations

2.1. Generating the Newton's method by inverse interpolation

2.2. Inverse interpolation

2.3. Generating the Newton's method by graphical way

2.4. The convergence of Newton's method

2.5. A priori error estimate

2.6. A posteriori error estimate

2.7. Other demonstrations of convergence of Newton's method

3. Newton's method for systems of linear equations

CHAPTER 2

The attraction basins for the real roots of nonlinear equations

In this chapter, the most relevant Newton-type iterative methods for solving nonlinear equations $f(x) = 0$ and a new method constructed by the author of the thesis, *Bisectrice-Newton method*, are presented.

In the following, we will refer to Newton's method by Classical Newton (NC) andto present Newton-type methods we use names and abbreviations established in the literature.

1. Newton-type methods for solving nonlinear equations

1.1. *Arithmetic mean Newton metod* (AN)

1.2. *Harmonic mean Newton metod* (HN)

1.3. *Midpoint Newton metod* (MN)

1.4. A new method, *Bisectrice Newton metod* (BN)

This method we introduced in **G. Ardelean** [**14**], and is given by the following calculation scheme:

$$x_{n+1} = x_n - f(x_n)\frac{f'(x_n)+f'(y_n)}{f'(x_n)f'(y_n)+\sqrt{(1+f'(x_n)^2)(1+f'(y_n)^2)}-1}, \tag{2.1}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},\ \ n = 0, 1, 2, ... \tag{2.2}$$

This method is third-order convergent for simple roots and its efficiency index is $3^{\frac{1}{3}} \approx 1.442$.

The base quality of our method consists in the fact that it has a larger convergence area, compared with similar methods (with the same order of convergence and the same efficiency index).

We construct our new iterative method based on a geometric observation. For this purpose consider the following figure (Figure 2.4).

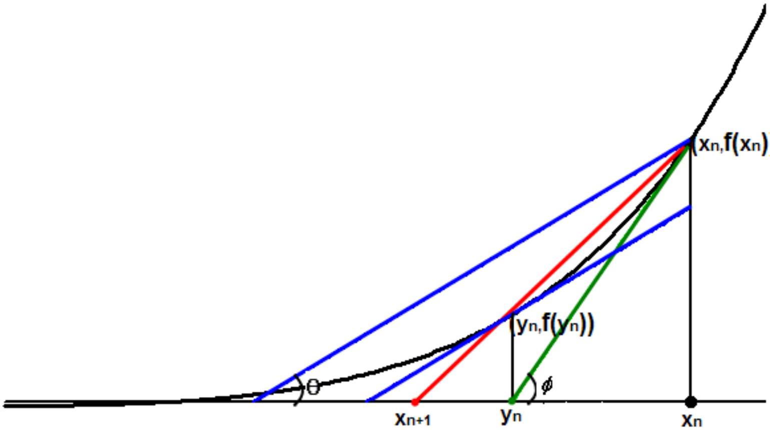


Figure 2.4. Bisectrix Newton's method, graphical view.

Let x_n be an approximation to a simple root α of $f(x) = 0$. Let's consider the angle between the tangent l_1 to the curve $y = f(x)$ at $(x_n, f(x_n))$ (the green line) and the line l_2 passing through the point $(x_n, f(x_n))$ and parallel to the tangent to the curve $y = f(x)$ at $(y_n, f(y_n))$, (the longer blue line) where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \, .$$

Then as the next approximation x_{n+1} we consider the point where the bisectrix of the angle mentioned above intersects the x-axis. Let θ and φ be the angles which the lines l_1 and l_2 make with the positive x-axis respectively. So we have $tan\varphi = f'(x_n)$ and $tan\theta = f'(y_n)$. It is clear that the tangent of the angle between the bisectrix and the positive x-axis or the slope of the bisectrix is

$$tan\frac{\varphi+\theta}{2} = \frac{1-\cos(\varphi+\theta)}{\sin(\varphi+\theta)} = \frac{1-cos\varphi cos\theta + sin\varphi sin\theta}{sin\varphi cos\theta + sin\theta cos\varphi}$$

$$= \frac{\frac{1}{cos\varphi cos\theta} - 1 + tan\varphi tan\theta}{tan\varphi + tan\theta} = \frac{tan\varphi tan\theta + \sqrt{(1+tan^2\varphi)(1+tan^2\theta)} - 1}{tan\varphi + tan\theta}$$

So we have

$$x_{n+1} = x_n - \frac{f(x_n)}{tan\frac{\varphi+\theta}{2}} = x_n - \frac{(tan\varphi + tan\theta)f(x_n)}{tan\varphi tan\theta + \sqrt{(1+tan^2\varphi))(1+tan^2\theta)} - 1}$$

and then

(2.3)
$$x_{n+1} = x_n - \frac{(f'(x_n)+f'(y_n))f(x_n)}{f'(x_n)f'(y_n) + \sqrt{(1+f'(x_n)^2)(1+f'(y_n)^2)} - 1} \qquad n \geq 0.$$

We call the method defined by equation (10) the *Bisectrix Newton's method (BN)*, where y_n is given by (2.2).

In the following we will prove that this method is third-order convergent. It is obvious that the efficiency index of the method also is $3^{\frac{1}{3}} \approx 1.442$.

1.5. The convergence of Bisectrice Newton method

To prove the convergence of *Bisectrice Newton* we use the following theorem:

Teorema 2.1.1. [**62**] *The fixed-point iteration*

(2.4)
$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \ldots$$

is of convergence order p if F is sufficiently smooth on an interval containing α*, a fixed point of F and satisfies*

(2.5)
$$F'(\alpha) = F^{(2)}(\alpha) = \cdots = F^{(p-1)}(\alpha) = 0 \quad \textit{and} \quad F^{(p)}(\alpha) \neq 0.$$

The convergence of *Bisectrice Newton method* results from the following theorem:

Teorema 2.1.2. ***(G. Ardelean*** [**14**]***)*** *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I. If $f(x)$ is sufficiently smooth in the neighborhood of α, and initial value x_0 is sufficiently close to α, then the method defined by the relations (2.1) şi (2.2) is third-order convergent and the corresponding error equation is*

$$e_{n+1} = (\frac{c_2^2}{A} + \frac{c_3}{2})e_n^3 + O(e_n^4), \tag{2.6}$$

where $A = \frac{1}{1+f'(\alpha)^2}$.

To prove the theorem we used symbolic computation in Maple 12.0 package.

2. The attraction basins for the real roots of nonlinear equations

The famous images of fractal basins of attraction for zeros of polynomials with complex coefficients is already known.

In the following, **G. Ardelean**[**13**], we present two new graphical models for the basins of attraction of real roots, generated by iterative methods for solving nonlinear equations. Even if the images generated are not so spectacular that in complex case, they shows us some informations about the behaviors of the iterative methods (as intervals of convergence and the speed of convergence).

2.1. Some "bizarre" results ?

2.2. Generating and viewing the basins of attraction

To present our variants for viewing the basins of attraction for the real roots of a nonlinear equation, we return to the classical Newton method (*classical Newton's method (CN))*, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{2.7}$$

where $f : I \to \mathbb{R}$ is a differentiable function. Let α be a root of the equation $f(x) = 0$, adică $f(\alpha) = 0$. It is known that if $f'(\alpha) \neq 0$, x_0 is sufficiently closed to α and under some conditions, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to α.

The application

$$F(x) = x - \frac{f(x)}{f'(x)} \tag{2.8}$$

is called *Iteration Function (I.F.)* of the method defined by (2.7).

Let $x_0 \in [a, b]$. Dacă starting from the point x_0, the sequence genereted by the relation (2.7) converges to α, then the point x_0 is said to be *attracted* by the root α.

The attraction basin for the rootα and the Iteration Function F is the set of all start points $x_0 \in [a, b]$, wich are attracted by α.

To present the graphical view of the basins of attraction we considered the *classical Newton's* method defined by the relation (2.7), and the nonlinear equation $f_1(x) = 0$, where

$$f_1 : [-8, 8] \to \mathbb{R}, \quad f_1(x) = 2xsinx - 2 \tag{2.9}$$

The roots of the equation $f_1(x) = 0$ in the interval $I_1 = [-8, 8]$ are $\alpha_1 \approx -6.439$, $\alpha_2 \approx -2.773$, $\alpha_3 \approx -1.114$, $\alpha_4 \approx 1.114$, $\alpha_5 \approx 2.773$ and $\alpha_6 \approx 6.439$.

We present two variants for graphical representation of the basins of attraction.

a) The "sinusoidal" model for the basins of attraction

To represent, for example, the basins of attraction for the roots of the equation $f_1(x) = 0$ on the interval $I_1 = [-8, 8]$ using classical Newton's method, we cover the interval I_1 by n+1 equidistant points $\{t_i\}_{i=0}^{n}$ (in this case $n = 500, t_0 = -8$ and $t_{500} = 8$). We solve the equation starting from each point x_0 in these points.

If x_0 attempts a root α of the equation, $\alpha \in I_1$,with a tolerance $\varepsilon = 10^{-5}$ in a maximum of 14 iterations then we draw a line between the points $(x_0, 0)$ and $(x_0, f_1(x_0))$ in a color associated to the root α.

If x_0 is not attracted by any root $\alpha \in I_1$, then we do'nt draw nothing. So, the white regions means that here the method does not converges to any root in I_1.

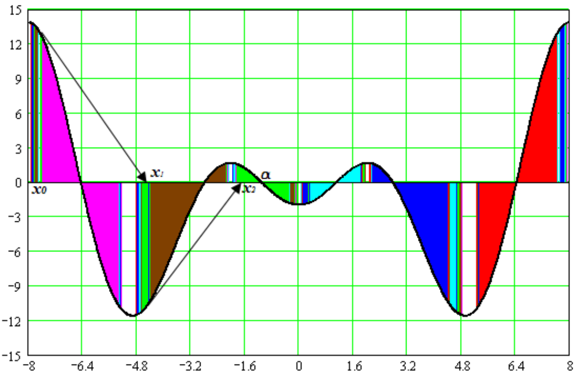


Figure 2.6."Sinusoidal" view for Newton's method and the equation $f_1(x) = 0$

b) The"bars" model for the basins of attraction

To represent, for example, the basins of attraction for the roots of the equation $f_1(x) = 0$ on the interval $I_1 = [-8, 8]$ using classical Newton's method, we cover the interval I_1 by n+1 equidistant points $\{t_i\}_{i=0}^{n}$ ($n = 500$). We solve the equation starting from each point x_0 in these points.

If x_0 attempts a root α of the equation, $\alpha \in I_1$,with a tolerance $\varepsilon = 10^{-5}$ in $k \leq 14$ iterations then we draw in x_0 a vertical bar with height equal to k and in a color associated to the root α.

If x_0 is not attracted by any root $\alpha \in I_1$, then we draw at x_0 a red bar under the Ox axes.

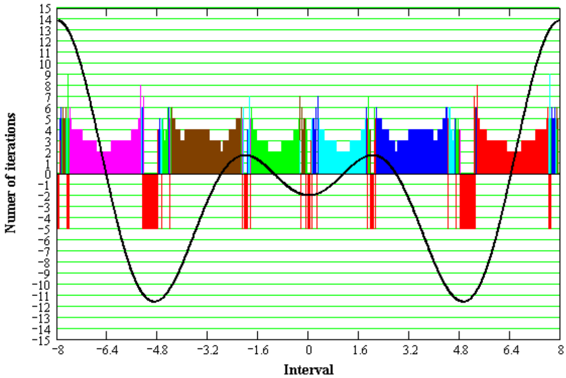


Figure 2.7 ."Bars" view for Newton's method and the equation $f_1(x) = 0$

This kind of representation is more relevant as the "sinusoidal" one because it shows us the number of iterations too. So, using this representation for the basins of attraction, we can to compare some iterative methods for solving nonlinear equations.

2.3. A comparison betwen the iterative methods by using the basins of attraction

To show the performances of our method, (*Bisectrice Newton (BN)*), for solving nonlinear equations, in comparison with other similar methods, we choosed in our study the methods:

(a) Classical Newton (CN)
(b) Arithmetic mean Newton (AN)
(c) Harmonic mean Newton (HN)
(d) Midpoint Newton (MN)
(e) Bisectrix Newton (BN)

To compare the methods (a)-(e), we choosed the test equations $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = 0$, where

$$f_1 : [-1, 2] \to \mathbb{R},\ f_1(x) = x^3 + 4x^2 - 10, \tag{2.10}$$

$$f_2 : [-2, 4] \to \mathbb{R},\ f_2(x) = (x-1)^6 - 1, \tag{2.11}$$

$$f_3 : [-3.5, 4] \to \mathbb{R},\ f_3(x) = \frac{x^4}{4}\sin x - 1. \tag{2.12}$$

We generated the basins of attraction for the roots of the equation $f_p(x) = 0$ on the interval $I_p,\ p = 1, 2, 3$, corresponding to the iterative methods CN, AN, HN , MN and BN, we cover the interval I_p by n+1 equidistant points $\{t_i\}_{i=0}^{n}$ $(n = 500)$. We denote the set of these points by T. We solve the equation starting from each point x_0 in T.

If x_0 attempts a root α of the equation, $\alpha \in I_p$,with a tolerance $\varepsilon = 10^{-5}$ in $k \le 14$ iterations then we draw at x_0 a vertical bar with height equal to k and in a color associated to the root α. If x_0 is not attracted by any root of the equation $f(x) = 0$, then we draw at x_0 a red bar under x-axes. Nothing to draw if x_0 is attracted by a root out of the interval I_p.

We used the "bars" variant for viewing the basins of attraction because this shows us the intervals of convergence for the corresponding method and number of iterations. So, using this representation for the basins of attraction, we can compare some iterative methods for solving nonlinear equations.

Now, for the equation $f_p(x) = 0$, the interval I_p and an iterative method we introduce :

Convergence Interval Index (CII) as

$$CII = \frac{Cpoints}{Tpoints}$$

where *Cpoints* is, among the points taken in T, the total number of points attracted by the roots of the equation $f(x) = 0$, *Tpoints* is the total number of poins taken in T and

Mean Number of Iterations (MNI) as

$$MNI = \frac{Niter}{Cpoints}$$

where $Niter$ is the total number of iterations corresponding to the points attracted by the roots of the equation.
The figures Figure 2.11, Figure 2.12 and Figure 2.13 show the basins of attraction corresponding to the iterative methods CN, AN, HN, MN and BN for the solutions of the equations $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = 0$ and the intervals I_1, I_2 and I_3.

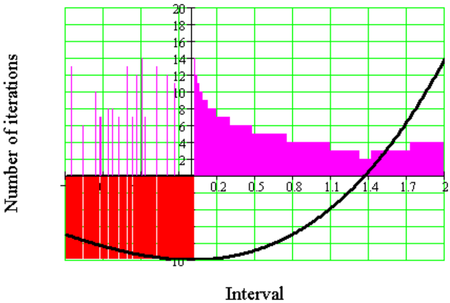
For each method, we use the same tolerance $\varepsilon = 10^{-5}$ and the same maxim number of iterations $k = 14$.

The numerical results are presented in Table 2.1, Table 2.2 and Table 2.3.

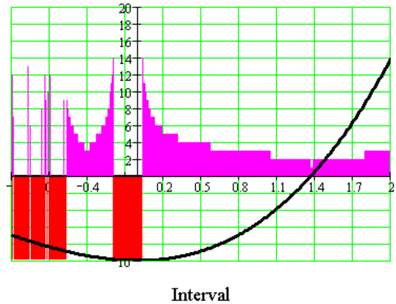
To generate the basins of attraction and to obtain the numerical results we used Mathcad 14.0 package.

The graphic representation of the basins of attraction for real roots allows us to see the behaviors of the iterative methods CN, AN, HN, MN and BN, and compare their intervals of convergence.

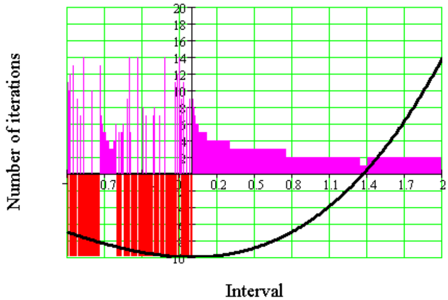
In Figure 2.11, Figure 2.12 and Figure 2.13 one can observe that our method(BN) has the largest intervals of convergence for each equation studied. Moreover, in the case of the equation $f_2(x) = 0$, in Table 2.2 one can see that our method has also the best convergence speed.



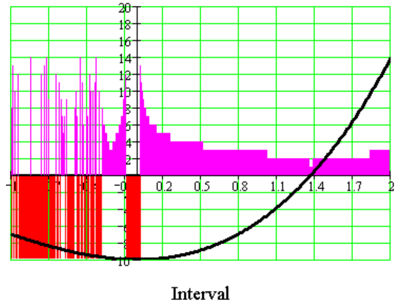
(a) the method CN



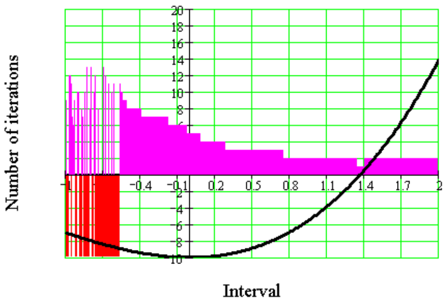
(b) the method AN



(c) the metod HN



(d) the metod MN

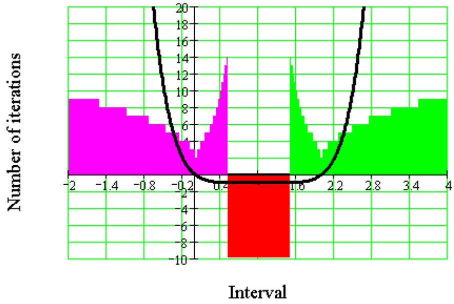


(e) the metod BN

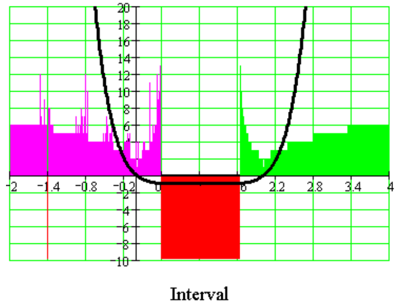
Table 2.1 Numerical results for $f_1(x)$

metoda	CII	MNI
CN	0.694	4.8
AN	0.798	3.8
HN	0.778	3.3
MN	0.784	3.8
BN	0.906	3.8

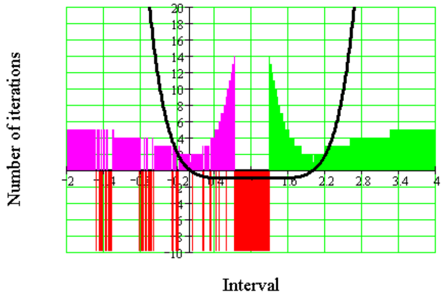
Figure 2.11 The attraction basins for $f_1(x)$



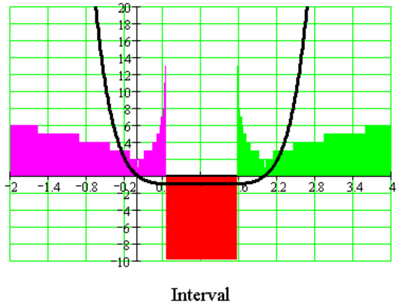
(a) the method CN



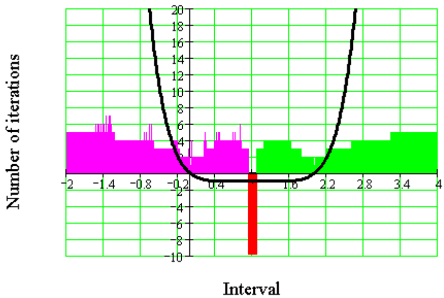
(b) the method AN



(c) the method HN



(d) the method MN

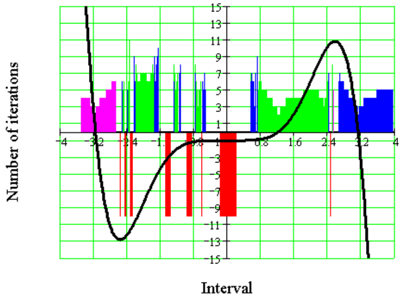


(e) the method BN

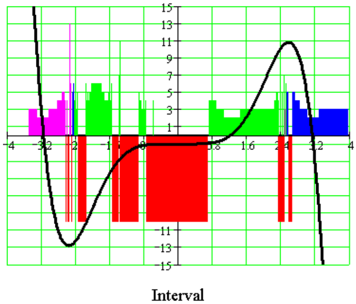
Table 2.2 Numerical results for $f_2(x)$

metoda	IAC	NMI
CN	0.838	6.9
AN	0.792	4.9
HN	0.828	4.3
MN	0.814	4.5
BN	0.978	3.8

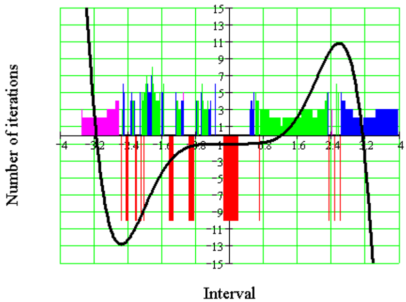
Figure 2.12 The attraction basins for $f_2(x)$



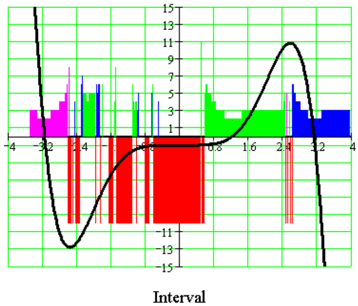
(a) the method CN



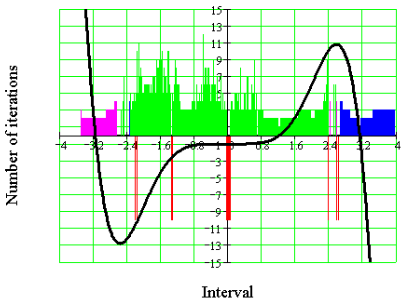
(b) the method AN



(c) the method HN



(d) the method MN



(e) the method BN

Table 2.3 Numerical results for $f_3(x)$

metoda	IAC	NMI
CN	0.900	5.1
AN	0.664	3.4
HN	0.900	3.4
MN	0.726	3.4
BN	0.974	3.7

Figure 2.13 The attraction basins for $f_3(x)$

CHAPTER 3

The attraction basins for the zeros of complex coefficients polynomials

An important particular case of nonlinear equations consists in determining zeros of polynomials. In this chapter we will refer to approximating zeros of polynomials with complex coefficients by using a few Newton type methods. At the end of the chapter, we present results of a comparative study of these methods, by using the basins of attraction **(G. Ardelean** [**11**]**)**.

1. Determination of polynomial zeros

2. An algorithm to approximate the polynomial zeros

3. The attraction basins for polynomial zeros

4. A comparison of iterative methods by using the basins of attraction

In the following we present the results of a comparative analysis of Newton type iterative methods for solving nonlinear equations. We chose to investigate six such methods, including classical method of Newton as a "standard", and three polynomials with complex coefficients for which we performed numerical experiments. The results of this study consists in ranking the investigated methods, in terms of performance, such as computer time used, the convergence area and velocity of convergence. The study is based on the basins of attraction for zeros of polynomials chosen, corresponding to the iterative methods analyzed.

4.1. Description of methods

The studied methods are:

(a) Classical Newton method(CN)

This method is defined by the relation

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}, \quad n = 0, 1, 2, \ldots \tag{3.13}$$

This method is quadratically convergent and with efficiency index $2^{\frac{1}{2}} = 1.414$.

(b) Arithmetic mean Newton method(AN) [**140**]

This method is defined by the relation

$$z_{n+1} = z_n - \frac{2P(z_n)}{P'(z_n) + P'(y_n)}, \quad n = 0, 1, 2, \ldots \tag{3.14}$$

where $y_n = z_n - \frac{P(z_n)}{P'(z_n)}$.

The method is third-order of convergence and is of efficiency index $3^{\frac{1}{3}} = 1.442$.

(c) **Harmonic mean Newton's method (HN)** [**107**]

The method is defined by

$$z_{n+1} = z_n - \frac{P(z_n)(P'(z_n) + P'(y_n))}{2P'(z_n)P'(y_n)}, \quad n = 0, 1, 2, \ldots \tag{3.15}$$

where $y_n = z_n - \frac{P(z_n)}{P'(z_n)}$.

This method is third-order of convergence and efficiency index is $3^{\frac{1}{3}} = 1.442$.

(d) **Midpoint Newton's method (MN)** [**107**]

The method is defined by

$$z_{n+1} = z_n - \frac{P(z_n)}{P'((z_n + y_n)/2)}\,, \quad n = 0, 1, 2, \ldots \tag{3.16}$$

where $y_n = z_n - \frac{P(z_n)}{P'(z_n)}$.

This method is third-order of convergence and of efficiency index $3^{\frac{1}{3}} = 1.442$ too.

(e) **Halley's method (HM) [132]**

This method is defined by

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\,\frac{L_P(z_n)}{1 - \frac{1}{2}\,L_P(z_n)}\right)\frac{P(z_n)}{P'(z_n)}, \quad n = 0, 1, 2, \ldots \tag{3.17}$$

where $L_P(z_n) = \frac{P''(z_n)P(z_n)}{[P'(z_n)]^2}$.

This method is third-order of convergence and of efficiency index $3^{\frac{1}{3}} = 1.442$.

(f) **Traub-Ostrowski method (TOM) [132]**

The method is defined by

$$z_{n+1} = z_n - \frac{P(y_n) - P(z_n)}{2P(y_n) - P(z_n)}\,\frac{P(z_n)}{P'(z_n)}\,, \quad n = 0, 1, 2, \ldots \tag{3.18}$$

where $y_n = z_n - \frac{P(z_n)}{P'(z_n)}$.

This method is of fourth-order of convergence and of efficiency index $4^{\frac{1}{3}} = 1.587$.

By Kung-Traub Conjecture, a method has the *optimal order* equals to 2^{d-1} and *optimal efficiency index* equals to $2^{(d-1)/d}$, where d is the informational usage. So, this method has optimal order $4 = 2^{3-1}$ (in this case $d = 3$) and optimal efficiency index $4^{\frac{1}{3}} = 2^{\frac{2}{3}}$.

The test polynomials used in our computer experiments are as follows:

1) $P_1(z) = z^7 - 1$ whose zeros are: $1,\ 0.623 \pm 0.782i,\ -0.223 \pm 0.975i,\ -0.901 \pm 0.434i$. This example is like in [16].

2) $P_2(z) = z^8 + (1+8i)z^7 + (-22+27i)z^6 + (-105+70i)z^5 + (-271+185i)z^4 + (-346+872i)z^3 + (1282+1658i)z^2 + (3060-2820i)z - 3600$.

The zeros are: 1, $1+i$, $2i$, $-2+2i$, -3, $-3-3i$, $-5i$, $5-5i$ (in a "spiral" placement that can be seen in Fig. 3. This example is from [6].

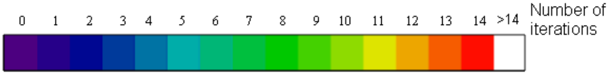
3) $P_3(z) = z^3 + (2-3i)z^2 - (5+5i)z - 8 - 12i$ with zeros: $-3+2i$, $-1-i$, $2+2i$.

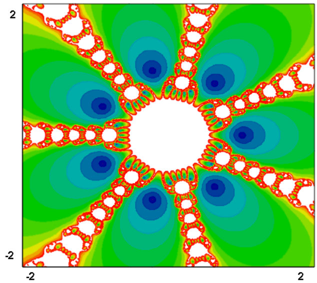
4.2. The Mathcad programs for generating the basins of attraction

4.3. Numerical results

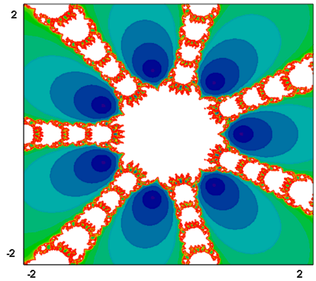
In practice, if the iterative method starting in $z_0 \in D$ reach a zero in k iterations ($k < 15$), then we mark this point z_0 with a color depending on k. For example, for $k = 14$ the associated color is red. For our polynomials $P_1(z)$, $P_2(z)$ and $P_3(z)$ we take the rectangles D_1, D_2 and respectively D_3 as follows: $D_1 = [2,2] \times [-2,2]$; $D_2 = [-15,20] \times [-20,15]$; $D_3 = [-300,300] \times [-300,300]$. In each of these cases, the rectangle contains all zeros of corresponding polynomial. To generate the pictures we employed Mathcad 14.0 and a computer laptop DELL Inspiron 1501, 1.6 GHz.

For the basins of attraction the following Figure 3.4, Fiure 3.5 and Figure 3.6 was obtained:

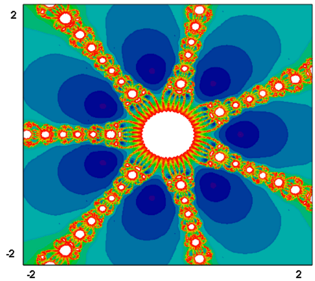




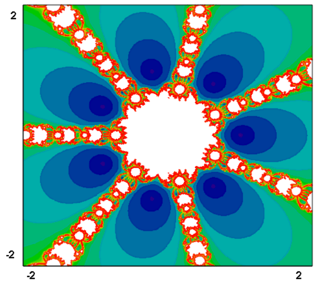
(a) Classical Newton method(CN)



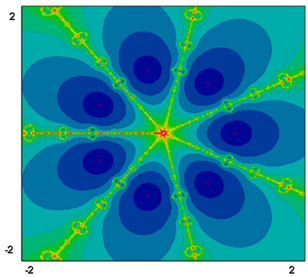
(b) Arithmetic mean Newton method(AN)



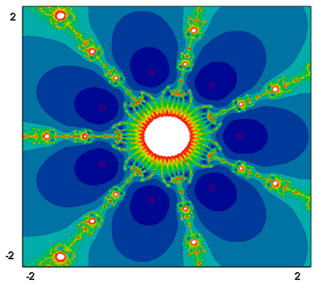
(c) Harmonic mean Newton method (HN)



(d) Midpoint Newton method(MN)

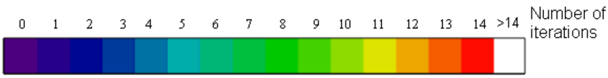


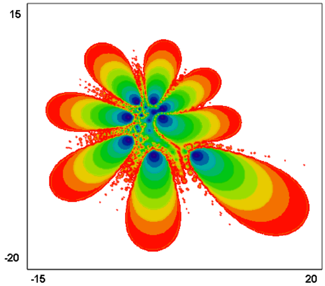
(e) Halley method(HM)



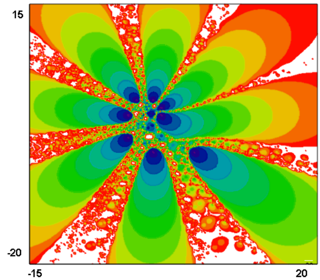
(f) Traub-Ostrowski method(TOM)

Figure 3.4 The attraction basins for the polynomial $P_1(z)$

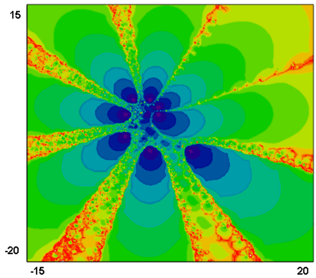




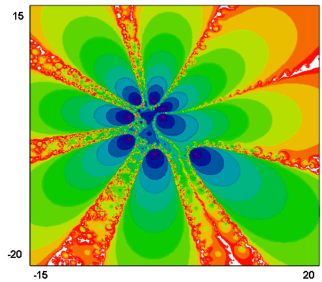
(a) Classical Newton method(CN)



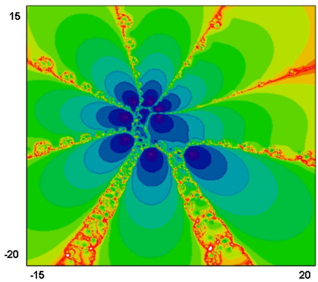
(b) Arithmetic mean Newton method(AN)



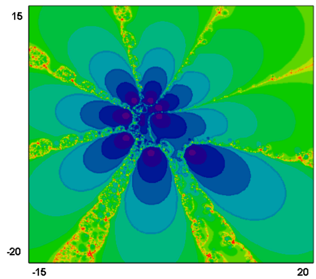
(c) Harmonic mean Newton method(HN)



(d) Midpoint Newton method(MN)

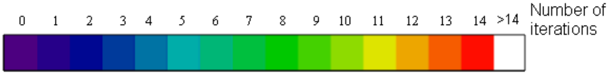


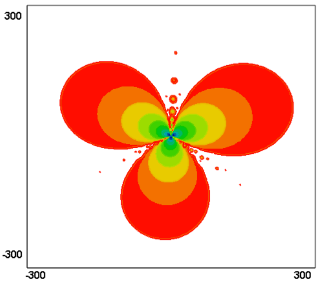
(e) Halley method(HM)



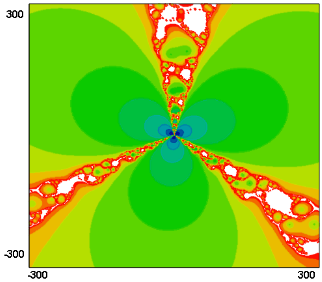
(f) Traub-Ostrowski method(TOM)

Figure 3.5 The attraction basins for the polynomial $P_2(z)$

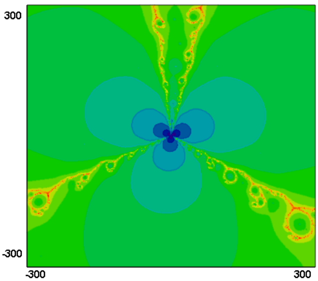




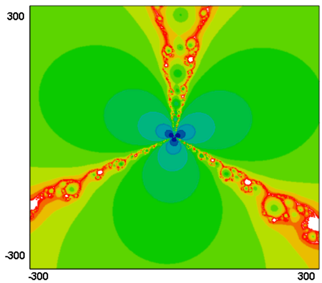
(a) Classical Newton method(CN)



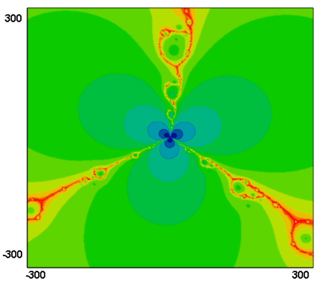
(b) Arithmetic mean Newton method(AN)



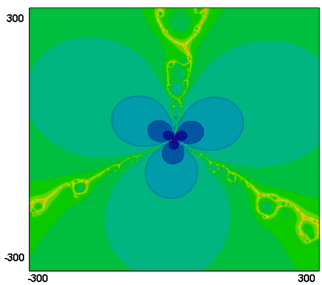
(c) Harmonic mean Newton method (HN)



(d) metoda Midpoint Newton (MN)



(e) Halley method(HM)



(f) Traub-Ostrowski method(TOM)

Figure 3.6 The attraction basins for the polynomial $P_3(z)$

The results of our computer experiments are presented in three suggestive diagrams (in Fig. 5, Fig. 6 and Fig. 7), that concentrate all informations concerning the behaviour of the investigated methods on the three example polynomials P_1, P_2 and P_3.

We think that the most relevant and important results are those in the diagram from Fig. 5 which show the computer time needed by each method to generate the basins of attraction for each polynomial P_1, P_2 and P_3. This time is presented as a relative time to Newton's method, to not depend on the computer used. The absolute values of time for Newton's method, on our computer was:

220 sec. for the polynomial P_1; 332 sec. for the polynomial P_2; 206 sec. for the polynomial P_3.

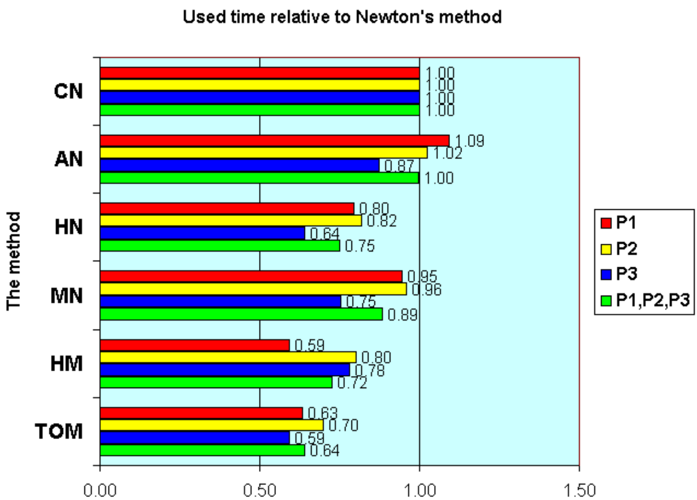


Figure 3.7 Computer time needed to generate the basins of attraction (Newton=1).

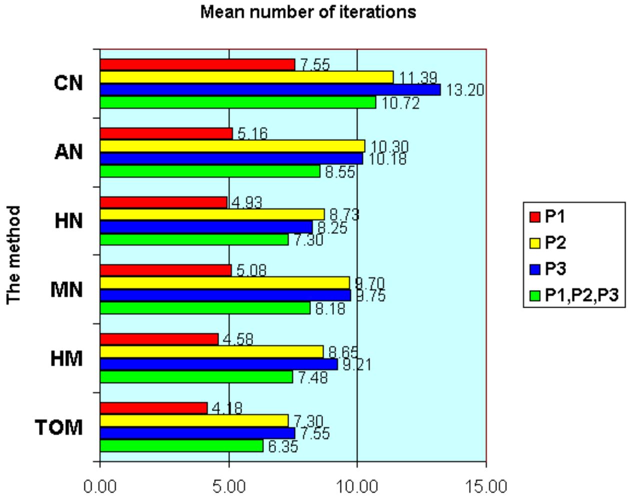


Figure 3.8 Mean number of iterations for convergent points.

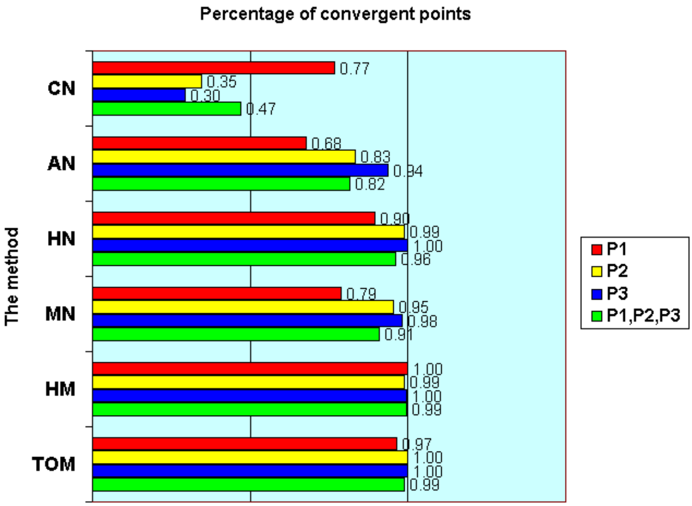


Figure 3.9 Number of convergent poins / Total number of starting points evaluated.

4.4. Conclusions

In each of shown diagrams, we included for each method a green bar corresponding to the mean of values presented for the three polynomials P_1, P_2 and P_3. This shows us that "the best" from all points of view is Traub-Ostrowski's method (TOM).It's normally, because this method has the greatest efficiency index, that is 1.587, and is an optimal efficiency index. We observe too that this method is only of fourth-order between the methods investigated.

About the methods of third-order (AN, HN, MN and HM) we consider that Halley's method (HM) to be the best of this group (is the best as the time parameter and percentage of convergent points).
The Arithmetic mean Newton's method (AN) seems to be "the last" from all points of view.

We specify that Classical Newton's method (CN) is not included in the "competition". This is only used as a standard.

The results of our study are presented în **(G. Ardelean** [**11**]**)**.

CHAPTER 4

Topological properties of the basins of attraction

1. Topological preliminaries

2. The left topology associated to a quasiorder relation

A *quasiorder relation* on a set is a binary reflexive and transitive relation.

Let M be a nonvoid set and let $\prec$ be a quasiorder relation on M. For each $w \in M$ let

$$\mathcal{O}_w = \{v/v \in M, v \prec w\}. \tag{4.19}$$

In [**133**] it is proved that the family of sets $\{O_w\}_{w \in M}$ is a base for a topology on M. This topology is called the *left topology* on M associated to the relation $\prec$. The *right topology* on a set of words is studied in [**32**].

Let $(M, J \prec)$ be the corresponding topological space.

Proposition 4.2.3. ***(G. Ardelean*** [**7**]***)*** *A subset A of M is an open set of the topological space $(M, J_\prec)$ iff from $y \in A$ and $x \prec y$ it follows $x \in A$.*

Corollary 4.2.4. ***(G. Ardelean*** [**7**]***)*** *A subset A of M is an open set of the topological space $(M, J_\prec)$ iff $A = Pred_\prec(A)$.*

Proposition 4.2.5. ***(G. Ardelean*** [**7**]***)*** *A subset B of M is a closed set of the topological space $(M, J_\prec)$ iff $x \in B$ and $x \prec y$ implies $y \in B$.*

Corollary 4.2.6. ***(G. Ardelean*** [**7**]***)*** *A subset B of M is a closed set of the topological space $(M, J_\prec)$ iff $B = Succ_\prec(B)$.*

Lemma 4.2.7. *(G. Ardelean* [7]*) If there exists $v \in M$ such that for every $w \in M$ follows $v \prec w$ or $w \prec v$, then the topological space $(M, K_{\prec})$ is connected.*

3. Topological features of the basins of attraction

Our function example is $f : [0.8, 6.2] \to R$.

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \tag{4.20}$$

and the equation

$$f(x) = 0. \tag{4.21}$$

It is obvious that the roots of the equation (4.21) are 1, 2, 3, 4, 5 and 6. To solve this equation we use the Newton's schema and associated iteration function

$$F(x) = x - \frac{f(x)}{f'(x)}\,.$$

Let α be one of the roots of the equation (4.21) and let $B(\alpha)$ be *the basin of attraction of the root* α.

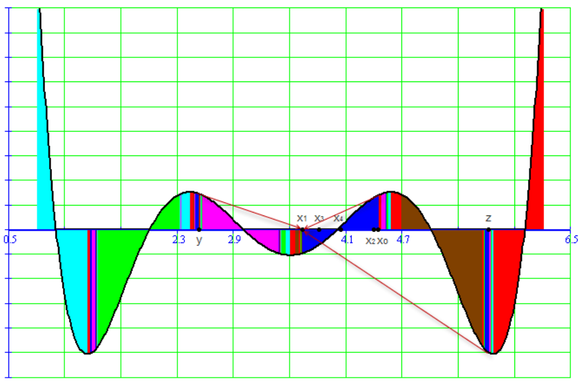


Figure 4.1 The basins of attraction for the zeros of the function (4.20)

As an example, we considered the start point $x_0 = 4.44$ and we have determinated the next five points generated by the iteration schema (**??**):

$x_1 = F(x_0) = 3.63448$ $x_2 = F(x_1) = 4.39176$ $x_3 = F(x_2) = 3.80975$
$x_4 = F(x_3) = 4.03653$ $x_5 = F(x_4) = 4.00031$.

As one can see in Figure 1, the point x_1 can be attempted by an iteration of Newton's method from the points $y = 2.52345$ and $z = 5.6169$, too.

So, we have $x_1 = F(y)$ and $x_1 = F(z)$. We determined too the points u, v and w such that $x_2 = F(u)$, $x_3 = F(v)$ and $x_4 = F(w)$, where $u = 2.48227$, $v = 5.61186$ and $w = 2.49599$.

For the values of $x_0, x_1, x_2, x_3, x_4, x_5$ and y, z, u, v, w and for the root $\alpha = 4$ we imagine the following tree structure scheme:

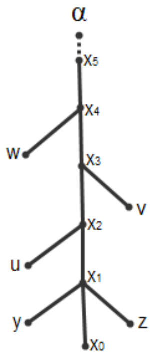


Figure 4.2 The structure of sequence starting from $x_0 = 4.44$

About the points $x_0, x_1, x_2, x_3, x_4, x_5, \ldots$ we can say that there are in the *base basin of attraction* of the root α, denoted $\overline{B}(\alpha)$, and about the points y, z, u, v, w we say that there are in the extended basin of attraction of the root α, denoted $B(\alpha)$. We can say that $\overline{B}(\alpha)$ is the largest interval that contains α and are included in $B(\alpha)$.

Let α be a real zero of the function $f : I \subseteq R \to R$. This zero can be determinated as a fixed point of an iterative method F such that for some values of the start point x_0, the sequence generated by

$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \ldots \tag{4.22}$$

is converging to α.

Let $B(\alpha)$ be the basin of attraction of the root α of the equation $f(x) = 0$, corresponding to the method F. On the set $B(\alpha)$ we define the next binary relation:

$$x, y \in B(\alpha), x \prec y \overset{\text{def}}{\Longleftrightarrow} \text{ there exists } k \in N \text{ such that } y = F^k(x). \tag{4.23}$$

So, we have $x \prec y$ iff y can be attempted from x by iterations of the method F.

By $F^k(x)$ we denoted $(\underbrace{F \circ F \circ \cdots \circ F}_{k \text{ times}})(x)$.

The relation $\prec$ is *reflexive* and *transitive*, so is a quasiorder relation on $B(\alpha)$.

Let $(B(\alpha), J_\prec)$ be the topological space with the left topology associated to the quasiorder relation $\prec$.

The family of sets

$$O_w\{v \in B(\alpha)/v \prec w\}$$

is the base of the topological space $(B(\alpha), J_\prec)$.

One can imagine for the basin of attraction $B(\alpha)$ the following tree-structure:

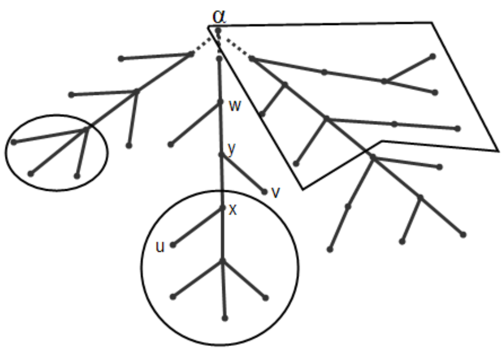


Figure 4.3 The tree-structure of the basin of attraction $B(\alpha)$

In this tree the knots are the elements (points) of $B(\alpha)$. On can pass from a knot to another only by iteration of the method F.

One considers that $t \prec \alpha$ for any $t \in B(\alpha)$ and that $\alpha \in B(\alpha)$. In the figure above, the sets with circular borders are examples of open sets, and the set with polygonal border is an example of closed set.

Let B be the union of all basins of attractions of the roots of the equations $f(x) = 0$ in the interval I and let $(B, \tau_\prec)$ be the topological space with the left topology associated to the quasiorder relation $\prec$ defined by (4.23) on B.

Some topological properties of the basin of attraction $B(\alpha)$ that results from the precedent considerations are the following proposition:

Teorema 4.3.8. ***(G. Ardelean*** **[16]*)*** 1. *The topological space* $(B(\alpha), J_\prec)$ *is connected.*

2. *The topological space* $(B(\alpha), J_\prec)$ *is compact.*

3. *The restriction of the application method at the set* $B(\alpha)$*,*

$$F : B(\alpha) \to B(\alpha)$$

is a continuous application in the sense of the topological space $(B(\alpha), J_\prec)$*.*

4. *The topological space* $(B, \tau_\prec)$ *is compact but is not connected. The only connected sets are the basins of attraction of the equation* $f(x) = 0$ *in the interval* I*.*

Finally, we present some topological considerations with respect to the complex case.

Let

$$f : D \subseteq C \to C \tag{4.24}$$

be a complex differentiable function on the rectangular domain D in the complex z plane that contains all the roots of the equation $f(x) = 0$. All the topological properties in real case are valid in the complex case too.

As an example, in this case we consider the function

$$f(x) = z^4 + 1 \tag{4.25}$$

and the domain $D = [-10, 10] \times [-10, 10]$.

In [**9**] we present a computer program to generate the basins of attraction for complex coefficients polynomials roots.

Some can associate the topological space $(B(\alpha), J_{\prec})$ with the topological space of *Hawaiian rings*, which is a continuum too (see [6]). The associate of the figures 4.4 and 4.5 is relevant.

The topological space of Hawaiian rings is the subspace $X \subset R^2$ that is the union of the circles C_n of radius $\dfrac{1}{n}$ and center $\left(\dfrac{1}{n}, 0\right)$ for $n = 1, 2, \ldots$.

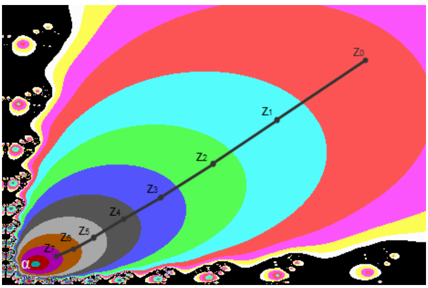


Figure 4.4 The basin of attraction of the root $\alpha = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i$

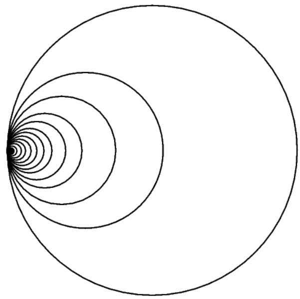


Figure 4.5 The Hawaiian rings

4. Conclusions

Our work connect too very different domains as the iterative methods for solving nonlinear equations and the topology. Our investigations developed the topological properties of the basins of attraction presented above.

We consider that the topological research of the basins of attraction can give more interesting and more relevant results.

CHAPTER 5

Application "Interactiv Bazin" for the basins of attraction

1. Description of application

"Interactive Basin" is an application which permits to generate the attraction basins for Newton methods applied to polynomials with complex coefficients. The soft gives an interactive graphical representation of attraction basins and gives information about the considered problem.

Therefore, after the launching, it is requested the necessary data to generate the attraction basins, namely the polynomial degree, the polynomial coefficients and the bounds of a domain in real plane, which contains the solutions.

After that, the attraction basins will be draw on the screen. The result is obtain by to Newton method.

The user can request information about any point in the established domain. This can be done by a left click on the selected point. These information are related to:

- the coordinates of selected point in real plan;

- the root to which Newton method converges. The start, the first iteration, is the selected point. Also, in case of convergence, it is displayed the number of iterations.

- the message "NU CONVERGE", in case of no convergence from the selected point;

- the polynomial value in the selected point:

- the obtained value which can be attained from the selected point by an iteration of Newton method;

Some of these information are viewed on screen by implemented dynamic elements, like dedicated cursor.

1.1. Complex plane and graphical screen

1.2. Input data

1.3. Attraction basins COLOURS/NUMBER OF ITERATIONS

Figure 5.3 contains all basic elements of *Interactiv Bazin* application. Here, can be viewed the attraction basins of $P_1(z)$ and remark that the colour of one pixel represents the necessary number of iterations by Newton method to reach the root of polynomial equation. The start point is z sitated in complex plan.

The white areas represent the set of divergences points of Newton method.

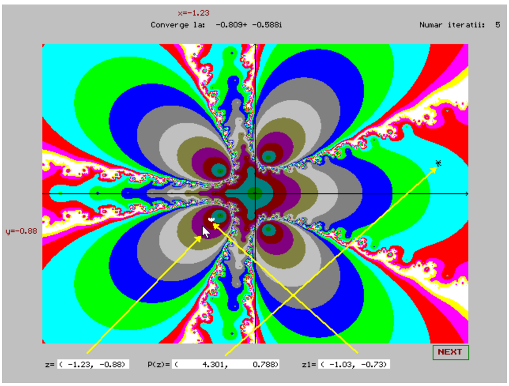


Figure 5.3. Attraction basins of $P_1(z)$ polynomial

On the screen appears the next elements:

- the three cursors:
  - the mouse cursor: 
    which points to a pixel on screen. This pixel is associate to an input data value z in complex plane;
  - the "angel"-cursor : 
    This cursor highlights on the screen the position related to $z1$ which represents the next Newton iteration for the given polynomial, i.e.

$$z1 = z - \frac{P(z)}{P'(z)}$$

  - the "daemon"-cursor: 
    This cursor highlights on screen the position of value $P(z)$, most often not visible because the value of $P(z)$ is outside of selected domain.
- At the bottom of the screen are displayed the values $z, P(z)$ and $z1$, which are update according to the position of the mouse;

After the initialize the value of z by positioning the mouse cursor on the screen and left click it will be displayed the next information:

- on top and left side of the screen are displayed the coordinates of z in complex plane;
- the root which Newton method converges from the initial point z and the number of iterations required to approximation with tolerance $\epsilon = 10^{-4}$ are displayed;
- if the method is not convergent, than it will be displayed the message "NU CONVERGE", this means that the cursor is in a white area on screen.

1.4. Basins of attraction COLOURS/ROOTS

As has been mentioned before, in this case the colours of areas on screen are associated with the roots of polynomial equation. The Newton method converges to the same root, from the point of an area with the same colour, and the white areas represent the set of point from which the Newton's method does not converge.

By the mouse, it can be select one screen point, which represents the value of z in complex plane and left click on it will be displayed the coordinates of z and the roots obtained by Newton's method, in case of convergence, or the message "NU CONVERGE" in case of divergence.

In Figure 5.6 are displayed the basins of attraction, obtained by Newton's method, for the $P_1(z)$ polynomial. For example, the green area is the set of all points from which the Newton's method converges to the root $z = -0.809 + 0.588i$, and the blue area is the set of all points from which the Newton iterations converge to the root $z = 0$.

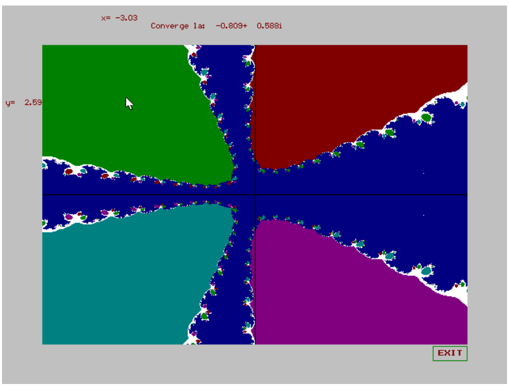


Figure 5.6 The colors are related to the roots

The white areas represent the set of divergence points related to Newton method.

An example is contains in Figure 5.7, where the message "NU CONVERGE" appears.

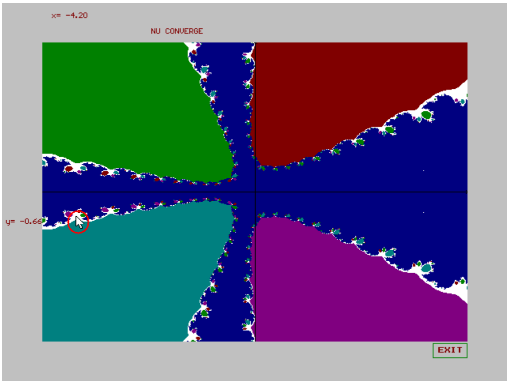


Figure 5.7 The white areas represent the points of divergence

Activation of "EXIT" button, launches the screen with values of polynomial roots, after that quit from application.

In **G. Ardelean** [**9**], we give an algorithm and the sours code written in Pascal, which can be used to found the roots, real and complex, of a given polynomial equation. Also, in **G. Ardelean**[**12**] we presented a new soft for view the graphical representation basins of attraction.

ANNEX - Mathcad and Pascal programs

Mathcad programs for generating the attraction basins

Pascal source programs

NEWTONPOL program

Newton Procedure

Pascal program sequence for computing the roots of the polynomial and generate the basins of attraction COLORS / NUMBER OF ITERATIONS

Pascal program sequence for computing the roots of the polynomial and generate the basins of attraction COLORS / ROOTS

The unit Ucomplex

Bibliography

[1] S. Abbasbandy, *Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method*, Appl. Math. Comput., **145** (2003), 887–893.

[2] O. Agratini, I. Chiorean, G. Coman, and R. Trîmbiţaş, *Analiză numerică şi teoria aproximării, Vol. III*, Presa Universitară Clujeană, 2002.

[3] L. V. Ahlfors, *Complex Analysis*, New York: McGraw-Hill, 1966.

[4] E. L. Allgover, K. Glasshoff, and H. O. Peitgen, *Numerical Solution of Nonlinear Equations: Lecture Notes in Mathematical 878*, New York: Springer-Verlag, 1981.

[5] S. Amat, S. Busquier, and J. M. Gutiérrez, *Geometric constructions of iterative functions to solve nonlinear equations*, J. Comput. Appl. Math., **157** (2003), 197–205.

[6] I. Andreadis and T. E. Karakasidis, *On a topological closeness of perturbed Mandelbrot sets*, Appl. Math. Comput., **215** (2010), 3674–3683.

[7] **G. ARDELEAN**, *Topologie, automate finite şi relaţia de dominare din lingvistica algebrică*, Studii şi Cercetări Matematice, **29** (1977), 577–585.

[8] **G. ARDELEAN**, *Some topological properties of a family of linear languages*, Topology, Vol. II(Proc. Fourth Colloq.,Budapest, 1978), 41-48, Colloq. Math. Soc. Janos Bolyai,23, North-Holland, Amsterdam, 1980, pp. 41–48.

[9] **G. ARDELEAN**, *On the determination of real and complex roots of a polynomial using the computer (in romanian)*, Carpathian J. Math., **07** (1991), 141–147.

[10] **G. ARDELEAN**, *Generating the basins of attraction for Newton's method*, Creative Math. and Inf., **17** (2008), 285–292.

[11] **G. ARDELEAN**, *A comparison between iterative methods by using the basins of attraction*, Appl. Math. Comput., **218** (2011), 88–95.

[12] **G. ARDELEAN**, *"Didactic bazin" - an application to view on the computer the basins of attraction*, 8-th International Conference on Applied Mathematics (2011), October 27–31, Baia Mare, Romania.

[13] **G. ARDELEAN**, *Graphical views for the basins of attraction of the real roots of a nonlinear equation*, Gaphical Models (2011), Manuscript number: GMOD–11–65(Under Review).

[14] **G. ARDELEAN**, *A New Third-Order Newton-Type Iterative Method for Solving Nonlinear Equations*, Appl. Math. Comput. (2011), Manuscript Number: AMC–D–11–02047R1(Under Review).

[15] **G. ARDELEAN**, *On "Some modifications of Newton's method with higher-order convergence for solving nonlinear equations"*, J. Comput. Appl. Math., (2011), Manuscript Number: CAM–D–11–00976(Under Review).

[16] **G. ARDELEAN**, *A topological structure on the basins of attraction for iterative methods*, Creative Math. and Inf., **20** (2011), 8–15.

[17] **G. ARDELEAN**, *Using symbolic computation in Mathematica for verifying the convrergence of the iterative methods*, 8-th International Conference on Applied Mathematics (2011), Otober 27–30, Baia Mare, Romania.

[18] **G. ARDELEAN**, *Prooving the convergence of the iterative methods by using symbolic computation in Maple*, Carpathian J. Math., **28** (2012), 1–8.

[19] I. K. Argyros, *On a theorem of L. V. Kantorovich concerning Newton's Method*, Appl. Math. Comput., **155** (2003), 223–230.

[20] I. K. Argyros, *Weak sufficient convergence conditions and applications for Newton methods*, J. Appl. Math. & Computing, **16** (2004), 1–17.

[21] I. K. Argyros, *Quasi-Newton methods for solving generalized equations*, Nonlinear Funct. Anal. & App., **11** (2006), 647–654.

[22] I. K. Argyros and F. Szidarovszky, *The Theory and Applications of Iteration Methods*, CRC Press, Boca Raton, 1993.

[23] J. Bak and D. J. Newman, *Complex Analysis*, New York: Springer-Verlag, 1982.

[24] M. Barnsley, *Fractals Everywhere*, New York: Academic Press, 1988.

[25] V. Berinde, *Remarks on the Convergence of the Newton-Raphson Method*, Rev. Anal. Numér. Théor. Approx., **24** (1995), 15–21.

[26] V. Berinde, *On some exit criteria for the Newton method*, Novi Sad J. Math., **27** (1997), 19–26.

[27] V. Berinde and M. Păcurar, *A Convergnce Theorem for Some Newton Type Methods under Weak Smoothness Conditions*, Analele Universităţii de Vest, Timişoara, Seria Matematică-Informatică, **2** (2008), 115–122.

[28] V. Berinde, *On the convergence of the Newton method*, Trans. of the Tech. Univ. of Kosice., **1** (1997), 68–77.

[29] V. Berinde, *Weak exit criteria for some Newton type methos*, Bulletins for Applied Mthematics, **77** (1995), 27–32.

[30] E. Bodewig, *Sur la méthod de L'aguerre pour l'approximation des racines des certaines équations et sur la critique d'Hermite*, Nederl. Acad. Vetensch. Proc., **49** (1946), 911–921.

[31] F. Cajori, *Historical note on the Newton-Raphson method of approximaton*, Amer. Math. Month., **18** (1911), 29–33.

[32] C.S. Calude, H. Jürgensen, and L. Staiger, *Topology on words*, Theoretical Computer Science, **410** (2009), 2323–2335.

[33] E. Cătinaş, *Inexact perturbed Newton methods and applications to a class of Krylov solvers*, J. Optim. Theory Appl., **108** (2001), 543–570.

[34] E. Cătinaş, *On the superlinear convergence of the successive approximations methods*, J. Optim. Theory Appl., **113** (2002), 473–485.

[35] E. Cătinaş, *The inexact, inexact-perturbed and quasi-Newton methods are equivalent models*, Math. Comp., **74** (2005), 291–301.

[36] E. Cătinaş, *Methods of Newton and Newton-Krylov type*, Risoprint, Cluj-Napoca, 2007.

[37] E. Cătinaş, *Estimating the radius of an attraction ball*, Appl. Math. Lett., **22** (2009), 712–714.

[38] F. Çilingir, *Mystery of the rational iteration arising from relaxed Newton's method*, Chaos, Solutions & Fractals, **32** (2007), 471–479.

[39] C. Chun, *Iterative Methods Improving Newton's Method by the Decomposition Method*, Computers&Mathematics with Applications, **50** (2005), 1559–1568.

[40] C. Chun, *A simply constructed third-order modifications of Newton's method*, J. Comput. Appl. Math., **219** (2008), 81–89.

[41] C. Chun and Y. Ham, *Some fourth-order modifications of Newton's method*, Appl. Math. Comput., **197** (2008), 654–658.

[42] O. Cira, *Lecţii de Mathcad 2001 Professional*, Editura Albastră, Cluj Napoca, 2003.

[43] O. Cira, *Metode numerice pentru rezolvarea ecuaţiilor algebrice*, Editura Academiei Române, Bucureşti, 2005.

[44] O. Cira and C. M. Cira, *Attraction basin for Broyden's method (numerical experiments)*, Bulletins for Applied and Computing Mathematics, PAMM, Iterunuversity Network in Central Europe, number 2440 in BAM-CVII, pages 41-48, 2005.

[45] O. Cira and Şt. Măruşter, *Metode numerice pentru ecuaţii neliniare*, Editura Matrix Rom, Bucureşti, 2008.

[46] O. Cira, *The Attraction Basins for Iterative Methods*, Proceedings of the Eigth Symposium of Mathematics and its Applications, Timişoara Research of the Romanian Academy and "Politehnica" University of Timişoara, 4-7 November (1999), 39–46.

[47] O. Cira, *Numerical Experiments on Attraction Basin*, Advanced Modelling and Optimization (AMO), **2(3)** (2001), 122–134.

[48] O. Cira and D. Bucerzan, *Graphics Reprezentation of Attraction Basins for Iterative Method of Aproximating the Roots of Nonlinear Equations*, Analele Universităţii "Aurel Vlaicu" din Arad, Fascicola Matematică-Informatică (2002), 181–186.

[49] G. Coman, *Analiză numerică*, Editura Libris, Cluj Napoca, 1995.

[50] S. D. Conte and C. de Boor, *Elementary Numerical Analysis, 3rd ed.*, 1980.

[51] Ion Gh. Şabac, *Matematici speciale*, Editura Didactică şi Pedagogică, Bucureşti, 1981.

[52] J. H. Curry, L. Garnett, and D. Sullivan, *On the iteration of a rational function: Computer experiments with Newton's method*, Communications in Mathematical Phisics, **91** (1983), 267–277.

[53] B. Demidovitch, *Elémentes de calcul numérique*, Editions Mir, Moscou, 1973.

[54] A. Diaconu, *On the acceleation of the convergence of certain iterative methods using a proceeding of Traub's type*, Pure Mathematics and Applications, **15**,No. 2-3, (2004), 135–147.

[55] A. Diaconu, *The approximation of the equation's solution in linear normed spaces using approximant sequences*, Studia Univ. "Babeş-Bolyai", Mathematics, **XLVIII**, No.1, (2003), 55–72.

[56] W. S. Dorn and D. D. Mc. Cracken, *Metode numerice cu programe în FORTRAN IV*, Editura Tehnică, Bucureşti, 1976.

[57] J. Dzunic, M. S. Petković, and L. D. Petković, *A family of optimal three-point methods for solving nonlinear equations using two parametric functions*, Appl. Math. Comput., **217** (2011), 7612–7619.

[58] B. Ezzell, *Graphics Programming in Turbo Pascal 6.0*, Addison Wesley, 1991.

[59] J. D. Foley and A. Van Dam, *Fundamentals of Interactive Computer Graphics*, Addison Wesley, London, 1990.

[60] W. Gautschi, *Families of algebraic test equations*, Calcolo, **16** (1979), 383–398.

[61] W. Gautschi, *Questions of numerical condition related to polynomials*, Studies in Numerical Analysis (G. H. Golub ed.) Washington, D.C.: MAA (1984), 140–177.

[62] W. Gautschi, *Numerical Analysis: An Introduction*, Birkhäuser, 1997.

[63] J. Glieck, *Chaos*, New York: Viking Press, 1987.

[64] J. M. Gutiérrez and M. A. Hernández, *An acceleration of Newton's method:Super-Halley method*, Appl. Math. Comput., **117** (2001), 223–239.

[65] E. Halley, *A new, exact and easy method for finding the roots of any equations generally, without any previous reduction (Latin)*, Phylos. Trans. Roy. Soc. London, **18** (1694), 136–148.

[66] E. Halley, *A new, exact and easy method for finding the roots of any equations generally, without any previous reduction*, Phylos. Trans. Roy. Soc. London, **3** (1809), 640–649.

[67] YM Ham, C. Chun, and S-G Lee, *Some higher-order modifications of Newton's method for solving nonlinear equations*, J. Comput. Appl. Math., **222** (2008), 477–486.

[68] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[69] P. Henrici, *Applied and Computational Complex Analysis*, New York: Wiley, 1974.

[70] H. H. H. Homeier, *On Newton-type methods with cubic convergence*, J. Comput. Appl. Math., **176** (2005), 425–432.

[71] H. H. H. Homeier, *On Newton-type methods for multiple roots with cubic convergence*, J. Comput. Appl. Math., **231** (2009), 249–254.

[72] A. Householder, *The numerical Treatment of Single Nonlinear Equation*, McGraw-Hill, New York, 1970.

[73] J. Hubbard, D. Schleicher, and S. Sutherland, *How to find all roots of complex polynomials by Newton's method*, Invent. math., **146** (2001), 1–33.

[74] Stoer J. and Bulirsch R., *Introduction to Numerical Analysis*, 1980.

[75] M. A. Jenkins and J. F. Traub, *A three-stage variable-schift iteration for polynomial zeros*, NM, **14** (1970), 252–263.

[76] G. Julia, *Mémoire sur l'itération des fonctions rationnelles*, Journal de Math. Pure et Appl., **8** (1918), 47–245.

[77] B. Kalantari, *Polinomiography and applications in art, education and science*, Comput. Graph., **28** (2004), 417–430.

[78] L. V. Kantorovich, *Functional Analysis in Applied Mathematics (în limba rusă)*, Uspechi Mat. Nauk, **3** (1948), 89–135.

[79] E. M. Kasenally, *A Generalized Minimum Backward Error Algorithm for Nonsymmetric Linear Systems*, SIAM J. Sci. Statist. Comput., **16** (1995), 698–719.

[80] E. M. Kasenally and V. Simoncini, *Analysis of a minimum perturbation algorithm for nonsymmetric linear systems*, SIAM J. Numer. Anal., **34** (1997), 48–66.

[81] C. T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, number 16 in Frontiers in Applied Mathematics, SIAM, Philadelphia, 1995.

[82] L. J. Kelley, *General Topology*, Van Nostrand Company, inc., 1955.

[83] D. Kincaid and W. Cheney, *Numerical Analysis: Mathematics of Scientific Computing*, AMS, 2009.

[84] N. Kollerstrom, *Thomas Simpson and 'Newton's method of approximation': an enduring myth*, BJHS, **25** (1992), 347–354.

[85] J. Kou, *The improvements of modified Newton's method*, Appl. Math. Comput., **189** (2007), 602–609.

[86] J. Kou and Y. Li, *Modified Chebyshev-Halley methods with sixth-order convergence*, Appl. Math. Comput., **188** (2007), 681–685.

[87] J. Kou, Y. Li, and X. Wang, *A family of fifth-order iterations composed of Newton and third-order methods*, Appl. Math. Comput., **186** (2007), 1258–1262.

[88] J. Kou, Y. Li, and X. Wang, *A family of fourth-order methods for solving non-linear equations*, Appl. Math. Comput., **188** (2007), 1031–1036.

[89] J. Kou, Y. Li, and X. Wang, *A variant of super-Halley method with accelerated fourth-order convergence*, Appl. Math. Comput., **186** (2007), 535–539.

[90] L.E. Kozma and **G. ARDELEAN**, *A possible visual image of the extended monogenous functions*, Creative Math. and Inf., **18** (2009), 182–187.

[91] L. E. Kozma, *Analiză matematică - funcţii complexe*, Ed. RISO-PRINT Cluj-Napoca, 2006.

[92] H. T. Kung and J. F. Traub, *Optimal order of one-point and multipoint iteration*, Journal of the ACM, **21** (1974), 643–651.

[93] A. Lupaş, *Metode numerice*, Editura Tehnică, Bucureşti, 1989.

[94] B. Mandelbrot, *The Fractal Geometry of Nature*, New York: Freeman, 1982.

[95] M. Marden, *Geometry of Polynomials*, 1966.

[96] Şt. Măruşter, *The solution by iteration of nonlinear equations in Hilbert spaces*, Proc. Amer. Math Soc., **63(1)** (1977), 69–73.

[97] Şt. Măruşter, *On the tangent parabola method for nonlinear equations in one variable*, Seminar of Informatics and Computational Mathematics, Depart. of Informatics, West Univ., Timişoara, Vol. 1 (1977), 1–9.

[98] Şt. Măruşter, *Numerical experiments on attraction basin*, Anals Computer Science Series, **2**, 162–171.

[99] Şt. Măruşter, *Metode numerice în rezolvarea ecuaţiilor neliniare*, Editura Tehnică, Bucureşti, 1981.

[100] I. Newton, *Method of Fluxions*, 1671(publicat in 1736).

[101] M. A. Noor, W. A. Khan, and A. Hussain, *A new modified Halley method without second derivatives for nonlinear equation*, Appl. Math. Comput., **189** (2007), 1268–1273.

[102] M. A. Noor, W. A. Khan, and A. Hussain, *A new modified Halley method without second derivatives for nonlinear equations*, Appl. Math. Comput., **189** (2007), 1268–1273.

[103] M. A. Noor and K. I. Noor, *Modified iterative methods with cubic convrgrnce for solving nonlinear equations*, Appl. Math. Comput., **184** (2007), 322–325.

[104] J. J. O'Connor and E. F. Robertson, *Joseph Raphson*, (1996).

[105] A. M. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[106] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Space*, Academic Press, New York, 1973.

[107] A. Y. Ozban, *Some New Variants of Newton's Method*, Appl. Math. Lett., **17** (2004), 677–682.

[108] H. Peitgen, D. Saupe, and V. F. Haeseler, *Cayley's problem and Julia sets*, Mathematics Intelligencer, **6** (1984), 11–20.

[109] H. O. Peitgen and P. H. Richter, *The Beauty of Fractals*, Springer-Verlag, New York, 1986.

[110] G. Peters and J. H. Wilkinson, *Practical problems arising in the solution of polynomial equations*, IMAJNA, **8** (1971), 16–35.

[111] L. D. Petković and M. S. Petković, *A note on some recent methods for solving nolinear equations*, Appl. Math. Comput., **185** (2007), 368–374.

[112] M. S. Petković and L. D. Petković, *A one parameter square root family of two-step root-finders*, Appl. Math. Comput., **188** (2007), 339–344.

[113] M. S. Petković and L. D. Petković, *Families of optimal multipoint methods for solving nonlinear equations*, Appl. Anal. Discrete Math., **4** (2010), 1–22.

[114] C. A. Pickover, *A note on chaos and Halley's methods*, ACMCOM, **31** (1988).

[115] S. Plaza and N. Romero, *Attracting cycles for relaxed Newton's method*, J. Comput. Appl. Math., **235** (2011), 3238–3244.

[116] I. Pop, *Topolgie Algebrică*, Editura Ştiinţifică, Bucureşti, 1990.

[117] V. Prejmerean, *Grafică pe calculator şi prelucrări de imagini*, Universitatea de Nord, Baia Mare, 2000.

[118] I. Păvăloiu, *Introducere în Teoria Aproximării Soluţilor Ecuaţiilor*, Editura Dacia, Cluj Napoca, 1976.

[119] I. Păvăloiu, *Rezolvarea ecuaţiilor prin iterpolare*, Editura Dacia, Cluj Napoca, 1981.

[120] I. Păvăloiu and E. Cătinaş, *On a Steffensen-Hermite method of order three*, Appl. Math. Comput., **215** (2009), 2663–2672.

[121] I. Păvăloiu and N. Pop, *Interpolare şi aplicaţii*, RISOPRINT Cluj Napoca, 2005.

[122] A. Raltson and P. Rabinowitz, *A First Course in Numerical Analysis*, New York:McGrow-Hill, 1978.

[123] J. Raphson, *Analysis aequationum universalis*, London, 1690.

[124] R. K. Roeder, *Topology For the Basins of Attraction of Newton's Method in two complex variables*, A Disertation Presented to the Faculty of the Graduate School of Cornell University (2005).

[125] Y. Saad and M. H. Schultz, *GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsimmetric Linear Equations*, SIAM J. Sci. Statist. Comput., **7** (1986), 856–869.

[126] M. L. Sahari and I. Djellit, *Fractal Newton basins*, Discrete Dynamics in Nature and Society, 28756(Article ID) (2006), 16 pages.

[127] M. L. Sahari and I. Djellit, *Fractalization du bassin d'attractions dans l'algorithme de Newton modifié*, Esaim:Proceedings, **20** (2007), 208–216.

[128] E. Schröder, *Über unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann., **2** (1870), 317–365.

[129] T. Simpson, *Essays ... on Mathematics*, London, 1740, 81.

[130] H. Susanto and N. Karjanto, *Newton's method basin of attraction revisited*, Appl. Math. Comput., **215** (2009), 1084–1090.

[131] S. Sutherland, *Finding roots of complex poynomials with Newton's method*, Preprint, Institute for Mathematical Sciences: SUNY Stony Brook (1990).

[132] J. F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1982.

[133] R. Valk, *Topologische Wortmengen, topologische Automaten, Zustandsendliche stetige Abbildungen*, Mitteilungen der Gessellschaft für Matematik und Datenverarbeitung, Bonn, **19** (1972).

[134] J. L. Varona, *Graphic and numerical comparison between iterative methods*, Math. Intell., **24** (2002), 37–46.

[135] M. Vlada, A. Posea, I. Nistor, and şi alţii, *Grafica pe calculator în limbajele Pascal şi C, Implementare şi aplicaţii*, Editura Tehnică, Bucureşti, 1992.

[136] *Website*, Maple, *http://www.maplesoft.com/*.

[137] *Website*, Matlab, *http://www.mathworks.com/*.

[138] *Website*, Mathcad, *http://www.ptc.com/*.

[139] *Website*, Mathematica, *http://www.wolfram.com/*.

[140] S. Weerakoon and T. G. I. Fernando, *A Variant of Newton's Method with Accelerated Third-Order Convergence*, Appl. Math. Lett., **13** (2000), 87–93.

[141] J. H. Wilkinson, *The perfidious polynomial*, Studies in Numerical Analysis (G. H. Golub ed.) Washington, D. C.:MAA (1984), 1–28.

[142] Z. Xiaojian, *A class of Newton's methods with third-order convergence*, Appl. Math. Lett., **20** (2007), 1026–1030.

[143] T. Yamamoto, *Historical developments in convergence analysis for Newton's and Newton-like methods*, J. Comput. Appl. Math., **124** (2000), 1–23.

[144] D. M. Young and R. T. Gregory, *A survey of Numerical Mathematics Vols. 1 and 2*, Reading Mass.:Addison-Wesley (Reprinted New York: Dover 1988)., 1972.