North University of Baia Mare
Faculty of Science

Ph.D. Thesis

FIXED POINTS THEOREMS IN METRIC SPACES ENDOWED WITH A GRAPH

Scientific advisor:
Prof. Univ. Dr. Vasile Berinde

Ph.D. Student:
Bojor Florin

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Introduction

The Fixed Point Theory is one of the most powerful and productive tools from the nonlinear analysis and it can be considered the kernel of the nonlinear analysis. The best known result from the Fixed Point Theory is Banach's Contraction Principle (1922), which can be considered the beginning of this theory. In a metric space setting it can be briefly stated as follows.

Theorem 0.0.1. *Let (X, d) be a complete metric space and $T : X \to X$ a contraction, i.e. a map satisfying*

$$d(Tx, Ty) \leqslant a d(x, y), \ \forall x, y \in X, \tag{0.1}$$

where $0 \leq a < 1$ is constant. Then:

(1) T has a unique fixed point p in X;
(2) The Picard iteration $(x_n)_{n \geq 0}$ defined by $x_{n+1} = Txn, n = 0, 1, 2, ...$ converges to p, for any $x_0 \in X$.

Following Petruşel and Rus [**26**], we say that T is a Picard operator (abbr., PO) if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$ and is a weakly Picard operator (abbr. WPO) if the sequence $(T^n x)_{n\in\mathbb{N}}$ converges , for all $x \in X$ and the limit (which may depend on x) is a fixed point of T. So, Banach's Principle Contraction says that any contraction on complete metric spaces is PO. One of the ways to generalize and extend the Banach's Principle Contraction is to substitute the condition 0.1 with a weaker condition or independent one.

Because any contraction is a continuous operator, it is natural to ask:*Are there contraction conditions which do not imply the continuity of the operator?*

The first answer of this question was give by R. Kannan [**20**] in 1968 who proved a fixed point theorem for operators which don't have to be continuous and replace the condition 0.1 with: there exists $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Tz) \leqslant b\left[d(x, Tx) + d(y, Ty)\right], \ \forall x, y \in X. \tag{0.2}$$

Following Kannan, Chatterjea [**10**] proved a fixed point theorem for operators which satisfies the condition: there exists $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Tz) \leqslant b\left[d(x, Ty) + d(y, Tx)\right], \ \forall x, y \in X. \tag{0.3}$$

It's well know, see Rhoades [**30**], that the three conditions (0.1), (0.2) and 0.3) are independent.

Using the above conditions (0.1), (0.2) and (0.3) L. Ćirić [**13**], S. Reich [**29**] and I.A. Rus [**32**] proved a fixed point theorem using a very general condition: there is a nonnegative numbers a, b, c with $a + b + c < 1$ such that

$$d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty), \ \forall x, y \in X \tag{0.4}$$

By combining the three independent conditions (0.1), (0.2) and (0.3) in an inspired manner, T. Zamrfescu [**34**] considered the following class of operators: there is $a \in [0, 1)$ and $b, c \in \left[0, \frac{1}{2}\right)$ such that for any $x, y \in X$ at least one of the following holds:

(z_1) $d(Tx, Ty) \leqslant ad(x, y)$;
(z_2) $d(Tx, Ty) \leqslant b[d(x, Tx) + d(y, Ty)]$;
(z_3) $d(Tx, Ty) \leqslant c[d(x, Ty) + d(y, Tx)]$.

Using the comparison function in 1983 I.A. Rus [**31**] gave another generalization of Banach Contraction Priciple replesing the condition (0.1) with the next condition: there is a comparison function. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx, Ty) \leqslant \varphi(d(x, y)), \ \forall x, y \in X. \tag{0.5}$$

A generalization of Kannan Theorem was made by R.M.T. Bianchini [**2**] who replace the condition (0.2) with: there is $a \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant a \max\{d(x, Tx), d(y, Ty)\}, \ \forall x, y \in X. \tag{0.6}$$

In 2003, W. A. Kirk, P. S. Srinivasan and P. Veeramani [**21**] introduced a new method to generalize the condition 0.1), and that is reducing the number of pairs $(x, y) \in X \times X$ which satisfied the contraction condition. For do this, they considered A and B is two nonempty and closed set of the metric space X and the operator $T : A \bigcup B \to A \bigcup B$ satisfied:

(1) $T(A) \subseteq B$, and $T(B) \subseteq A$;
(2) $d(Tx, Ty) \leqslant ad(x, y), \ \forall x \in A, \ \forall y \in B$..

In these condiţitions the operator T has an unique fixed point . In the same paper W. A. Kirk, P. S. Srinivasan and P. Veeramani extended the above condition to p sets, where $p \geq 2$.

One year later, A. C. M. Ran and M. C. B. Reurings [**28**] considered operators om partial ordered metric spaces, and the contraction condition (0.1) was replace with: there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant ad(x, y), \ \forall x, y \in X, \ x \leqslant y, \tag{0.7}$$

where " $\leq$ " is the partial order on space X.

In 2008, J. Jachymski [**17**] had the excellent ideea to use the metric spaces endowed with a graph instead of partial metric space, and the contraction condition (0.1) will be satisfied only for the edges of the graph. If G is a directed graph and $E(G)$ is the set of of its edges then the contraction condition is:

$$d(Tx,Ty) \leqslant ad(x,y), \ \forall (x,y) \in E(G). \tag{0.8}$$

Starting from this results, the goal of the thesis is to make a methodically study of fixed point theorem in metric spaces endowed with a graph. The contraction conditions which we used is: Kannan, Zamfirescu, Ćirić-Reich-Rus, $\varphi-$contractions and Bianchini.

The study material has been organized in four chapters, connected to each other on various threads. The 1^{st} Chapter, Preliminaries, is far from exhaustively introducing all the concepts and basic results of metrical fixed point theory, being rather a brief overview of the prerequisites of this particular book. Among the six sections, referring to basic notations and notions, the concepts of metric and metric space, graphs, comparison functions, classes of operators on metric spaces and appropriate fixed point theorem as well as fixed point theorem in partially ordered metric spaces, and for cyclical operators.

The personal contributions in the first chapter are: Definition 1.4.3.

In the second chapter, we present the concept of Banach G-contraction, concept who was first introduce by J. Jachymski in [**17**] and fixed point theorems for this type of contractions. Further on, we prove that: Fixed point theorem in partially ordered metric spaces, Edelstein's Theorem, Fixed point theorems for cyclical operators and Fixed point alternative are consequences of Jachymski's theorem 2.1.2. After that we prove Hyers-Ulam stability of two functional equation.

The personal contributions in the second chapter are: Theorem 2.2.4, 2.2.5 and 2.2.6, Corollaries 2.2.1, 2.2.2, 2.2.3, 2.2.4, Lemma 2.2.1, Conclusion 2.2 and the proof of Theorem 2.2.2 and 2.2.3.

In the 3^{st} Chapter we extend clasical fixed point theorem by Kannan, Zamfirescu, Reich-Ćirić-Rus, Bianchini for metric spaces endowed with a graph. From this we conclude fixed point theorem in partial ordered metric spaces and fixed point theorem for cyclical operators. For each new concept we show the extention is not trivial, giving some example.

The personal contributions in the third chapter are:
Definitions 3.1.1,3.2.1,3.3.1,3.4.1,3.5.1,
Theorems 3.1.1, 3.2.1, 3.3.1, 3.4.1, 3.4.2, 3.4.3, 3.5.1,
Corollaries 3.1.1, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.3.1, 3.3.2, 3.3.3, 3.3.4, 3.4.1, 3.4.2, 3.5.1,
Lemmas 3.1.1, 3.2.1, 3.3.1, 3.3.2, 3.5.1, 3.5.2,
Examples 3.1.3, 3.1.1, 3.1.4, 3.1.5, 3.1.6, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.3.1, 3.3.2,3.3.3, 3.4.1,3.4.2, 3.5.2, 3.5.3, 3.5.4,

Observations 3.1.1, 3.2.1, 3.2.2,
the proof of Theorems 3.1.2, 3.2.2, 3.3.2, 3.5.2.

The last Chapter of this thesis, we extend the fixed point theorems from Chapter 3 in two different context:

a) working in metric spaces G-complete;

b) working on set endowed with two metrics(Maia type theorem).

The personal contributions in this chapter are: Definition 4.1.2,
Example 4.1.1, Theorems 4.1.1 and 4.2.2.

CHAPTER 1

Preliminaries

In this chapter we present some basic notions, like the concept of metric spaces, Banach space, concepts regarding comparison functions, the notion of fixed point structure accompanied by examples. The fourth section is dedicated to some concepts regarding graphs, like the path between two vertices and the connectivity of graph. And this is:

4. Graphs

The next definitions and examples is well-know. (see [**15**] and [**19**]).

Definition 1.4.1. *An **directed graph** G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, disjoint from $V(G)$ together with an incidence function ψ_G that associates with each edge of G an ordered pair of (not necessarily distinct) vertices of $V(G)$.*

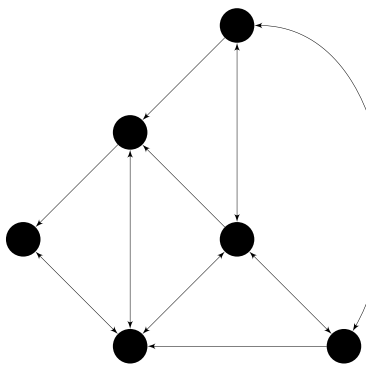

Fig. 1 *Directed Graph G*

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$V\left(G^{-1}\right) = V(G) \text{ şi } E\left(G^{-1}\right) = \{(b,a) \mid (a,b) \in E(G)\}$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$V\left(\widetilde{G}\right) = V(G) \text{ şi } E\left(\widetilde{G}\right) = E(G) \cup E\left(G^{-1}\right) \quad (1.9)$$

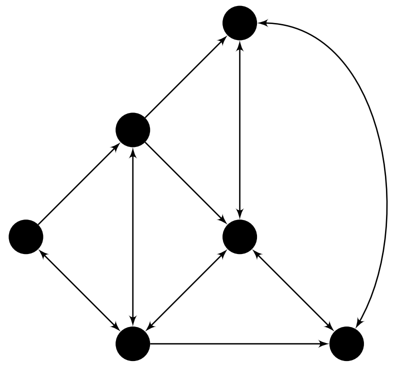

Fig. 3 *The converse of graph G*

Now we recall a few basic notions concerning connectivity of graphs.

Definition 1.4.2. *If x and y are vertices in a graph G, then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$.*

A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected.

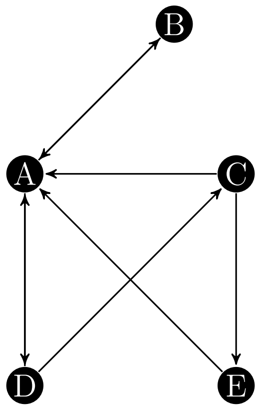

Fig. 6 Connected graph

Definition 1.4.3. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. We say that the graph G is $T-$connected if for all vertices x, y of G with $(x, y) \notin E(G)$, there exists a path in G,$(x_i)_{i=0}^{N}$ from x to y such that $x_0 = x, x_N = y$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, ..., N-1$.*

CHAPTER 2

Fixed point theorems for Banach G-contractions

The main topic of this thesis is concerned with contractive operators on metric spaces endowed with a graph. The first author which deal with this type of operators it was J. Jachymski [**17**]. In this chapter we will present the Jachymski results and we will prove that the fixed point theorems in partial ordered metric spaces, Edelestein's theorem, fixed point theorem for cyclical operators and Fixed point alternative are consequences of Jachymski's theorem 2.1.2. Then we will prove two apllications of Fixed point alternative to Hyers-Ulam stability for generalized Cauchy equation and first order linear differential equation.

The personal contributions in the second chapter are: Theorem 2.2.4, 2.2.5 and 2.2.6, Corollaries 2.2.1, 2.2.2, 2.2.3, 2.2.4, Lemma 2.2.1, Conclusion 2.2 and the proof of Theorem 2.2.2 and 2.2.3.

1. Fixed point theorems for Banach G-contractions

Throughout this section we assume that (X,d) is a metric space, and G is a directed graph such that $V(G)=X$ and $\Delta=\{(x,x)\,|x\in X\}\subseteq G$.

Definition 2.1.1 ([**17**], Def. 2.1)**.** *We say that a mapping $f:X\to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.*

$$\forall x,y\in X\left((x,y)\in E(G)\Rightarrow(f(x),f(y))\in E(G)\right)\tag{2.10}$$

and f decreases weights of edges of G in the following way:

$$\exists\alpha\in(0,1)\,,\forall x,y\in X\left((x,y)\in E(G)\Rightarrow d(f(x),f(y))\leqslant\alpha d(x,y)\right)\tag{2.11}$$

Example 2.1.1 ([**17**], Ex. 2.1)**.** *Any constant function $f:X\to X$ is a Banach G-contraction since $E(G)$ contains all loops. (In fact, $E(G)$ must contain all loops if we wish any constant function to be a $G-$contraction.)*

Example 2.1.2 ([**17**], Ex. 2.2)**.** *Any Banach contraction is a G_0-contraction, where $V(G_0)=X$ and $E(G_0)=X\times X$.*

Proposition 2.1.1 ([**17**], P. 2.1)**.** *If $f:X\to X$ is a G-contraction, then atunci f is both a G^{-1}-contraction and a $\widetilde{G}$-contraction.*

Theorem 2.1.1 ([**17**], T. 3.1)**.** *The following statements are equivalent:*

(i) G is weakly connected;

(ii) for any G-ontraction $f: X \to X$ given $x, y \in X$, the sequences $(f^n x)_{n\in\mathbb{N}}$ and $(f^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;
(iii) for any G-contraction $f: X \to X$, card $(Fix(f)) \leqslant 1$.

Corollary 2.1.1 ([**17**], C. 3.1)**.** *Let (X, d) be complete. The following statements are equivalent:*

(i) G is weakly connected;
(ii) for any G-contraction $f: X \to X$, there is $x^* \in X$ such that $\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$.

Theorem 2.1.2 ([**17**], Th 3.2)**.** *Let (X, d) be complete, and let the triple (X, d, G) have the following property:*

(2.12)
$$\text{for any } (x_n)_{n\in\mathbb{N}} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G)$$

for all $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$.

Let $f: X \to X$ be a G-contraction, and $X_f = \{x \in X \,|(x, fx) \in E(G)\}$. Then the following statements hold.

1. *card $(Ff) =$ card $\left\{[x]_{\widetilde{G}} \,|x \in X_f\right\}$.*
2. *$Ff \neq \emptyset$ iff $X_f \neq \emptyset$.*
3. *f has a unique fixed point iff there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*
4. *For any $x \in X_f$, $f\big|_{[x]_{\widetilde{G}}}$ is a PO.*
5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*
6. *If $X' := \cup\left\{[x]_{\widetilde{G}} \,|x \in G\right\}$ then $f\,|_{X'}$ is a WPO.*
7. *If $f \subseteq E(G)$, then f is a WPO.*

2. Aplicaţii ale Banach G-contracţiilor

Conclusion 2.1. *Banach's Contraction Principle is a consequence of Theorem2.1.2.*

Conclusion 2.2. *Theorem for contraction on partial ordered metric spaces given by Ran and Reurings* [**28**] *is a consequence of Theorem 2.1.2.*

On the other hand, Theorem 2.1.2 yields directly the following well-known fixed point theorem which is quite different from the above results.

Theorem 2.2.1 (Edelstein, [**16**])**.** *Let (X, d) be complete and ϵ-chainable for some $\epsilon >$ 0, i.e., given $x, y \in X$ there is $N \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^{N}$ such that $x_0 = x, x_N = y$ and $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, ..., N$. Let $T: X \to X$ be such that*

$$\exists \alpha \in (0,1), \forall x, y \in X \,(d(x, y) < \varepsilon \Rightarrow d(Tx, Ty) \leqslant \alpha d(x, y)). \tag{2.13}$$

then T is a PO.

Fixed point theorem of W.A. Kirk, P.S. Srinivasan and P. Veeramani [**21**] yields directly from Theorem 2.1.2.

Theorem 2.2.2 ([**21**])**.** *Let $p \in \mathbb{N}$, $p \geq 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space, and suppose $T : \{A_i\}_{i=1}^{p} \to \{A_i\}_{i=1}^{p}$ satisfies the following conditions*

1 $T(A_i) \subseteq A_{i+1}$, where $A_{p+1} = A_1$;

2

(2.14)

$\exists k \in (0, 1$ such that $) \, d(Tx, Ty) \leqslant kd(x, y)$, *for* $x \in A_i$, $y \in A_{i+1}$, for$1 \leq i \leq p$

Then T has a unique fixed point.

Using the Jachymski's fixed point theorem we will prove The fixed point alternative the form introduced by V. Radu in [**27**].

Theorem 2.2.3 (Fixed point alternative)**.** *Suppose we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with the Lipschitz constant a. Then, for each given element $x \in \Omega$, either*

$$d\left(T^n x,\, T^{n+1} x\right) = \infty,\ \forall n \geqslant 0,$$

or there exists a natural number n_0 such that

i. $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geqslant n_0$;

ii. The sequence $(T^n x)_{n \geqslant 0}$ *is convergent to a fixed point* $y*$ *of* T;

iii. $y*$ *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega \,|\, d(T^{n_0} x, y) < \infty\}$;

iv. $d(y, y*) \leqslant \frac{1}{1-a} d(y, Ty)$ *for all* $y \in \Delta$.

As announced in the begining of this chapter, our main aim in the following is to present Hyers-Ulam stability of two functional equation.

First is a generalized Cauchy equation:

$$f\left(a_1(x_1) + a_2(x_2) + ... + a_n(x_n)\right) = m_1 f(x_1) + m_2 f(x_2) + ... + m_n f(x_n), \tag{2.15}$$

where $a_i : X \to X$, $i = \overline{1, n}$ there are a given mappings, $f : X \to Y$ is the unknown function and $m_1, m_2, ..., m_n$ are n real numbers whit some property. The stability of this equation was proved by author in [**3**].

From now on, let X be a real vector and $(Y, \|\cdot\|)$ be a real Banach space and for real numbers $m_1, m_2, ..., m_n$, we set:

$$m_1 + m_2 + ... + m_n = S.$$

For a given function $g : X \to Y$ and $a_i : X \to X$, $i = \overline{1, n}$, n given functions we set:

$$Dg(x_1, x_2, ..., x_n) := g\left(a_1(x_1) + a_2(x_2) + ... + a_n(x_n)\right) - m_1 g(x_1) - m_2 g(x_2) - ... - m_n g(x_n)$$

and

$$A(x) := a_1(x) + a_2(x) + ... + a_n(x) \tag{2.16}$$

for all $x, x_1, x_2, ..., x_n \in X$.

For completeness, we will first present the next lemma:

Lemma 2.2.1. *Let X and Y two real vector spaces and at least two of the functions $a_i : X \to X$, $i = \overline{1,n}$ are bijective. If a function $f : X \to Y$ satisfies the functional equation (2.15), $S \neq 1$ and $A(0) = 0$ then the function f is an additive function.*

Utilizing the fixed point alternative, we now obtain the generalized Hyers-Ulam-Rassias stability of the equation (2.15).

Let $\varphi : X^n \to [0, \infty)$ be a function such that

$$\lim_{k\to\infty} \frac{\varphi\left(A^{ki}(x_1), A^{ki}(x_2), ..., A^{ki}(x_n)\right)}{|S|^{ki}} = 0 \tag{2.17}$$

for all $x_1, x_2, ..., x_n \in X$, where $i \in \{\pm 1\}$.

Theorem 2.2.4. *Suppose that $S \neq 0$ and the function $A : X \to X$ is a bijective and additive function and commuting with all function $a_i, i = \overline{1,n}$. Assume that, the function*

$$x \mapsto \psi(x) = \varphi\left(A^{-1}(x), A^{-1}(x), ..., A^{-1}(x)\right)$$

has the property $\exists\, M = M(i) \in (0,1)$ such that

$$\psi\left(A^i(x)\right) \leqslant |S|^i M \psi(x), \ \forall x \in X, i \in \{\pm 1\}. \tag{2.18}$$

If a function $f : X \to Y$ satisfies the functional inequality

$$\|Df(x_1, x_2, ..., x_n)\| \leqslant \varphi(x_1, x_2, ..., x_n), \ \forall\, x_1, x_2, ..., x_n \in X. \tag{2.19}$$

then there exists a unique solution $L : X \to Y$ for the equation (2.15) such that the inequality

$$\|f(x) - L(x)\| \leqslant \frac{M^{\frac{1-i}{2}}}{(1-M)} \psi(x), \tag{2.20}$$

holds for all $x \in X$.

From **Theorem 2.2.4**, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability of the functional equation (2.15).

Corollary 2.2.1. *Let X be a real vector space and Y be a real Banach space and $m_1, m_2, ...m_n \in \boldsymbol{R}$ such that $m_1 + m_2 + ... + m_n \overset{not}{=} S \neq 0$. Assume that $a_i : X \to X$, $i = \overline{1,n}$, are n homogeneous functions such that, the function*

$$A : X \to Y, A(x) = a_1(x) + a_2(x) + ... + a_n(x) = bx, \ \forall\, x \in X$$

where $b \in R \backslash \{0\}$ and ε and p two fixed real numbers such that $\epsilon \geq 0$ and $p \neq \log_{|b|} |S|$.

If a function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, x_2, ...x_n)\| \leqslant \varepsilon (\|x_1\|^p + ... + \|x_n\|^p), \tag{2.21}$$

$\forall x_1, x_2, ..., x_n \in X$. *then there exists a unique solution $L : X \to Y$ of equation (2.15) such that the inequality*

$$\|f(x) - A(x)\| \leqslant \frac{n\varepsilon}{||S| - |b|^p|} \|x\|^p \tag{2.22}$$

holds for all $x \in X$.

The following corollary is the Hyers-Ulam stability of equation (2.15).

Corollary 2.2.2. *Let X be a real vector space and Y be a real Banach space and $m_1, m_2, ...m_n \in \boldsymbol{R}$ such that $m_1 + m_2 + ... + m_n \overset{not}{=} S \neq 0$. Assume that $a_i : X \to X$, $i = \overline{1, n}$, are n homogeneous functions such that, the function*

$$A : X \to Y, A(x) = a_1(x) + a_2(x) + ... + a_n(x) = bx, \ \forall x \in X$$

where $b \in R\setminus\{0\}$ and $\theta \geq 0$ a fixed real number.

If a function $f : X \to Y$ satisfies the inequality

$$\|f(a_1(x_1) + ... + a_n(x_n)) - m_1 f(x_1) - ... - m_n f(x_n)\| \leqslant \theta, \tag{2.23}$$

$\forall x_1, x_2, ..., x_n \in X$.

then, there exists a unique solution $L : X \to Y$ of equation (2.15) such that the inequality

$$\|f(x) - A(x)\| \leqslant \frac{\theta}{||S| - 1|} \tag{2.24}$$

holds for all $x \in X$.

In the following we will prezent the second application of The fixed point alternative to Hyers-Ulam stability of differential equation

$$y'(x) + f(x)y(x) + g(x) = 0 \tag{2.25}$$

in some conditions, other than [**18**].

Using the idea of Cădariu and Radu [**14**], we will prove the Hyers-Ulam-Rassias stability for the equation (2.25) on the intervals $I = [a, b)$ where $-\infty < a < b \leq \infty$.

Theorem 2.2.5 (F. Bojor, [**6**])**.** *Let $f, g : I \to \mathbb{R}$ be continuous functions and let for a positive constant M, $|f(x)| \geq M$ for all $x \in I$. Assume that $\psi : I \to [0, \infty)$ is an integrable function with the property $\exists P \in (0, 1)$ so that*

$$\int_a^x |f(t)|\psi(t)\,dt \leq P\psi(x) \tag{2.26}$$

for all $x \in I$. If a continuously differentiable function $y : I \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.27}$$

for all $x \in I$, then there exists a unique solution $S : I \to \mathbb{R}$ of the equation (2.25) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.28}$$

for all $x \in I$ and $S(a) = y(a)$.

Theorem 2.2.6 (F. Bojor, [**6**])**.** *Let $f, g : J \to \mathbb{R}$ be continuous functions and let for some positive constant M, $|f(x)| \geq M$ for all $x \in J$. Assume that $\psi : J \to [0, \infty)$ is an integrable function with the property $\exists\, P \in (0,1)$ so that*

$$\int_x^a |f(t)|\psi(t)\, dt \leq P\psi(x) \tag{2.29}$$

for all $x \in J$. If a continuously differentiable function $y : J \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.30}$$

for all $x \in J$, then there exists a unique solution $S : J \to \mathbb{R}$ of the equation (2.25) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.31}$$

for all $x \in J$ and $S(a) = y(a)$.

The Hyers-Ulam-Rassias stability equation (2.25) on $\mathbb{R}$ will be proved by Theorem 2.2.5 and Theorem 2.2.6.

Corollary 2.2.3 (F. Bojor, [**6**])**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and let for some positive constant M, $|f(x)| \geq M$ for all $x \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \to [0, \infty)$ is an integrable function with the property $\exists\, P \in (0,1)$ so that*

$$\left|\int_0^x |f(t)|\psi(t)\, dt\right| \leq P\psi(x) \tag{2.32}$$

for all $x \in \mathbb{R}$. If a continuously differentiable function $y : \mathbb{R} \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.33}$$

for all $x \in \mathbb{R}$, then there exists a unique solution $S : \mathbb{R} \to \mathbb{R}$ of the equation (2.25) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.34}$$

for all $x \in \mathbb{R}$ and $S(0) = y(0)$.

Using the Theorem 2.2.5 it can be shown the Hyers-Ulam stability for the equation (2.25) on $I = [a, b)$ where $-\infty < a < b \leq \infty$.

Corollary 2.2.4 (F. Bojor,[**6**]). *Let ε, $M > 0$ and let $f : I \to [M, \infty)$ and $g : I \to \mathbb{R}$ be continuous. If a continuously differentiable function $y : I \to \mathbb{R}$ verifies the relation*

$$|y'(x) + f(x)\, y(x) + g(x)| \leq \varepsilon \tag{2.35}$$

for all $x \in I$, then there exists a unique solution $S : I \to \mathbb{R}$ of the equation (2.25) which verifies the relations:

$$|y(x) - S(x)| \leq \frac{\varepsilon}{M(2q-1)} \tag{2.36}$$

for all $x \in I$, where $q \in \left(\frac{1}{2}, 1\right)$ and $S(a) = y(a)$.

The equation (2.25) is not Hyers-Ulam stable on the intervals $J = (-\infty, a]$ in general, as we can see in the following example.

Example 2.2.1 (F. Bojor,[**6**]). *Let us consider the equation (2.25) where $f(x) = x^2$ and $g(x) = 0$. The solution of this equation $S : J \to \mathbb{R}$ which verifies the condition $S(a) = p$ is*

$$S(x) = p \cdot e^{\frac{a^3 - x^3}{3}}. \tag{2.37}$$

A continuously differentiable function $y : J \to \mathbb{R}$ which verifies the inequality (2.35) is

$$y(x) = p \cdot e^{\frac{a^3 - x^3}{3}} + \varepsilon \cdot e^{-\frac{x^3}{3}} \int_a^x e^{\frac{t^3}{3}} dt. \tag{2.38}$$

Considering the equation (2.25) being Hyers-Ulam stable, there exists $k > 0$ so that

$$|y(x) - S(x)| \leq \varepsilon \cdot k \tag{2.39}$$

for all $x \in J$. By substitution we have

$$\left| \int_a^x e^{\frac{t^3}{3}} dt \right| \leq k e^{\frac{x^3}{3}} \tag{2.40}$$

for all $x \in J$. Now letting $x \to -\infty$ it generates a contradiction. So the equation (2.25) is not Hyers-Ulam stable.

CHAPTER 3

Fixed point for generalized contractions in metric spaces endowed with a graph

In this chapter, working in metric spaces endowed with a graph, we will generalize the notions Kannan, Zamfirescu, Círić-Reich-Rus, Bianchini operators and φ-contractions and we will prove fixed point theorems for this type of operators. This theorems generalize the clasical theorem for this type of operators.

Throughout this chapter we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$ and $\Delta = \{(x, x) \,|x \in X\} \subseteq G$. The personal contributions in this chapter are:
Definitions 3.1.1,3.2.1,3.3.1,3.4.1,3.5.1,
Theorems 3.1.1, 3.2.1, 3.3.1, 3.4.1, 3.4.2, 3.4.3, 3.5.1,
Corollaries 3.1.1, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.3.1, 3.3.2, 3.3.3, 3.3.4, 3.4.1, 3.4.2, 3.5.1,
Lemmas 3.1.1, 3.2.1, 3.3.1, 3.3.2, 3.5.1, 3.5.2,
Examples 3.1.2, 3.1.1, 3.1.3,3.1.4, 3.1.5, 3.1.6, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.3.1, 3.3.2,3.3.3, 3.4.1,3.4.2, 3.5.2, 3.5.3, 3.5.4,
Observations 3.1.1, 3.2.1, 3.2.2,
the proof of Theorems 3.1.2, 3.2.2, 3.3.2, 3.5.2.

1. Fixed point theorems for G-Kannan operators

In this section, by using the idea of Jakhymski [**17**], we will consider the following concept:

Definition 3.1.1 (F. Bojor, [**7**])**.** *Let (X, d) be a metric space and G a graph. The mapping $T : X \to X$ is said to be a G-Kannan mapping if:*

1. $\forall x, y \in X$ (If $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$).
2. *there exists $a \in \left[0, \frac{1}{2}\right)$ such that:*

$$d(Tx, Ty) \leqslant a [d(x, Tx) + d(y, Ty)]$$

for all $(x, y) \in E(G)$.

Remark 3.1.1. *If T is a G-Kannan mapping, then T is both a G^{-1}-Kannan mapping and a $\widetilde{G}$-Kannan mapping.*

Example 3.1.1. *Any Kannan mapping is a G_0-Kannan mapping, where the graph G_0 is defined by $E(G_0) = X \times X$.*

Example 3.1.2 (F. Bojor, [**7**]). *Let* $X = \{0,1,3\}$ *and the euclidean metric* $d(x,y) = |x-y|, \forall x,y \in X$. *The mapping* $T : X \to X$, $Tx = 0$, *for* $x \in \{0,1\}$ *and* $Tx = 1$, *for* $x = 3$ *is a G-Kannan mapping with constant* $a = \frac{1}{3}$, *where* $G = \{(0,1);(1,3);(0,0);(1,1));(3,3)$ *but is not a Kannan mapping because* $d(T0,T3) = 1$ *and* $d(0,T0) + d(3,T3) = 2$.

Lemma 3.1.1 (F. Bojor, [**7**]). *Let* (X,d) *be a metric space endowed with a graph* G *and* $T : X \to X$ *be a G-Kannan mapping with constant* a. *If the graph* G *is weakly* $T-$*connected then, given* $x,y \in X$, *there is* $r(x,y) \geqslant 0$ *such that*

$$d(T^n x, T^n y) \leqslant ad\left(T^{n-1}x, T^n x\right) + \left(\frac{a}{1-a}\right)^n r(x,y) + ad\left(T^{n-1}y, T^n y\right) \tag{3.41}$$

for all $n \in \mathbb{N}^*$.

The main result of G-Kannan operators is given by the following theorem.

Theorem 3.1.1 (F. Bojor, [**7**]). *Let* (X,d) *be a complete metric space endowed with a graph* G *and* $T : X \to X$ *be a G-Kannan mapping. We suppose that:*

(i.) G *is weakly* $T-$*connected;*
(ii.) *for any* $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T *is a PO.*

The next example shows that the condition (ii.) is a necessary condition for G-Kannan mapping to be a PO.

Example 3.1.3 (F. Bojor, [**7**]). *Let* $X := [0,1]$ *be endowed with the Euclidean metric* d_E. *Define the graph* G *by*

$$E(G) = \{(x,y) \in (0,1] \times (0,1] \,|\, x \geqslant y\} \cup \{(0,0),(0,1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0,1], \text{ and } T0 = 1$$

It is easy to verify (X,d) *is a complete metric space,* G *is weakly* $T-$*connected and* T *is a G-Kannan mapping with* $a = \frac{3}{7}$. *Clearly,* $T^n x \to 0$ *for all* $x \in X$, *but* T *has no fixed points.*

The next example shows that the graph G must be $T-$connected so the G-Kannan mapping T to be a PO.

Example 3.1.4 (F. Bojor, [**7**]). *Let* $X = \mathbb{N}\setminus\{0,1\}$ *be endowed with the Euclidean metric and define* $T : X \to X$, $Tx = 2x$. *Consider the graph* G *define by:*

$$V(G) = X \text{ and } E(G) = \left\{\left(2^k n, 2^k(n+1)\right) : n \in X, k \in \mathbb{N}\right\} \cup \Delta$$

Then T is G-Kannan operator with $a = \frac{2}{5}$ because

$$\begin{aligned} d_E\left(T2^k n, T2^k(n+1)\right) &= 2^{k+1} \leqslant \frac{2}{5}2^k(2n+1) \\ &= \frac{2}{5}\left[d_E\left(2^k n, T2^k n\right) + d_E\left(2^k(n+1), T2^k(n+1)\right)\right] \end{aligned}$$

for all $n \in X$ and for all $k \in \mathbb{N}$.

Then (X, d) is a complete metric space, G is weakly connected but not weakly $T-$connected because $(2,4) \notin E\left(\widetilde{G}\right)$ and the only path in $\widetilde{G}$ from 2 to 4 is $y_0 = 2, y_1 = 3, y_2 = 4$ and $(3, T3) = (3, 6) \notin E\left(\widetilde{G}\right)$.

Clearly, $T^n x$ not converge for all $x \in X$ and T has no fixed points.

From Theorem 3.1.1, we obtain the following corollary concerning the fixed point of Kannan operator in partially ordered metric spaces.

Corollary 3.1.1 (F. Bojor, [**7**])**.** *Let $(X, \leqslant)$ be a partially ordered set and d be a metric on X such that the metric space (X, d) is complete. Let $T : X \to X$ be an increasing operator such that the following three assertions hold:*

(i.) There exist $a \in \left[0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leqslant a[d(x, Tx) + d(y, Ty)]$ for each $x, y \in X$ with $x \leqslant y$;

(ii.) For each $x, y \in X$, incomparable elements of $(X, \leqslant)$, there exists $z \in X$ such that $x \leqslant z$, $y \leqslant z$ and $z \leqslant Tz$;

(iii.) If an increasing sequence (x_n) converges to x in X, then $x_n \leqslant x$ for all $n \in \mathbb{N}$.

Then T is a PO.

In the next we show the fixed point theorem for cyclic Kannan mapping proved in [**25**] by Petric is a consequence of the Theorem 3.1.1.

Theorem 3.1.2. *Let $A_1, A_2, ..., A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (X, d) and suppose $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is a cyclical operator, and there exists $a \in \left[0, \frac{1}{2}\right)$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $i \in \{1, 2, ..., p\}$, we have*

$$d(Tx, Ty) \leqslant a[d(x, Tx) + d(y, Ty)].$$

Then T is a PO.

Examples of G-Kannan operators.

Example 3.1.5 (F. Bojor, [**7**])**.** *Let $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ endowed with euclidean metric d_E and $T : X \to X$,*

$$Tx = \begin{cases} -\left|\sin x\right|, & x \in \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \\ 0, & x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \end{cases}.$$

Let the graph G with

$$V(G) = X \quad \text{and} \quad E(G) = \{(0,x) : x \in X\}.$$

The graph G is weakly T-connected and ii. from Theorem 3.1.1 is satisfied.

For all $x \in X$ we have: if $(0,x) \in E(G)$ then $(T0, Tx) = (0, Tx) \in E(G)$ and T is G-Kannan operator with constant

$$a = \frac{3}{2\pi + 1} < \frac{1}{2}.$$

By Theorem 3.1.1, T is a PO and the unique fixed point of T is $x^* = 0$.

The operator T is not clasical Kannan operator.

Example 3.1.6 (F. Bojor, [**7**])**.** *Let $X = [0,5]$ endowed with euclidean metric d_E and $T : X \to X$,*

$$Tx = \begin{cases} x - 4; \; x \in (4,5] \\ \dfrac{x}{4}; \quad x \in [0,4] \end{cases}$$

Let the graph G defined by

$$V(G) = X \quad \text{and} \quad E(G) = \{(4,x) : x \in (4,5]\} \cup [0,4] \times [0,4] \cup \Delta.$$

The graph G is weakly T-connected and ii. from Theorem 3.1.1 is satisfied.

For all $x \in X$ we have: if $(4,x) \in E(G)$ then $(T4, Tx) = \left(\frac{1}{4}, \frac{x}{4}\right) \in E(G)$ and T is a G-Kannan operator with constant $a = \frac{1}{3}$.

By Theorem 3.1.1, T is a PO and the unique fixed point of T is $x^* = 0$.

The operator T is not a G-Banach contraction.

2. Fixed point theorems for G-Zamfirescu operators

In this section we will extend Zamfirescu's fixed point theorem on metric spaces endowed with a graph.

Definition 3.2.1 (F. Bojor, [**9**])**.** *Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be a G-Zamfirescu mapping if:*

1. $\forall x, y \in X$ (If $(x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$).
2. *there exist the real numbers α, β and γ satisfying $0 \le \alpha < 1$, $0 \le \beta < \frac{1}{2}$ and $0 \le \gamma < \frac{1}{2}$, such that, for each $(x,y) \in E(G)$, at least one of the following is true:*
   (z_1) $d(Tx, Ty) \le \alpha d(x,y)$;
   (z_2) $d(Tx, Ty) \leqslant \beta [d(x, Tx) + d(y, Ty)]$;
   (z_3) $d(Tx, Ty) \leqslant \gamma [d(x, Ty) + d(y, Tx)]$.

Example 3.2.1. *If T is a G-Zamfirescu mapping, then T is both a G^{-1}-Zamfirescu mapping and a $\tilde{G}$-Zamfirescu mapping.*

Example 3.2.2. *Let* $X = \{0, 1, 2, 3\}$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$. *The mapping* $T : X \to X$, $Tx = 0$, *for* $x \in \{0, 1\}$ *and* $Tx = 1$, *for* $x \in \{2, 3\}$ *is a G-Zamfirescu mapping satisfying* (z_1) *from Def 3.2.1 with constant* $\alpha = \frac{2}{3}$, *where*

$$V(G) = X \text{ and } E(G) = \{(0,1); (0,2); (2,3); (0,0); (1,1)); (2,2); (3,3)\},$$

but is not a Zamfirescu mapping because

- $d(T1, T2) = 1$ *and* $d(1, 2) = 1$ *so* (z_1) *from Def 3.2.1 is not satisfied;*
- $d(T1, T2) = 1$ *and* $d(1, T1) + d(2, T2) = 2$ *so* (z_2) *from Def 3.2.1 is not satisfied;*
- $d(T1, T2) = 1$ *and* $d(1, T2) + d(2, T1) = 2$ *so* (z_3) *from Def 3.2.1 is not satisfied.*

Lemma 3.2.1 (F. Bojor, [**9**]). *Let* (X, d) *be a metric space and* $T : X \to X$ *be a G-Zamfirescu mapping. Then*

$$d(Tx, Ty) \leqslant 2\delta d(x, Ty) + \delta d(x, y) \tag{3.42}$$

for all $(x, y) \in E(G)$, *where* $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$.

Remark 3.2.1 (F. Bojor, [**9**]). *In a similar manner with Lemma 3.1.1 we obtain*

$$d(Tx, Ty) \leqslant 2\delta d(x, Tx) + d(x, y) \tag{3.43}$$

valid for all $(x, y) \in E(G)$.

Lemma 3.2.2 (F. Bojor, [**9**]). *Let* (X, d) *be a metric space endowed with a graph* G *and* $T : X \to X$ *be a G-Zamfirescu mapping such that the graph* G *is weakly* $T-$*connected.*

If $(x, y) \notin E\left(\widetilde{G}\right)$ *then there is* $r(x, y) \geqslant 0$ *such that*

$$d(T^n x, T^n y) \leqslant n\delta^n r(x, y) \tag{3.44}$$

for all $n \in \mathbb{N}^*$, *where* $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$.

The main result of this paper is given by the following theorem.

Theorem 3.2.1 (F. Bojor, [**9**]). *Let* (X, d) *be a complete metric space endowed with a graph* G *and* $T : X \to X$ *be a G-Zamfirescu mapping. We suppose that:*

(i.) G *is weakly* $T-$*connected;*

(ii.) for any $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T *is a PO.*

The next example shows that the condition (ii.) is a necessary condition for G-Zamfirescu mapping to be a PO.

Example 3.2.3 (F. Bojor, [**9**]). *Let $X := [0,1]$ be endowed with the Euclidean metric d_E. Define the graph G by*

$$E(G) = \{(x,y) \in (0,1] \times (0,1] \mid x \geqslant y\} \cup \{(0,0), (0,1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0,1], \text{ and } T0 = \frac{1}{4}$$

It is easy to verify (X,d) is a complete metric space, G is weakly $T-$connected and T is a G-Zamfirescu mapping satisfies (z_2) with $\beta = \frac{3}{7}$. Clearly, $T^n x \to 0$ for all $x \in X$, but T has no fixed points.

The next example shows that the graph G must be $T-$connected so the G-Zamfirescu mapping T to be a PO.

Example 3.2.4 (F. Bojor, [**9**]). *Let $X = \{3,4,5,...\} = \mathbb{N}\setminus\{0,1,2\}$ be endowed with the Euclidean metric d_E. Define the graph G by*

$$V(G) = X \text{ and } E(G) = \left\{\left(2^k n, 2^k (n+1)\right) : k \in \mathbb{N}, n \in \mathbb{N}\setminus\{0,1,2\}\right\} \cup \Delta.$$

Set

$$Tx = 2x$$

Then (X,d) is a complete metric space. The graph G is weakly connected because for all $m,n \in \mathbb{N}$ with $m < n$ we have that the sequence $x_0 = m, x_1 = m+1, ..., x_{n-m} = n$ is a path in G from m to n. The graph G is not weakly $T-$connected because $(3,5) \notin E(G)$, and the only path from 3 to 5 is $y_0 = 3, y_1 = 4, y_2 = 5$ and $(4, T4) = (4,8) \notin E(G)$. For all $(x,y) \in E(G)$ we have that

$$(Tx, Ty) = (2x, 2y) \in E(G)$$

and the mapping T satisfies (z_2) with $\beta = \frac{1}{3}$ because for $(x,y) \in E(G)$ there is $k \in \mathbb{N}$ and $n \in \{3,4,5,...\}$ such that $(x,y) = \left(2^k n, 2^k (n+1)\right)$ and

$$\begin{aligned} d(Tx,Ty) &= \left|2^{k+1} n - 2^{k+1}(n+1)\right| = 2^{k+1} = \frac{1}{3} 2^k \cdot 6 < \frac{1}{3} 2^k (2n+1) \\ &= \frac{1}{3}\left(2^k n + 2^k (n+1)\right) = \frac{1}{2}\left(d(x,Tx) + d(y,Ty)\right). \end{aligned}$$

The property ii. from Theorem 3.1.1 is satisfies because every convergent sequence is a constant sequence. Clearly, $(T^n x)_{n\in\mathbb{N}}$ is not convergent for all $x \in X$.

Corollary 3.2.1 (F. Bojor, [**9**]). *Let (X,d) be a complete metric space endowed with a graph G and $T : X \to X$ be a Banach G-contraction. We suppose that:*

(i.) G is weakly $T-$connected;

(ii.) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Then T is a PO.

Corollary 3.2.2 (F. Bojor, [**9**]). *Let (X,d) be a complete metric space endowed with a graph G and $T : X \to X$ be a $G-$Kannan mapping. We suppose that:*

(i.) *G is weakly $T-$connected;*
(ii.) *for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.*

Then T is a PO.

Corollary 3.2.3 (F. Bojor, [**9**]). *Let (X,d) be a complete metric space endowed with a graph G and $T : X \to X$ satisfies the condition Banach (z_3) from Theorem3.1.1 (We can call T a G-Chatterjea mapping) . We suppose that:*

(i.) *G is weakly $T-$connected;*
(ii.) *for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.*

Then T is a PO.

From Theorem 3.2.1, we obtain the following corollary concerning the fixed point of Zamfirescu mapping in partially ordered metric spaces.

Corollary 3.2.4 (F. Bojor, [**9**]). *Let $(X, \leqslant)$ be a partially ordered set and d be a metric on X such that the metric space (X,d) is complete. Let $T : X \to X$ be an increasing operator such that the following three assertions hold:*

(i.) *There exist the real numbers α, β and γ satisfying $0 \le \alpha < 1$, $0 \le \beta < \frac{1}{2}$ and $0 \le \gamma < \frac{1}{2}$, such that, for each $x, y \in X$ with $x \leqslant y$, at least one of the following is true:*
(z_1) $d(Tx,Ty) \le \alpha d(x,y)$;
(z_2) $d(Tx,Ty) \leqslant \beta [d(x,Tx) + d(y,Ty)]$;
(z_3) $d(Tx,Ty) \leqslant \gamma [d(x,Ty) + d(y,Tx)]$.
(ii.) *For each $x, y \in X$, incomparable elements of $(X, \leqslant)$, there exists $z \in X$ such that $x \leqslant z$, $y \leqslant z$ and $z \leqslant Tz$;*
(iii.) *If an increasing sequence (x_n) converges to x in X, then $x_n \leqslant x$ for all $n \in \mathbb{N}$.*

Then T is a PO.

In the next we show the fixed point theorem for cyclic Zamfirescu operators proved in [**25**] by Petric and Zlatanov is a consequence of the Theorem 3.2.1.

Let $p \geqslant 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space X. A mapping $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a *cyclical operator* if

$$T(A_i) \subseteq A_{i+1}, \quad \text{for all } i \in \{1, 2, ..., p\} \tag{3.45}$$

where $A_{p+1} := A_1$.

Theorem 3.2.2 ([**25**], T 3.1)**.** *Let $A_1, A_2, ..., A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (X, d) and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator, and there exist real numbers $a \in [0,1)$, $b \in \left[0, \frac{1}{2}\right)$ and $c \in \left[0, \frac{1}{2}\right)$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $i \in \{1, 2, ..., p\}$, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$;

(z_2) $d(Tx, Ty) \leqslant b[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leqslant c[d(x, Ty) + d(y, Tx)]$.

Then T is a PO.

3. Fixed point theorems for G-Ćirić-Reich-Rus operators

In this section, by using the idea of Jachymski [**17**], we will consider the following concept:

Definition 3.3.1 (F. Bojor, [**5**])**.** *Let (X, d) be a metric space. The operator $T : X \to X$ is said to be a G-Ćiriċ-Reich-Rus operator if:*

1. $((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G))$, $\forall x, y \in X$.
2. *there exists nonnegative numbers a, b, c with $a + b + c < 1$, such that, for each $(x, y) \in E(G)$, we have:*

$$d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

Example 3.3.1 (F. Bojor, [**5**])**.** *Let $X = \{0, 1, 2, 3\}$ and the euclidean metric $d(x, y) = |x - y|$, $\forall x, y \in X$. The operator $T : X \to X$, $Tx = 0$, $for\ x \in \{0, 1\}$ and $Tx = 1$, $for\ x \in \{2, 3\}$ is a G-Ćiriċ-Reich-Rus operator with constants $a = \frac{1}{3}, b = 0$ and $c = \frac{1}{3}$, where $G = \{(0,1); (0,2); (2,3); (0,0); (1,1)); (2,2); (3,3)\}$, but is not a Ćiriċ-Reich-Rus operator because $d(T1, T2) = 1$, $d(1, 2) = 1$, $d(1, T1) = 1$ and $d(2, T2) = 1$.*

Lemma 3.3.1 (F. Bojor, [**5**])**.** *Let (X, d) be a metric endowed with the graph G and $T : X \to X$ be a G-Ćiriċ-Reich-Rus operator. If $x \in X$ satisfies the condition $(x, Tx) \in E(G)$ then we have*

$$d\left(T^n x, T^{n+1} x\right) \leqslant \alpha^n d(x, Tx), \tag{3.46}$$

for all $n \in \mathbb{N}$, where $\alpha = \frac{a+b}{1-c}$.

Lemma 3.3.2 (F. Bojor, [**5**])**.** *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Ćiriċ-Reich-Rus operator such that the graph G is $T-$connected. For all $x \in X$ the sequence $(T^n x)_{n \in \mathbb{N}}$ is a Cauchy sequence.*

The main result of this section is given by the following theorem.

Theorem 3.3.1 (F. Bojor, [**5**])**.** *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-Ćiriċ-Reich-Rus operator. We suppose that:*

(i.) *G is T−connected;*

(ii.) *for any* $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T is a PO.

The next example shows that the condition (ii.) is a necessary condition for *G*-Ċirič-Reich-Rus operator to be a PO.

Example 3.3.2 (F. Bojor, [**5**]). *Let* $X := [0,1]$ *be endowed with the Euclidean metric* $d(x,y) = |x-y|$, *for all* $x, y \in X$. *Define the graph G by*

$$E(G) = \{(x,y) \in (0,1] \times (0,1] \mid x \geqslant y\} \cup \{(0,0),(0,1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0,1], \text{ and } T0 = 1$$

It is easy to verify (X,d) *is a complete metric space, G is weakly T−connected and T is a G-Ċirič-Reich-Rus operator with constants* $a = \frac{3}{4}$, $b = \frac{1}{16}$, $c = \frac{1}{8}$. *Clearly,* $T^n x \to 0$ *for all* $x \in X$, *but T has no fixed points.*

The next example shows that the graph G must be $T-$connected so the G-Ċirič-Reich-Rus operator T to be a PO.

Example 3.3.3 (F. Bojor, [**5**]). *Let* $X := [0,\infty)$ *be endowed with the Euclidean metric* $d(x,y) = |x-y|$, *for all* $x, y \in X$ *and*

$$T : X \to X, \quad Tx = x + 5$$

Define the graph G by

$$E(G) = \left(\left[0,\frac{1}{2}\right) \cup \{1\}\right) \times \left(\left[0,\frac{1}{2}\right) \cup \{1\}\right) \cup \left[\frac{1}{2},1\right) \times \left[\frac{1}{2},1\right) \cup \left\{\left(0,\frac{1}{2}\right)\right\}$$

Then (X,d) *is a complete metric space, G is weakly connected but not weakly T−connected and T is a G-Ċirič-Reich-Rus operator with* $a = \frac{1}{2}, b = c = \frac{1}{6}$. *Clearly,* $T^n x$ *not converge for all* $x \in X$ *and T has no fixed points.*

Corollary 3.3.1 (F. Bojor, [**5**]). *Let* (X,d) *be a complete metric space endowed with a graph G and* $T : X \to X$ *be an operator. We suppose that:*

(i.) *G is weakly T−connected;*

(ii.) *there exists nonnegative numbers a and b satisfying* $a + 2b < 1$ *such that, for each* $(x,y) \in E(G)$, *we have* $d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)]$;

(iii.) *for any* $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T is a PO.

Corollary 3.3.2 (F. Bojor, [**5**]). *Let* (X,d) *be a complete metric space endowed with a graph G and* $T : X \to X$ *be a Banach G-contraction. We suppose that:*

(i.) *G is weakly* $T-$*connected;*

(ii.) *for any* $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T *is a PO.*

Corollary 3.3.3 (F. Bojor, [**5**])**.** *et* (X, d) *be a complete metric space endowed with a graph* G *and* $T : X \to X$ *be a* $G-$*Kannan mapping. We suppose that:*

(i.) *G is weakly* $T-$*connected;*

(ii.) *for any* $(x_n)_{n\in\mathbb{N}}$ *in* X, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$ *then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Then T *is a PO.*

From Theorem 3.3.1, we obtain the following corollary concerning the fixed point of C̀iric̀-Reich-Rus operator in partially ordered metric spaces.

Corollary 3.3.4 (F. Bojor, [**5**])**.** *Let* $(X, \leqslant)$ *be a partially ordered set and* d *be a metric on* X *such that the metric space* (X, d) *is complete. Let* $T : X \to X$ *be an increasing operator such that the following three assertions hold:*

(i.) *There exist the real numbers* $a, b, c > 0$ *with* $a + b + c < 1$, *such that, for each* $x, y \in X$ *with* $x \leqslant y$ *we have*

$$d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

(ii.) *For each* $x, y \in X$, *incomparable elements of* $(X, \leqslant)$, *there exists* $z \in X$ *such that* $x \leqslant z$, $y \leqslant z$ *and* $z \leqslant Tz$;

(iii.) *If an increasing sequence* (x_n) *converges to* x *in* X, *then* $x_n \leqslant x$ *for all* $n \in \mathbb{N}$.

Then T *is a PO.*

In the next we show the fixed point theorem for cyclic C̀iric̀-Reich-Rus operator proved in [**23**] by Petric is a consequence of the Theorem 3.3.1.

Theorem 3.3.2 (M. Petric, [**23**])**.** *Let* $A_1, A_2, ..., A_p, A_{p+1} = A_1$ *be nonempty closed subsets of a complete metric space* (X, d) *and suppose* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *is a cyclical operator, and there exist nonnegative numbers* a, b, c *satisfying* $a + b + c < 1$ *such that for each pair* $(x, y) \in A_i \times A_{i+1}$, *for* $i \in \{1, 2, ..., p\}$, *we have*

$$d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

Then T *is a PO.*

4. Teoreme de punct fix pentru (G, φ) −contracţii

By using the idea of Jakhymski [**17**], we will say that:

Definition 3.4.1 (F. Bojor, [**4**])**.** *Let (X,d) be a metric space and G a graph. The mapping $T : X \to X$ is said to be a $(G,\varphi) -$ contraction if:*

1. *$\forall\, x, y \in X\, ((x,y) \in E\,(G) \Rightarrow (Tx,Ty) \in E\,(G))$.*
2. *there exists a comparison function $\varphi : R_{+} \to \mathbb{R}_{+}$ such that:*

$$d\,(Tx,Ty) \leqslant \varphi\,(d\,(x,y))$$

for all $(x,y) \in E\,(G)$.

Remark 3.4.1. *If T is a $(G,\varphi) -$ contraction, then T is both a $(G^{-1},\varphi) -$ contraction and a $\left(\tilde{G},\varphi\right) -$ contraction. This is consequence of symmetry of d and 1.*

Example 3.4.1. *Any $G -$ contraction is a $(G,\varphi) -$ contraction, where the comparison function is $\varphi : \mathbb{R}_{+} \to \mathbb{R}_{+},\ \varphi\,(t) = at$.*

The first main result of this section is a fixed point theorem for $(G,\varphi) -$ *contraction* on an complete metric space endowed with a graph.

Theorem 3.4.1 (F. Bojor, [**4**])**.** *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be an operator. We suppose that:*

(i.) *G is weakly connected;*

(ii.) *for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $d\,(x_n,x_{n+1}) \to 0$ there exists $k,\, n_0 \in \mathbb{N}$ such that $(x_{kn},x_{km}) \in E\,(G)$ for all $m,n \in \mathbb{N}$ $m,n \geqslant n_0$;*

$(iii.)_a$ *T is orbitally continuous*

or

$(iii.)_b$ *T is orbitally G-continuous and there exists a subsequence $(T^{n_k}x_0)_{k\in\mathbb{N}}$ of $(T^{n}x_0)_{n\in\mathbb{N}}$ such that $(T^{n_k}x_0,x^*) \in E\,(G)$ for each $k \in N$;*

(iv.) *there exists a comparison function $\varphi : \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that T is a $(G,\varphi) -$ contraction;*

(v.) *the metric d is complete.*

Then T is a PO.

Now if we improve the properties of the operator T then we can drop some of the conditions of the graph G. From now on we will consider that the function φ is a $(c) -$ *comparison* function.

In the following we will show that the convergence of successive approximations for $(G,\varphi) -$ *contraction* is closely related to the connectivity of a graph. We say that sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, elements of X, are Cauchy equivalent if each of them is a Cauchy sequence and $d\,(x_n,y_n) \to 0$.

Theorem 3.4.2 (F. Bojor, [**4**])**.** *The following statements are equivalent:*

(i) *G is weakly connected;*

(ii) *for any* $(G,\varphi)-$ *contraction* $T:X\to X$, *given* $x,y\in X$, *the sequences* $(T^n x)_{n\in\mathbb{N}}$ *and* $(T^n y)_{n\in\mathbb{N}}$ *are Cauchy equivalent;*

(iii) *for any* $(G,\varphi)-$ *contraction* $T:X\to X$, $\operatorname{card}(\operatorname{Fix}T)\leqslant 1$.

As an immediate consequence of Theorem 3.4.2, we obtain the following:

Corollary 3.4.1 (F. Bojor, [**4**]). *Let* (X,d) *be a complete metric space and* G *a graph weakly connected. For any* $(G,\varphi)-$ *contraction* $T:X\to X$, *there is* $x^*\in X$ *such that* $\lim\limits_{n\to\infty}T^n x=x^*$ *for all* $x\in X$.

The next example shows that one cannot improve Corollary 2.2.3 by adding that x^* is a fixed point of T.

Example 3.4.2 (F. Bojor, [**4**]). *Let* $X:=[0,1]$ *be endowed with the Euclidean metric* d_E. *Define the graph* G *by*

$$E(G)=\{(x,y)\in(0,1]\times(0,1]\,|x\geqslant y\}\cup\{(0,0),(0,1)\}$$

Set

$$Tx=\frac{x}{4}\ \text{ for }\ x\in(0,1]\,,\text{ and } T0=\frac{1}{4}$$

It is easy to verify G *is weakly connected and* T *is a* $(G,\varphi)-$ *contraction with* $\varphi(t)=\frac{t}{4}$. *Clearly,* $T^n x\to 0$ *for all* $x\in X$, *but* T *has no fixed points.*

The proofs of our fixed point theorems depend on the following:

Proposition 3.4.1 (F. Bojor, [**4**]). *Assume that* $T:X\to X$ *is a* $(G,\varphi)-$*contraction such that for some* $x_0\in X$, $Tx_0\in[x_0]_{\tilde G}$. *Let* $\tilde G_{x_0}$ *be the component of* $\tilde G$ *containing* x_0. *Then* $[x_0]_{\tilde G}$ *is T-invariant and* $T|_{[x_0]_{\tilde G}}$ *is a* $\left(\tilde G_{x_0},\varphi\right)-$ *contraction. Moreover, if* $x,y\in[x_0]_{\tilde G}$, *then* $(T^n x)_{n\in\mathbb{N}}$ *and* $(T^n x)_{n\in\mathbb{N}}$ *are Cauchy equivalent.*

Theorem 3.4.3 (F. Bojor, [**4**]). *Let* (X,d) *be complete, and let the triple* (X,d,G) *have the following property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n\to x$ and $(x_n,x_{n+1})\in E(G)$ for $n\in\mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n},x)\in E(G)$ *for* $n\in\mathbb{N}$.

Let $T:X\to X$ *be a* $(G,\varphi)-$ *contraction, and* $X_T=\{x\in X\,|(x,Tx)\in E(G)\}$. *Then the following statements hold.*

(1) $\operatorname{card}Fix\,T=\operatorname{card}\{[x]_{\tilde G}\,|x\in X_T\}$.

(2) $\operatorname{Fix}T\neq\emptyset$ iff $X_T\neq\emptyset$.

(3) *T has a unique fixed point iff there exists* $x_0\in X_f$ *such that* $X_T\subseteq[x_0]_{\tilde G}$.

(4) *For any* $x\in X_T$, $T\Big|_{[x]_{\tilde G}}$ *is a PO.*

(5) *If* $X_T\neq\emptyset$ *and* G *is weakly connected, then* T *is a PO.*

(6) *If* $X':=\cup\{[x]_{\tilde G}\,|x\in G\}$ *then* $T\,|_{X'}$ *is a WPO.*

(7) *If* $T\subseteq E(G)$, *then* T *is a WPO.*

Corollary 3.4.2 (F. Bojor, [**4**]). *Let* (X,d) *be complete and* ε*-chainable for some* $\varepsilon > 0$, *i.e., given* $x, y \in X$, *there is* $N \in \mathbb{N}$ *and a sequence* $(x_i)_{i=0}^{N}$ *such that*

$$x_0 = x, x_N = y \text{ and } d(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, ..., N.$$

Let $T : X \to X$ *be a function and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a* $(c) - comparison\ function$ *such that*

$$\forall x, y \in X \left(d(x,y) < \varepsilon \Rightarrow d(Tx,Ty) \leq \varphi(d(x,y))\right) \tag{3.47}$$

Then T *is a PO.*

5. Fixed point theorems for G-Bianchini operators

In this section, by using the idea of Jakhymski [**17**], we will consider the following concept:

Definition 3.5.1 (F. Bojor, [**8**]). *Let* (X,d) *be a metric space and* G *a graph. The mapping* $T : X \to X$ *is said to be a G-Bianchini mapping if:*

1. $\forall x, y \in X$ (If $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$).
2. *there exists* $a \in [0,1)$ *such that:*

$$d(Tx,Ty) \leqslant a \max\{d(x,Tx), d(y,Ty)\}$$

for all $(x,y) \in E(G)$.

Example 3.5.1. *Any Bianchini mapping is a* G_0*-Bianchini mapping, where the graph* G_0 *is defined by* $E(G_0) = X \times X$.

Example 3.5.2 (F. Bojor, [**8**]). *Let* $X = \{0,1,2\}$ *and the euclidean metric* $d(x,y) = |x-y|, \forall\, x,y \in X$. *The mapping* $T : X \to X$, $Tx = 0$, *for* $x \in \{0,1\}$ *and* $Tx = 1$, *for* $x = 2$ *is a G-Bianchini mapping with constant* $a = \frac{1}{2}$, *where* $G = \{(0,1);(0,0);(1,1));(2,2)\}$, *but is not a Bianchini mapping because* $d(T0,T2) = 1$ *and* $max\{d(0,T0), d(2,T2)\} = \max\{0,1\} = 1$. *It's easy to see the Picard iteration* $(T^n x)_{n\in\mathbb{N}}$ *converge to 0 for all* $x \in X$.

Remark 3.5.1. *If* T *is a G-Bianchini mapping, then* T *is both a* G^{-1}*-Bianchini mapping and a* $\tilde{G}$*-Bianchini mapping.*

Notation 5.1. *Let* X *be a nonempty set,* G *be an graph with* $V(G) = X$, $T : X \to X$ *be a mapping and* $n_0 \in \mathbb{N}$. *We set*

$$X_{T^{n_0}} := \left\{x \in X \,\middle|\, \left(T^{n_0}x, T^{n_0+1}x\right) \in E\left(\tilde{G}\right)\right\}.$$

Lemma 3.5.1 (F. Bojor, [**8**]). *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Bianchini mapping with constant a. If there exist $n_0 \in \mathbb{N}$ and $x \in X$ such that $x \in X_{T^{n_0}}$ then:*

$$d\left(T^n x, T^{n+1}x\right) \leqslant a^{n-n_0} d\left(T^{n_0}x, T^{n_0+1}x\right) \tag{3.48}$$

for all $n \in \mathbb{N}, n \geq n_0$.

Lemma 3.5.2 (F. Bojor, [**8**]). *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Bianchini mapping with constant a. If the graph G is weakly connected and*

$$\bigcup_{n\in\mathbb{N}} X_{T^n} = X \tag{3.49}$$

then, given $x, y \in X$, there is $r(x,y) \geqslant 0$ and $k(x,y) \in \mathbb{N}$ such that

$$d\left(T^n x, T^n y\right) \leqslant a^{n-k(x,y)} r(x,y) \tag{3.50}$$

for all $n \in \mathbb{N}^*, n \geq k(x,y)$.

The main result of this paper is given by the following theorem.

Theorem 3.5.1 (F. Bojor, [**8**]). *Let (X,d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-Bianchini mapping. We suppose that:*

(i.) *G is weakly connected;*
(ii.) $\bigcup_{n\in\mathbb{N}} X_{T^n} = X$;
(iii.) *for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.*

Then T is a PO.

The next example shows that the condition $(iii.)$ is a necessary condition for G-Bianchini mapping to be a PO.

Example 3.5.3 (F. Bojor, [**8**]). *Let $X := [0,10]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$, for all $x, y \in X$. Define the graph G by*

$$E(G) = \{(x,y) \in (0,10] \times (0,10] \,|\, x \geqslant y\} \cup \{(0,0), (0,10)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0,10], \text{ and } T0 = \frac{5}{2}$$

It is easy to verify (X,d) is a complete metric space, G is weakly connected and T is a G-Bianchini mapping with $a = \frac{1}{3}$. For all $x \in (0,10]$ we have $(x,Tx) = \left(x, \frac{x}{4}\right) \in E(G)$ because $x \geqslant \frac{x}{4}$ thus $(0,10] \subset X_{T^0}$ and $(T0, T^2 0) = \left(\frac{5}{2}, \frac{5}{8}\right) \in E(G)$ so $0 \in X_{T^1}$, consequently $\bigcup_{n\in\mathbb{N}} X_{T^n} = X$. Clearly, $T^n x \to 0$ for all $x \in X$, but T has no fixed points.

The next example shows that the graph G must be weakly connected so the G-Bianchini mapping to be a PO.

Example 3.5.4 (F. Bojor, [**8**])**.** *Let* $X := [0,1]$ *be endowed with the Euclidean metric* $d(x,y) = |x-y|$, *for all* $x, y \in X$ *and set*

$$Tx = \frac{x}{4} \text{ for } x \in (0,1], \text{ and } T0 = 1$$
$$E(G) = \{(x,y) \in [0,1) \times [0,1) \mid x \geqslant y\} \cup \{(1,1)\}$$

Then (X,d) *is a complete metric space,* G *is not weakly connected because there not path in* $\tilde{G}$ *from 0 to 1, the conditions* (ii) *and* (iii) *there are true and* T *is a* G*-Bianchini mapping with* $a = \frac{1}{3}$*. Clearly,* $T^n x$ *converge to 0 for all* $x \in X$ *and* T *has no fixed points.*

As a consequence of the above theorem, we have the following corollaries concerning the fixed point of Bianchini mapping in partial ordered metric spaces.

Corollary 3.5.1 (F. Bojor, [**8**])**.** *Let* $(X, \leqslant)$ *be a partially ordered set and* d *be a metric on* X *such that the metric space* (X,d) *is complete. Let* $T : X \to X$ *be an increasing operator such that the following three assertions hold:*

1. *There exist* $a \in [0,1)$ *such that* $d(Tx,Ty) \leqslant a \cdot \max\{d(x,Tx) + d(y,Ty)\}$ *for each* $x, y \in X$ *with* $x \leqslant y$*;*
2. *For each* $x \in X$ *there exists* $n_x \in \mathbb{N}$ *such that* $T^{n_x}x$ *and* $T^{n_x+1}x$ *is comparable elements of* $(X, \leqslant)$*;*
3. *If an increasing sequence* (x_n) *converges to* x *in* X*, then there is a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *with* $x_{k_n} \leqslant x$ *for all* $n \in \mathbb{N}$.

Then T *is a PO.*

In the next we show the fixed point theorem for cyclic Bianchini mapping proved in [**24**] by Petric is a consequence of the Theorem 3.5.1.

Theorem 3.5.2 (M. Petric, [**23**])**.** *Let* $A_1, A_2, ..., A_p, A_{p+1} = A_1$ *be nonempty closed subsets of a complete metric space* (X,d) *and suppose* $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ *is a cyclical operator, and there exists* $a \in [0,1)$ *such that for each pair* $(x,y) \in A_i \times A_{i+1}$*, for* $i \in \{1,2,...,p\}$*, we have*

$$d(Tx,Ty) \leqslant a \cdot \max\{d(x,Tx) + d(y,Ty)\}.$$

Then T *is a PO.*

CHAPTER 4

Some extensions of fixed point theorems in metric spaces endowed with a graph

In the 4^{st} Chapter of the book, fixed point theorems from Chapter 3 in two different context:

a) working in metric spaces G-complete;

b) working on set endowed with two metrics(Maia type theorem).

The personal contributions in this chapter are: Definition 4.1.2,
Example 4.1.1, Theorems 4.1.1 and 4.2.2.

1. Fixed point theorems for generalized contractions in metric spaces G-complete

In 1971 Ćirić [**12**] extended the fixed point theorems for contractive operators in orbitally complete metric spaces .

Definition 4.1.1 (Ćirić [**12**]). *Let T maps M to M; a space M is said to be $T-$orbitally complete if every sequence $\{T^{n_i}x,\, i \in \mathbb{N}\}$, which is a Cauchy secquvence, has a limit point in M.*

Using this concept, many authors [**11**], [**12**], [**1**], [**33**] introduced and proved fixed point theorems.

Throughout this chapter we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$ and $\Delta = \{(x, x)\,|x \in X\} \subseteq G$.

Definition 4.1.2. *A metric space (X, d) is said to be G-complete if any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ with property*

$$(x_n, x_{n+1}) \in E\left(\widetilde{G}\right), \text{ for all } \; n \in \mathbb{N}$$

is convergent.

It easy to see that any complete metric space is a metric space G-complete for any graph G but the converse is not true as it see in the next example.

Example 4.1.1. *The set $X = [0, \infty) \cap \mathbb{Q}$ with euclidean metric is a metric space, and let the graph G defined by*

$$V(G) = X \text{ şi } E(G) = \{(x, 0)\,|x \in X\}$$

If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with property

$$(x_n, x_{n+1}) \in E\left(\widetilde{G}\right), \text{ for all } n \in \mathbb{N}$$

then the sequence $(x_n)_{n\in\mathbb{N}}$ has a constant subsequence 0 so is convergent to 0. But the metric space X is not complete because $(e_n)_{n\in\mathbb{N}^*}$, $e_n = \left(1+\frac{1}{n}\right)^n$ is not convergent in X.

In the next we will extend Theorem 3.1.1 [**7**] to metric space G-complete.

Theorem 4.1.1. *Let (X,d) be a metric space endowed with a graph G such that the space (X,d) is $\widetilde{G}$-complete. If:*

i. $T : X \to X$ is an G-Kannan operator with constant a;
ii. the graph G is weakly T-connect;
iii. for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Then T is PO.

In the same manner we can prove fixed point theorems for G-Zamfirescu operators, G-Ćirić-Reich-Rus operators and G-Bianchini operators in metric space G-complete.

2. Maia'stype fixed point theorems for generalized contractions in metric spaces endowed with a graph

In [**22**] Maia proved the next theorem:

Theorem 4.2.1 (Maia, [**22**])**.** *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to X$ a mapping. Assume that:*

i. $d(x,y) \leqslant \rho(x,y)$, $\forall x,y \in X$;
ii. (X,d) is a complete metric space;
iii. $T : (X,d) \to (X,d)$ is continuous;
iv. $T : (X,\rho) \to (X,\rho)$ is an a-contraction with $a \in [0,1)$;

Then T is a PO.

The Maia's type fixed point theorem for G-Kannan operators is:

Theorem 4.2.2. *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to X$ a mapping. Assume that:*

i. $d(x,y) \leqslant \rho(x,y)$, $\forall x,y \in X$;
ii. (X,d) is a complete metric space;
iii. $T : (X,d) \to (X,d)$ is continuous;
iv. $T : (X,\rho) \to (X,\rho)$ is a G-Kannan operator with constant $a \in \left[0,\frac{1}{2}\right)$;
v. the graph G is weakly T-connected.

Then T is a PO.

3. List of papers

Papers in print

(1) *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal., **75** (2012), 3895 – 3901.

(2) *Note on the stability of first order liniar differential equation*, Opuscula Math., **32** (2012), no. 1, 67 – 74.

(3) *Fixed point of φ-contraction in metric spaces endowed with a graph*, Ann. Univ. Craiova, Math. Comput. Sci. Ser., **37** (2010), no. 4, 85 – 92.

(4) *Generalized additive Cauchy equations and their Ulam-Hyers stability*, Creative Math & Info., **18** (2009), no. 2, 129 – 135.

(5) *On the stability of quartic type functional equation*, Creative Math & Info., **17** (2008), no. 2, 319 – 325.

Accepted papers

(1) *Fixed points of Bianchini mappings in metric spaces endowed with a graph*, Carpathian J. Math.

(2) *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. Stiint. Univ. Ovidius Constanta Ser. Mat.

Send papers

(1) *Fixed point theorem of Zamfirescu mappings in metric spaces endowed with a graph*

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