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Doctoral thesis summary

TRIPLED FIXED POINTS FOR OPERATORS IN PARTIALLY ORDERED METRIC SPACES

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Introduction

The subject of the thesis is part of the non-lineal sub-domanin, to be more specific of the fixed point theory. The fundamental result from the metric theory of the fixed point is the Principle of Banach-Caccioppoli-Picard Contraction [**17**], which in states that in a complete metric space, an operator $F : X \to X$ which fulfills the contraction condition:

$$d(F(x), F(y)) \leq ad(x, y) \; \forall \; x, y \in X, a \in [0, 1), \tag{0.1}$$

has a unique fixed point which can be obtained through the Picard iterations.

Starting from this result, in the last 50 years, a very vast theory was developed, as we can see in the following works: [**106**], [**118**],[**119**], [**120**], [**121**], [**122**] [**27**], [**28**], [**29**], [**30**]. Among these contribution, we can also mention the contribution of the Romanian school of fixed point theory from Cluj, school lead by Prof. Univ. Dr. Ioan Rus. These contributions refer to the fixed point theory and also at the applications of this theory in developing functional equations, of differential equations, of integral equations, of the integral-differential equations, and others. An important moment in the evolution of the fixed point theory is market by the work [**108**] from 2004 of **Ran** and **Reurings**, who consider the contradiction condition (0.1) in a metric space partially ordonat and which does not have to satisfied by any $x, y \in X$, but only by the compatible elements in the sense of the order relation defined on X:

$$\exists \; 0 < c < 1 : \; d(F(x), F(y)) \leq cd(x, y) \; \forall \; x \geq y. \tag{0.2}$$

Ran and Reurings have obtained the existence and the uniqueness of the fixed point for such contractions for the metric spaces partially ordered, also imposing a monotony condition on F and indicated applications as regards solving some matricial ecuations. This new research direction has attracted many reasearchers and had very important results. Among the persons that have obtained results in this direction, we mention **Nieto** and **R. Rodríguez-López** in the work [**97**],who established the existance and the uniqueness of the fixed point in the conditions in which the operator $F : X \to X$ verify the contraction condition (0.2), is monotone nondecreasing, but is not necessary continuous. For this to be possible, the authors have introduced the additional condition on the space: X:

$$\text{if there is a nondecreasing sequence } \{x_n\} \to x, \text{ than } x_n \leq x \text{ for any } n. \tag{0.3}$$

If the operator $F : X \to X$ is monotone nonincreasing, then in the condition (0.3) it is considered a nonincreasing sequence.

Synthesizing the results from the work of Nieto [**97**], **Bhaskar** and **Lakshmikantham**, in the work [**61**] have propposed the study of the coupled fixed points for the operators $F : X \times X \to X$, in the presence of a contaction condition with the following form:

$$d\left(F(x,y),F(u,v)\right) \le \frac{k}{2}\left[d(x,u)+d(y,v)\right], \text{ pentru } x \ge u, y \le v. \tag{0.4}$$

The thematic of the coupled points according to the work of Bhaskar and Lakshmikantham, was received with great interest. Only in the Scopus data base there are over [85] works: [**2**], [**3**], [**4**], [**5**], [**10**], [**8**], [**11**], [**12**], [**13**], [**14**], [**16**], [**18**], [**19**], [**20**], [**21**], [**22**], [**23**], [**24**], [**25**], [**43**], [**46**], [**48**], [**49**], [**50**], [**53**], [**54**], [**55**], [**57**], [**59**], [**65**], [**66**], [**67**], [**68**], [**69**], [**70**], [**71**], [**72**], [**75**], [**78**], [**79**], [**82**], [**86**], [**87**], [**88**], [**90**], [**93**], [**94**],[**95**], [**96**], [**97**], [**98**], [**99**], [**102**], [**103**], [**123**], [**124**], [**125**], [**126**], [**127**], [**128**], [**129**], [**130**], [**131**], [**132**], [**133**], [**134**], [**136**], [**137**], [**140**], [**141**], [**142**], [**135**], [**100**], [**101**], [**77**], [**52**], [**26**], [**40**],[**114**], [**91**], [**15**], [**58**], [**62**], works in which there were obtained results as regards the existance, the existance and the uniqueness of the coupled fixed points for operators that verify the different types of contactions, ans also the applications for these results in solving some differential and integral equations.

Starting from these results we wanted to study the tripled fixed points, which refer to the operators $F : X \times X \times X \to X$, motivated by the fact that, through the coupled fixed points technique we cannot solve a system with the following form:

$$\begin{cases} x^2 + 2yz - 6x + 3 = 0 \\ y^2 + 2zx - 6y + 3 = 0 \\ z^2 + 2xy - 6z + 3 = 0 \end{cases} . \tag{0.5}$$

In the articles [**31**] **"Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces"** and [**32**] **"Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces"** the authors **Berinde-Borcut** respectivelly **Borcut-Berinde** have introduced the concepts **tripled fixed point** and **tripled coincidence point** and have obtained results on the existence, the existance and the uniqueness of the fixed tripled point, respectivelly of the coincidence tripled point for operators defined on metric space partially ordonat which verify the contraction condition of the following type:

there is a $j, k, l \in [0,1)$ cu $j + k + l < 1$, so that

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \le jd\left(x,u\right) + kd\left(y,v\right) + ld\left(z,w\right), \tag{0.6}$$

$\forall x \geq u, y \leq v, z \geq w$, respectivelly a condition of the folowing form

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq jd\left(g\left(x\right),g\left(u\right)\right)+kd\left(g\left(y\right),g\left(v\right)\right)+ +ld\left(g\left(z\right),g\left(w\right)\right), \tag{0.7}$$

for any $x, y, z, u, v, w \in X$ cu $g\left(x\right) \leq g\left(u\right),\ g\left(y\right) \geq g\left(v\right),\ g\left(z\right) \leq g\left(w\right)$.

Starting from these results, the main purpose of this thesis is to present the concepts and the main results as regards the existence, the existence and the uniqueness of the tripled fixed points, respectivelly of the tripled concidence points for operators defined on metric spaces partially ordered. As base for this new theory, a minimal package from the fixed point theory, the coupled fixed points and the coupled coincidence points was presented.

The work is structured on 4 chapters and ends with the list of the bibliographic references.

Chapter I, named **Basic elements from the fixed point theory**, is a chapter that includes the minimal package of notions and results from the fixed point theory, examples and applications. This chapter is made up of 4 paragraphs.

In **Paragraph 1.1. Notions from the fixed point theory** are presented various basic notions from the fixed point theory, meaning: **the metric space, the complet metric space, the ordered metric space, the space partially ordered**, **fixed point**, **continuous operator, Lipschitz operator, a-contraction, φ-contraction.** The sub-paragraph 1.1.3 includes three fixed point theorem [the Contraction Principles], which are fundamental for this work. These theorems are: **the Banach-Caccioppoli-Picard Theorem, the Matkowski Theorem, the Ran-Reurings Theorem**. The last theorem is actually the Banach-Caccioppoli-Picard Contraction Theory applied to the partially ordered metric spaces.

Paragraph 1.2: The coupled fixed points for operators defined on partially ordered metric spacesincludes three sub-paragraphs. The sub-paragraph 1.2.1 includes the following concepts: produced metric space $X \times X$ partially ordered, mixed-monotone operator $F : X \times X \to X$, coupled fixed point for F, sometric composition of two operators and their properties. In the paragraphs 1.2.2 and 1.2.3 are presented 8 theorems regarding the existence and the uniqueness of the coupled fixed points for mixed-monotone and continuous operators,defined on metric spaces partially odonate, operators type Picard (extensions or generalizations of the contraction condition) or φ-contractions. In the absence of the continuity of F must be imposed conditions on the space X,, meaning the fact that the row $\{x_n\}$ converge nondecreasing to an element $x \in X$, than we have $x_n \leq x, \forall n$. The same thing happens for rows that convergent nonincreasing. In order to have the uniqueness of the coupled fixed point, at the ypothesis of the existence theorems must exists a comparison condition of the elements from $X \times X$, condition connected to the order relation with which the space $X \times X$. is provided with.

Paragraph 1.3: Coupled concidence points for operators defined on partially ordered metric spaces. This paragraph is structured on three sub-paragraphs, in sub-paragraph 1.3.1 being presented the following concepts: mixed-g-monotone operator, coupled coincidence point and the comutativity of the operators F şi g. In paragraphs 1.3.2 and 1.3.3 are mentioned 2 existence theorems and 3 uniqueness theorems of the concident coupled points for mixed-g-monotone operators of the type φ-contraction[Matkowski].

Paragraph 1.4: Examples. Applications. In this paragraph are presented the exampled that ilustrate the applicative importance of the results from this chapter. As application, we studied the existence and the uniqueness of the solution to the problem at the limit with periodical values, application of the theorem 1.2.38 taken from [**61**]

The contribution of the authors to this chapter are: the Example 1.4.59; the Example 1.4.60; the Examplel 1.4.61; the Example 1.4.62.

Chapters II and III form the nucleus of the thesis and are entirely the contribution of the author. These two chapters contain 12 definitions, 27 theorems, 3 corollary, 4 propositions, 12 examples and 1 applications. Each chapter has three paragraphs, which are devided in sub-pararaphs.

Chapter II. Tripled fixed points for operators in partially ordered metric spaces.

Paragraph 2.1. Tripled fixed points for mixed-monotone operators : The firts sub-paragraph is dedicated to define the new concepts. For $(X, d, \leq)$ a complete metric space partially ordonat is define as follows: the metric space produced $X \times X \times X = X^3$ partially ordonat, the metrics on this space, and for the opeator $F : X^3 \to X$ we define: the mixed-monotone property, the tripled fixed point for mixed-monotone operators as follows: $(x, y, z) \in X^3$ is a tripled fixed point if

$$x = F(x, y, z), y = F(y, x, y), z = F(z, y, x).$$

In the sub-paragraphs 2.1.2 respectivelly 2.1.3 are presented the existence theorems, respectivelly the uniqueness theorems of the tripled fixed points for operator type Picard. In order to have the uniqueness of the tripled fixed point, at the existence hypothesis is introduced a suplimentary condition to the comparison of the elements, which must be fulfilled by the space X^3.

Paragraph 2.2 Tripled fixed points for monotone operators: The structure of this paragraph is similar to the one of the previous paragraph, with the mention that the operators are monotone, and the definition of the tripled fixed point is completely different, as follows: $(x, y, z) \in X^3$ is a tripled fixed point for the monotone operator F if

$$x = F(x, y, z), y = F(y, x, z), z = F(z, y, x).$$

The results regarding the existence and the uniqueness of the tripled fixed points for monotone operators are presented in the sub-paragraphs 2.2 and 2.3.

Paragrah 2.3 Examples. Applications. In this paragraph are presented examples of mixed-monotone operators (monotone) that have one or more tripled fixed points.

As application to the fixed points theory, we present the solution to the integral equations:

$$x(t) = \int_0^T G(t, s)[f(s, x(s)) + g(s, x(s)) + h(s, x(s))]ds + a(t), t \in [0, T], T > 0.$$

We consider that the space $X = C([0, T], \mathbb{R})$ of the continuous operators defined on $[0, T]$ with real values, with the metric

$$d(u, v) = \max_{0 \leq t \leq T} |u(t) - v(t)|\,, \text{ for } u, v \in X,$$

and the operator $F : X^3 \to X$ defined as follows:

$$F(x_1, x_2, x_3)(t) = \int_0^T G(t, s)[f(s, x_1(s)) + g(s, x_2(s)) + h(s, x_3(s))]ds + a(t), t \in [0, T],$$

for any $x_1, x_2, x_3 \in X$, fulfills the condition of the Theorem 2.1.15.

Chapter III. Tripled coincidence points for operators defined on partially ordered metric spaces, chapter in which is introduced the concept of tripled coincidence point for mixed-g-monotone operators, and also for g-monoton operators. As well as Chapter II, Chapter III is structured in three paragraphs, and the first two paragraphs are divided each in three sub-paragraphs.

Paragraph 3.1 Tripled coincidence points for mixed-g-monotone operators. In sub-paragraph 3.1.1 are presented the following definitions: for the mixed-g-monotony of the operator $F : X^3 \to X$, where $g : X \to X$ is any function; for the comutativity of the operator F with the function g; for the tripled coincidence point of the operator F and of the function g as follows $(x, y, z) \in X^3$ is the tripled coincidence point if

$$F(x, y, z) = g(x), F(y, x, y) = g(y), F(z, y, x) = g(z).$$

In the sub-paragraphs 3.1.2 and 3.1.3 are presented the existence and the uniqueness theorems for the tripled coincidence points.

Paragraph 3.2 Tripled coincidence points for g-monotone operators. The structure of this paragraph is the same as of paragraph 3.1, but with a different content, meaning that the operators are g-monotone and so the definition of the tripled coincidence point is completely different. If $F : X^3 \to X$ is a g-monotone operator, than $(x, y, z) in X^3$ is the tripled coincidence point for F and g if

$$F(x, y, z) = g(x), F(y, x, z) = g(y), F(z, y, x) = g(z).$$

The examples of tripled coincidence points for the operator F and the function g are presented in paragraph 3.3.

Chapter IV Conclusions In this chapter are presented the conclusions of this thesis and are mentioned a few future research directions, starting from the results presented in this work.

CHAPTER 1

Basic elements from the fixed point theory

The fixed point theory was created for operators defined on different spaces, as for example: metric spaces, Banach spaces, topologic spaces, partially ordered metric spaces, uniform spaces, etc. In this work we will debate the fixed point theory, especially for operators defined on partially ordered metric spaces. This is why in this chapter we discuss only a few basic notions, strictly necessary in chapters II and III.

1. Notions of fixed point theory

In this paragraph are presented the fundamental notions of fixed point theory (**fixed point, complet metric space, space partially ordonat, Lipschitz operator, a-contraction, φ-contraction**), ans also three principles of existence, existence and uniqueness of the fixed point (**Banach-Caccioppoli-Picard Contraction Principle**, **Matkowski Principle of φ-Contraction** , **Ran-Reurings Principle**). The bibliographic works used in order to write this paragraph are the following: [**6**], [**17**], [**27**], [**28**], [**29**], [**30**], [**44**], [**45**], [**56**], [**60**], [**84**], [**108**], [**106**], [**121**], [**119**], [**120**], [**118**], [**122**].

2. Coupled fixed points for operators in partially ordered metric spaces

In this paragraph we will present the basic theory of coupled fixed points, theory introduced by Bhaskar and Lakshmintham in the work [**61**] inl 2006. This work is fundamental for many scientific works and is fundamental also for this work. This paragraph has three sub-paragraphs in which we presented definitions for: the produced space $X \times X$ partially ordered, the mixed-monotone operator $F : X \times X \to X$, the fixed coupled point for the operator F, the simetric composition of the operators F, G and its properties [sub-paragraph 1.2.1], and also the existence theorems of the coupled fixed points for the operator F [sub-paragraph 1.2.2] and also uniqueness theorems for the coupled fixed points for the operator F [sub-paragraph 1.2.3].

The bibliographic references studied for this sub-paragraph are: [**16**], [**18**], [**19**], [**21**], [**22**], [**23**], [**27**], [**61**],[**19**], [**29**], [**30**], [**42**], [**43**], [**97**], [**65**], [**98**], [**123**], [**124**], [**94**].

3. Coupled coincidence points for operators in partially ordered metric spaces

In this paragraph is presented thetheory of coupled coincidence points in a minimal form, basedon the work [**85**], work published in 2009. The authors of this work, Lakshmikantham and Ćirić generalize the notion of coupled fixed point and introduce the concept of coupled coincidence point. The bibliographic works studied for this paragraph are the following: [**85**], [**2**], [**8**], [**11**], [**20**], [**21**], [**23**], [**122**], [**118**], [**120**], [**119**], [**121**], [**27**], [**29**], [**30**], [**16**], [**43**], [**18**], [**48**], [**49**], [**50**], [**65**], [**61**], [**70**], [**71**], [**72**].

CHAPTER 2

Tripled fixed points for operators in partially ordered metric spaces

The organization of this chapter is the following: introduction of the new concepts, giving basic results regarding the existence, the existence and the uniqueness of the tripled fixed points for mixed-monotone operators, and also for i monotonoe operators, defined on partially ordered metric spacesand we ilustrate the theoretical results with examples and applications.

The chapter on the integrality contains the author's contributuins, contributions made of **7 definitions**, **19 theorems**, **4 propositions**, **6 examples**, **1 application**.

The results of this chapter are included in the following works:

[31] **Berinde, V.,Borcut, M.,** ***Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*****, Nonlinear Anal., 74, (2011) 4889-4897**;

[34] **Borcut, M.,** ***Tripled fixed point theorems for monotone contractive type mappings in partially ordered metric spaces*** **Carpathian J. Math. (Acceptat)**.

[36] **Borcut, M.,** ***Tripled fixed point theorems for operators which verify the contraction-type condition Chatterjea in partially ordered metric spaces*****, Aplied Mathematics and Computation, (Submitted)**;

[37] **Borcut, M.,** ***Tripled fixed point theorems for operators which verify the contraction-type condition Kannan in partially ordered metric spaces*****, Mathematics and Computers in Simulation, (Submitted)**;

[38] **Borcut, M.,** ***Tripled fixed point theorems in partially ordered metric spaces*****, Hacettepe Journal of Mathematics and Statistics (Submited)**;

The bibliographic references on which this chapter is based on are: [**31**], [**38**], [**34**], [**16**], [**18**], [**19**], [**21**], [**22**], [**23**], [**27**], [**61**],[**19**], [**29**], [**30**], [**42**], [**43**], [**97**], [**65**], [**98**], [**123**], [**124**], [**6**], [**17**], [**27**], [**28**], [**29**], [**30**], [**44**], [**45**], [**56**], [**60**], [**84**], [**108**], [**106**], [**121**], [**119**], [**120**], [**118**], [**122**], [**2**], [**3**], [**4**], [**5**], [**137**], [**8**], [**10**], [**11**], [**65**], [**66**], [**67**], [**68**], [**69**], [**71**], [**72**], [**79**], [**82**], [**88**], [**90**], [**95**], [**102**], [**104**], [**94**].

1. Tripled fixed points for mixed-monotone operators

This paragraph is dedicated to the theory of tripled fixed points for mixed-monotone operators, where in the first sub-paragraph are presented the definitions for: the space produced $X \times X \times X$ partially ordered, mixed-monotony of the operator F, tripled fixed point, metric of the space produced $X \times X \times X$, the simetric composition of two operators and their properties. In sub-paragraphs 2.1.2 and 2.1.3 are presented the existence theorems and the uniqueness theorems of the tripled fixed points.

1.1. Definitions

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric spaces. Further, we endow the product space $X \times X \times X$ with the following partial order:

for $(x, y, z), (u, v, w) \in X \times X \times X, (u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$

Definition 2.1.1 (Borcut, [**38**]). *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x, monotone nonincreasing in y and monotone nondecreasing in z, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, z_2 \leq z_1 \Rightarrow F(x, y, z_2) \geq F(x, y, z_1)$$

Definition 2.1.2 (Berinde-Borcut, [**31**]). *Call an element $(x, y, z) \in X \times X \in X$ a tripled fixed point of the mapping F if*

$$F(x, y, z) = x, F(y, x, y) = y, F(z, y, x) = z.$$

.

Definition 2.1.3 (Berinde-Borcut, [**31**]). *Let (X, d) be a complete metric space set. It is called metric on $X \times X \times X$, the mapping $d : X \times X \times X \to X$ with*

$$d[(x, y, z), (u, v, w)] = d(x, u) + d(y, v) + d(z, w).$$

To simplify the demonstration we define the composition of symmetric operators with three variables.

Definition 2.1.4 (Borcut, [**38**]). *Let X, Y, Z be nonempty sets and $F : X \times X \times X \to X$, $G : Y \times Y \times Y \to Z$. We define the symmetric composition (or, the s-composition for short) of F and G by $G * F : X \times X \times X \to Z$,*

$$(G * F)(x, y, z) = G(F(x, y, z), F(y, x, y), F(z, y, x))\,(x, y, z \in X).$$

For each nonempty set X, denote by P_x the projection mapping

$$P_X : X \times X \times X \to X, P(x, y, z) = x \text{ for } x, y, z \in X.$$

The symmetric composition has the following properties:

Proposition 2.1.5 (Borcut, [**38**]). *(Associativity). If $F : X \times X \times X \to Y$, $G : Y \times Y \times Y \to Z$ and*

$$H : Z \times Z \times Z \to W, \text{ then } (H * G) * F = H * (G * F).$$

Proposition 2.1.6 (Borcut, [**38**]). *(Identity Element). If $F : X \times X \times X \to Y$, then*

$$F * P_X = P_Y * F = F.$$

Proposition 2.1.7 (Borcut, [**38**]). *(Mixed Monotonicity). If $(X, \leq)$, $(Y, \leq)$, $(Z, \leq)$ are partially ordered sets and the mappings $F : X \times X \times X \to Y, G : Y \times Y \times Y \to Z$ are mixed monotone, then $G * F$ is mixed monotone.*

Proposition 2.1.8 (Borcut, [**38**]). *If $(X, \leq)$ is a partially ordered set and F is mixed monotone, then $F^n = F * F^{n-1} = F^{n-1} * F$ is mixed monotone for every n.*

1.2. Existence theorems

Theorem 2.1.9 (Borcut, [**38**]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)] \tag{2.8}$$

$$\text{for each } x \geq u, y \leq v, z \geq w.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0), \tag{2.9}$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

The previous result is still valid for F not necesarily continuous. Instead, we require that the underlying metric space X has an additional property. We discuss this in the following theorem.

Theorem 2.1.10 (Borcut, [**38**]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$*

be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq \frac{k}{3}\left[d\left(x,u\right)+d\left(y,v\right)+d\left(z,w\right)\right]$$
$$\text{for each } x \geq u, y \leq v, z \geq w.$$

Assume that the following property:

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y_n \geq y$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \geq F\left(y_0,x_0,y_0\right) \text{ and } z_0 \leq F\left(z_0,y_0,x_0\right),$$

then there exist $x, y, z \in X$ *such that*

$$x = F\left(x,y,z\right), y = F\left(y,x,y\right) \text{ and } z = F\left(z,y,x\right).$$

Theorem 2.1.11 (Berinde-Borcut, [**31**])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j,k,l \in [0,1)$ with $j+k+l < 1$ for which*

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,w\right), \tag{2.10}$$

$\forall x \geq u, y \leq v, z \geq w$. *If there exist* $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \geq F\left(y_0,x_0,y_0\right) \text{ and } z_0 \leq F\left(z_0,y_0,x_0\right),$$

then there exist $x, y, z \in X$ *such that*

$$x = F\left(x,y,z\right), y = F\left(y,x,y\right) \text{ and } z = F\left(z,y,x\right).$$

The previous result is still valid for F not necessarily continuous. Instead, we require that the underlying metric space X has an additional property. We discuss this in the following theorem.

Theorem 2.1.12 (Berinde-Borcut, [**31**])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists the constants $j,k,l \in [0,1)$ with $j+k+l < 1$ such that* (2.12) *is satisfied for each $x \geq u, y \leq v, z \geq w$. Assume that X has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n,

(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \geq y$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \geq F\left(y_0,x_0,y_0\right) \text{ and } z_0 \leq F\left(z_0,y_0,x_0\right),$$

then there exist $x, y, z \in X$ *such that*

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

1.3. The existence and the uniqueness theorems

In [**61**] and [**124**] the authors also considered some additional conditions to ensure the uniqueness of the coupled fixed point and appropriate conditions to ensure that for the coupled fixed point (x, y) we have $x = y$.

Similarly, one can prove that the tripled fixed point is in fact unique, provided that the product space $X \times X \times X$ endowed with the partial order mentioned earlier has an additional property.

Theorem 2.1.13 (Borcut, [**38**]). *By adding to the hypothesis of Theorem 2.1.9 the condition: for every* $(x, y, z), (x_1, y_1, z_1) \in X \times X \times X$*, there exists a* $(u, v, w) \in X \times X \times X$ *that is comparable to* (x, y, z) *and* (x_1, y_1, z_1)*, we obtain the uniqueness of the tripled fixed point of* F*.*

Assuming that every tripled of elements of X have either an upper bound or a lower bound in X, one can in fact show that even the components of the tripled fixed points are equal.

Theorem 2.1.14 (Borcut, [**38**]). *In addition to the hypothesis of Theorem 2.1.9 (resp., Theorem 2.1.10) suppose that every tripled of elements of* X *has an upper bound or lower bound in* X*. Then* $x = y = z$*.*

Theorem 2.1.15 (Borcut, [**38**]). *In addition to the hypothesis of Theorem 2.1.9 (resp. Theorem 2.1.10) suppose that* $x_0, y_0, z_0 \in X$ *are comparable. Then* $x = y = z$*.*

Theorem 2.1.16 (Berinde-Borcut, [**31**]). *By adding to the hypothesis of Theorem 2.1.11 the condition: for every* $(x, y, z), (x_1, y_1, z_1) \in X \times X \times X$*, there exists a* $(u, v, w) \in X \times X \times X$ *that is comparable to* (x, y, z) *and* (x_1, y_1, z_1)*, we obtain the uniqueness of the tripled fixed point of* F*.*

Theorem 2.1.17 (Berinde-Borcut,[**31**]). *In addition to the hypothesis of Theorem 2.1.11 (resp., Theorem 2.1.12) suppose that every tripled of elements of* X *has an upper bound or lower bound in* X*. Then* $x = y = z$*.*

Theorem 2.1.18 (Berinde-Borcut, [**31**]). *In addition to the hypothesis of Theorem 2.1.11 (resp. Theorem 2.1.12) suppose that* $x_0, y_0, z_0 \in X$ *are comparable. Then* $x = y = z$*.*

2. Tripled fixed points for monotone operators

In this paragraph we will present the results regarding the existence and the uniqueness of the tripled fixed points for monotone operators defined on partially ordered metric spaces. The produced space $X \times X \times X$ on which the operator F, is defined, will be defined as partially ordered as in the previous paragraph, the metric with which we will operate is the same, but the operator is not mixed-monotone, but monotone, and the tripled fixed point has a different form.

2.1. Definitions

Let$(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X = X^3$ the following partial order: for $(x, y, z), (u, v, w) \in X^3$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 2.2.19 (Borcut, [**34**]). *Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$. We say that F monotone if $F(x, y, z)$ is monotone nondecreasing in x, y and z, thet is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \leq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2.2.20 (Borcut, [**34**]). *An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if*

$$F(x, y, z) = x, F(y, x, z) = y, \text{ and } F(z, y, x) = z.$$

Remark 2.2.21. *The tripled fixed notion in this context is different from the one from the previous section.*

2.2. Existence theorems

Theorem 2.2.22 (Borcut, [**34**]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a continuous and monotone mapping. Assume that there exist $k \in [0, 1)$, for which*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)] \tag{2.11}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, z_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ *such that*

$$x = F(x, y, z), y = F(y, x, z) \text{ and } z = F(z, y, x).$$

The previous result is still valid for F not necessarily continuous. Instead, we require that the underlying metric space X has an additional property. We discuss this in the following theorem.

Theorem 2.2.23 (Borcut, [**34**])**.** *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X^3 \to X$ *be a mapping having the monotone property on* X*. Assume that there exists the constant* $k \in [0, 1)$ *, such that*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)]$$

for each $x \geq u, y \leq v, z \geq w$*. Assume that* X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$*, then* $x_n \leq x$ *for all* n*.*

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, z_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ *such that*

$$x = F(x, y, z), y = F(y, x, z) \text{ and } z = F(z, y, x).$$

Theorem 2.2.24 (Borcut, [**34**])**.** *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X^3 \to X$ *be a continuous and monotone mapping. Assume that there exist the constants* $j, k, l \in [0, 1)$ *with* $j + k + l < 1$ *for which*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{2.12}$$

$\forall x \geq u, y \leq v, z \geq w$*. If there exist* $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, z_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ *such that*

$$x = F(x, y, z), y = F(y, x, z) \text{ and } z = F(z, y, x).$$

Theorem 2.2.25 (Borcut, [**34**])**.** *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X^3 \to X$ *be a mapping having the monotone property on* X*. Assume that there exists the constants* $j, k, l \in [0, 1)$ *with* $j+k+l < 1$ *such that* (2.12) *is satisfied for each* $x \geq u, y \leq v, z \geq w$*. Assume that* X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$*, then* $x_n \leq x$ *for all* n*.*

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, z_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ *such that*

$$x = F(x, y, z), y = F(y, x, z) \text{ and } z = F(z, y, x).$$

2.3. Existence and uniqueness theorems

In this sub-paragraph we will present the uniqueness theorems of the tripled fixed points for monotone operators define on partially ordered metric spaces. If at the existence theorems hypothesis is added a comparison condition of the elements from the produced space $X \times X \times X$, then the tripled fixed poit willbe unique.

Theorem 2.2.26 (Borcut, [**34**]). *By adding to the hypothesis of Theorem 2.2.22 the condition: for every* $(x, y, z), (x_1, y_1, z_1) \in X^3$*, there exists a* $(u, v, w) \in X^3$ *that is comparable to* (x, y, z) *and* (x_1, y_1, z_1)*, we obtain the uniqueness of the tripled fixed point of F.*

In the following Theorem are presented the conditions in which the tripled fixed point is unique and ots components are equal.

Theorem 2.2.27 (Borcut, [**34**]). *In addition to the hypothesis of Theorem 2.2.22 (resp., Theorem 2.2.23) suppose that every tripled of elements of* X *has an upper bound or lower bound in* X*. Then* $x = y = z$*.*

Theorem 2.2.28 (Borcut, [**34**]). *By adding to the hypothesis of Theorem 2.2.24 the condition: for every* $(x, y, z), (x_1, y_1, z_1) \in X^3$*, there exists a* $(u, v, w) \in X^3$ *that is comparable to* (x, y, z) *and* (x_1, y_1, z_1)*, we obtain the uniqueness of the tripled fixed point of F.*

Theorem 2.2.29 (Borcut, [**34**]). *In addition to the hypothesis of Theorem 2.2.24 (resp., Theorem 2.2.25) suppose that every tripled of elements of* X *has an upper bound or lower bound in* X*. Then* $x = y = z$*.*

3. Examples and applications

In this paragraph we will present examples of operators which fulfill the conditions of some theorems on the existence, the existence and the uniqueness presented in this chapter, as well as an application on solving the integral equation

$$(2.13)\quad x(t) = \int_0^T G(t,s)[f(s,x(s)) + g(s,x(s)) + h(s,x(s))]ds + a(t), t \in [0,T], T > 0.$$

CHAPTER 3

Tripled coincidenc points for operators in partially ordered metric spaces

The purpose of this chapter is to present the concident tripled points theory for mixed-monotone operators, as well as for monotone operators, define on partially ordered metric spaces. The content is the following: definitions of the new concepts, from the results regarding the existence, the existence and the uniqueness of the coincidence tripled points, ans also examples.

This chapter **containes only the author's contributions**, contributions that include **5 definitions**, **8 theorems**, **3 corollary**, **6 examples**.

The results ofthis chapter are included in the following articles:

[32] **Borcut, M., Berinde, V., *Tripled coincidenc theorems for contractive type mappings in partially ordered metric spaces* , Aplied Mathematics and Computation, 218 (10) (2012) pp. 5929-5936**;

[38] **Borcut, M., *Tripled coincidence point theorems for contractive type mappings in partially ordered metric spaces*, Aplied Mathematics and Computation, 218 (2012) pp. 7339-7346** ;

[35] **Borcut, M., *Tripled coincidence point theorems for monotone contractive type mappings in partially ordered metric spaces*, Creative Mathematics and Informatics, (acceptat)**;

[39] **Borcut, M., *Tripled coincidence point theorems for monoton ϕ-contractive type mappings in partially ordered metric spaces*Filomat J. (submitted)**.

The bibliograhic works on which the present chapter is based on are the following: [**32**], [**33**], [**35**], [**39**], [**85**], [**2**], [**8**], [**11**], [**20**], [**21**], [**23**], [**122**], [**118**], [**120**], [**119**], [**121**], [**27**], [**29**], [**30**], [**16**], [**43**], [**18**], [**48**], [**49**], [**50**], [**65**], [**61**], [**70**], [**71**], [**72**]. [**31**], [**38**], [**34**], [**16**], [**18**], [**19**], [**21**], [**22**], [**23**], [**27**], [**61**],[**19**], [**29**], [**30**], [**42**], [**43**], [**97**], [**65**], [**98**], [**123**], [**124**], [**6**], [**17**], [**27**], [**28**], [**29**], [**30**], [**44**], [**45**], [**56**], [**60**], [**84**], [**108**], [**106**], [**121**], [**119**], [**120**], [**118**], [**122**], [**2**], [**3**], [**4**], [**5**], [**137**], [**8**], [**10**], [**11**], [**65**], [**66**], [**67**], [**68**], [**69**], [**71**], [**72**], [**79**], [**82**], [**88**], [**90**], [**95**], [**102**], [**104**], [**94**].

1. Tripled coincidenc points for mixed-g-monotone operators

In this paragraph are presented the definitions of the new concepts and the existence and uniqueness theorems of the coincidence tripled coincidence points. The new introduced concepts are: mixed-g-monotony, coincidence point for a mixed-g-monotone operator [Sub-paragraph 3.1.1]. The results regarding the existence and uniqueness of the coincidence points are presented in sub-paragraphs 3.1.2 and 3.1.3.

1.1. Definitions

Let$(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 3.1.1 (Borcut, [**35**]). *Let $(X, \leq)$ be a partially ordered set and two mappings*
$F : X \times X \times X \rightarrow X, g : X \rightarrow X$. We say that F has the mixed-g-monotone property if $F(x, y, z)$ it is g-monotone nondecreasing in x, it is g-monotone nonincreasing in y and is g-monotone nondecreasing in z, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, g(z_2) \leq g(z_1) \Rightarrow F(x, y, z_2) \geq F(x, y, z_1).$$

Definition 3.1.2 (Borcut, [**35**]). *Call an element $(x, y, z) \in X \times X \in X$ a tripled coincidence point of the mappings F and g if*

$$F(x, y, z) = g(x),\ F(y, x, y) = g(y), F(z, y, x) = g(z).$$

Definition 3.1.3 (Borcut, [**35**]). *Let X be a non-empty set and $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ two mappings. We say F and g are commutative (or that F and g commute) if::*

$$g(F(x, y, z)) = F(g(x),\ g(y), g(z)), \forall x, y, z \in X.$$

1.2. Existence theorems

Theorem 3.1.4 (Borcut, [**35**]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ be such that F has the mixed-g-monotone property. Assume that there exists $j, k, l \in [0, 1)$ with $j + k + l < 1$, such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(g(x), g(u)) + kd(g(y), g(v)) + \tag{3.14}$$
$$+ld(g(z), g(w)),$$

for all $x, y, z, u, v, w \in X$ with $g(x) \leq g(u),\ g(y) \geq g(v),\ g(z) \leq g(w)$.

Suppose $F(X\times X\times X)\subseteq g(X)$, g *is continuous and commutes with* F *and also suppose either*

(a) F *is continuous or*

(b) X *has the following property:*

$$(i)\ \text{if a non-decreasing sequence}\ \{x_n\}\to x,\ \text{then}\ x_n\leq x\ \text{for all}\ n,$$

$$(ii)\ \text{if a non-increasing sequence}\ \{y_n\}\to y,\ \text{then}\ y_n\geq y\ \text{for all}\ n.$$

If there exist $x_0,y_0,z_0\in X$, *such that*

$$g(x_0)\leq F(x_0,y_0,z_0)\,,g(y_0)\geq F(y_0,x_0,z_0)\ \text{and}\ g(z_0)\leq F(z_0,y_0,x_0)$$

then there exist $x,y,z\in X$ *such that*

$$g(x)=F(x,y,z)\,,g(y)=F(y,x,y)\ \text{and}\ g(z)=F(z,y,x)\,.$$

Corollary 3.1.5 (Borcut, [**35**]). *Let* $(X,\leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $F:X\times X\times X\to X$ *and* $g:X\to X$ *be such that* F *has the mixed-g-monotone property. Assume that there exists* $a\in[0,1)$, *such that*

$$d(F(x,y,z)\,,F(u,v,w))\leq\frac{a}{3}\left[d(g(x),g(u))+d(g(y),g(v))+d(g(z),g(w))\right]\tag{3.15}$$

for all $x,y,z,u,v,w\in X$ *with* $g(x)\leq g(u)$, $g(y)\geq g(v)$, $g(z)\leq g(w)$.

Suppose $F(X\times X\times X)\subseteq g(X)$, g *is continuous and commutes with* F *and also suppose either*

(a) F *is continuous or*

(b) X *has the following property:*

$$(i)\ \text{if a non-decreasing sequence}\ \{x_n\}\to x,\ \text{then}\ x_n\leq x\ \text{for all}\ n,$$

$$(ii)\ \text{if a non-increasing sequence}\ \{y_n\}\to y,\ \text{then}\ y_n\geq y\ \text{for all}\ n.$$

If there exist $x_0,y_0,z_0\in X$, *such that*

$$g(x_0)\leq F(x_0,y_0,z_0)\,,g(y_0)\geq F(y_0,x_0,z_0)\ \text{and}\ g(z_0)\leq F(z_0,y_0,x_0)$$

then there exist $x,y,z\in X$ *such that*

$$g(x)=F(x,y,z)\,,g(y)=F(y,x,y)\ \text{and}\ g(z)=F(z,y,x)\,.$$

Theorem 3.1.6 (Borcut, [**35**]). *Let* $(X,\leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $F:X\times X\times X\to X$ *and* $g:X\to X$ *be such that* F *has the mixed-g-monotone property. Assume there is a function* $\varphi:[0,\infty)\to[0,\infty)$ *with* φ *non-decreasing,* $\varphi(t)<t$ *and* $\lim\limits_{r\to t_+}\varphi(r)<t$ *for each* $t>0$ *such that*

$$d(F(x,y,z)\,,F(u,v,w))\leq\tag{3.16}$$

$$\leq \varphi\left(\max\left\{d(g(x),g(u));d(g(y),g(v));d(g(z),g(w))\right\}\right)$$

$$\text{for all } x,y,z,u,v,w \in X \text{ with } g(x) \leq g(u),\ g(y) \geq g(v),\ g(z) \leq g(w).$$

Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i) \text{ if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x \text{ for all } n,$$

$$(ii) \text{ if a non-increasing sequence } \{y_n\} \to y, \text{ then } y_n \geq y \text{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0,y_0,z_0),\ g(y_0) \geq F(y_0,x_0,y_0) \text{ and } g(z_0) \leq F(z_0,y_0,x_0)$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

Corollary 3.1.7 (Borcut, [**35**]). *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be such that F has the mixed-g-monotone property. Assume that there exists $k \in [0,1)$, such that*

$$d\left(F(x,y,z), F(u,v,w)\right) \tag{3.17}$$

$$\leq k\left(\max\left\{d(g(x),g(u));d(g(y),g(v));d(g(z),g(w))\right\}\right)$$

$$\text{for all } x,y,z,u,v,w \in X \text{ with } g(x) \leq g(u),\ g(y) \geq g(v),\ g(z) \leq g(w).$$

Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i) \text{ if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x \text{ for all } n,$$

$$(ii) \text{ if a non-increasing sequence } \{y_n\} \to y, \text{ then } y_n \geq y \text{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0,y_0,z_0),\ g(y_0) \geq F(y_0,x_0,y_0) \text{ and } g(z_0) \leq F(z_0,y_0,x_0)$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

1.3. Existence and uniqueness theorems

In [**31**], [**61**], [**85**] and other related papers, the authors also considered some additional conditions to ensure the uniqueness of the coupled fixed point, of the coupled coincidence or of the tripled fixed point, respectively.

So, we state and prove the corresponding result regarding the uniqueness of tripled coincidenc points.

Theorem 3.1.8 (Borcut, [**35**]). *By adding to the hypotheses of Theorem 3.1.4 the condition: for every* $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$*, there exists a* $(u, v, w) \in X^3$ *such that* $(F(u, v, w), F(v, u, w), F(w, v, u))$ *is comparable to* $(g(x), g(y), g(z))$ *and to* $(g(x^*), g(y^*), g(z^*))$*, then* F *and* g *have a unique tripled coincidenc point.*

Theorem 3.1.9 (Borcut, [**35**]). *By adding to the hypotheses of Theorem 3.1.6 the condition: for every* $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$*, there exists a* $(u, v, w) \in X^3$ *such that* $(F(u, v, w), F(v, u, w), F(w, v, u))$ *is comparable to* $(g(x), g(y), g(z))$ *and to* $(g(x^*), g(y^*), g(z^*))$*, then* F *and* g *have a unique tripled coincidenc point.*

2. Tripled coincidence points for monotone operators

In this paragraph are presented the results regarding the existence, the existence and uniqueness of the coincidence tripled points for monotone operators, and the definition of the tripled coincidence point is completely different, as appears in paragraph 2.2.

2.1. Definitions

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric spaces. We endow the product space $X \times X \times X$ with the following partial order:

for $(x,y,z), (u,v,w) \in X \times X \times X, (u,v,w) \leq (x,y,z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$

Definition 3.2.10 (Borcut, [**35**]). *Let $(X, \leq)$ be a partially ordered set and two mappings*
$F : X \times X \times X \to X, g : X \to X$. We say that F has the g-monotone property if $F(x,y,z)$ it is g-monotone nondecreasing in x, y, z, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1,y,z) \leq F(x_2,y,z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x,y_1,z) \leq F(x,y_2,z),$$

$$z_1, z_2 \in X, g(z_2) \leq g(z_1) \Rightarrow F(x,y,z_1) \leq F(x,y,z_2).$$

Definition 3.2.11 (Borcut, [**35**]). *Call an element $(x,y,z) \in X \times X \in X$ a tripled coincidenc point of the mappings F and g if*

$$F(x,y,z) = g(x), \; F(y,x,z) = g(y), F(z,y,x) = g(z).$$

2.2. Existence theorems

Theorem 3.2.12 (Borcut, [**35**]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be such that F has the g-monotone property. Assume that there exists $j, k, l \in [0,1)$ with $j + k + l < 1$, such that*

$$d(F(x,y,z), F(u,v,w)) \leq jd(g(x), g(u)) + kd(g(y), g(v)) \tag{3.18}$$
$$+ld(g(z), g(w)),$$

for all x,y,z,u,v,w $\in X$ with $g(x) \leq g(u), g(y) \geq g(v), g(z) \leq g(w)$.Suppose $F(X^3) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property: if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n, If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0,y_0,z_0), g(y_0) \leq F(y_0,x_0,z_0) \text{ and } g(z_0) \leq F(z_0,y_0,x_0)$$

then there exist $x, y, z \in X$ *such that* $g(x) = F(x, y, z), g(y) = F(y, x, z)$ *and* $g(z) = F(z, y, x)$.

Theorem 3.2.13. *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \times X \to X$ *and* $g : X \to X$ *be such that* F *has the* g*-monotone property. Assume there is a function* $\varphi : [0, \infty) \to [0, \infty)$ *with* $\varphi(t) < t$ *and* $\lim_{r \to t} \varphi(r) < t$ *for each* $t > 0$, *such that*

(3.19)
$$d(F(x, y, z), F(u, v, w)) \leq \varphi(\max\{d(g(x), g(u)); d(g(y), g(v)); d(g(z), g(w))\})$$

$$\text{for all } x, y, z, u, v, w \in X \text{ with } g(x) \leq g(u),\ g(y) \geq g(v),\ g(z) \leq g(w).$$

Suppose $F(X^3) \subseteq g(X)$, g *is continuous and commutes with* F *and also suppose either*

(a) F *is continuous or*

(b) X *has the following property: if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n, *If there exist* $x_0, y_0, z_0 \in X$ *such that*

$$g(x_0) \leq F(x_0, y_0, z_0), g(y_0) \leq F(y_0, x_0, z_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $x, y, z \in X$ *such that* $g(x) = F(x, y, z), g(y) = F(y, x, z)$ *and* $g(z) = F(z, y, x)$.

Corollary 3.2.14. *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \times X \to X$ *and* $g : X \to X$ *be such that* F *has the* g*-monotone property. Assume that there exist* $k \in [0, 1)$, *such that*

$$d(F(x, y, z), F(u, v, w)) \tag{3.20}$$

$$\leq k(max\{d(g(x), g(u)); d(g(y), g(v)); d(g(z), g(w))\})$$

$$\text{for all } x, y, z, u, v, w \in X \text{ with } g(x) \leq g(u),\ g(y) \geq g(v),\ g(z) \leq g(w).$$

Suppose $F(X^3) \subseteq g(X)$, g *is continuous and commutes with* F *and also suppose either*

(a) F *is continuous or*

(b) X *has the following property: if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n, *If there exist* $x_0, y_0, z_0 \in X$ *such that*

$$g(x_0) \leq F(x_0, y_0, z_0), g(y_0) \leq F(y_0, x_0, z_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $x, y, z \in X$ *such that* $g(x) = F(x, y, z), g(y) = F(y, x, z)$ *and* $g(z) = F(z, y, x)$.

2.3. Existence and uniqueness theorems

Theorem 3.2.15 (Borcut, [**35**]). *By adding to the hypotheses of Theorem 3.2.12 the condition: for every* $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$, *there exists a* $(u, v, w) \in X^3$ *such that* $(F(u, v, w), F(v, u, w), F(w, v, u))$ *is comparable to* $(g(x), g(y), g(z))$ *and to* $(g(x^*), g(y^*), g(z^*))$, *then* F *and* g *have a unique tripled coincidence point.*

Theorem 3.2.16. *By adding to the hypotheses of Theorem 3.2.13 the condition: for every* $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$, *there exists a* $(u, v, w) \in X^3$ *such that* $(F(u, v, w), F(v, u, w), F(w, v, u))$ *is comparable to* $(g(x), g(y), g(z))$ *and to* $(g(x^*), g(y^*), g(z^*))$, *then* F *and* g *have a unique tripled coincidence point.*

3. Examples.

In this paragraph we will give examples of mixed-monotone operators, as well as of monotone operators, which have tripled coincidence points, and also verifies the existence and uniqueness theorem hypothesis presented in paragraphs 3.1 and 3.2.

CHAPTER 4

Conclusions

As presented at the begining of this work, the role of the fixed point theory is a major one in developing the science and the technique, through the pure theoretical contributions, as well as through the applicative contributions. Without doubt, the central role of this theory is **the Banach-Caccioppoli-Picard Principle**, the starting principle of the entire fixed point theory. In the past century, the study of this theory was made mainly for operators defined on complete metric spaces.

In 2004, Ran and Reueings, in their work [**108**], apply the Banach-Caccioppoli-Picard Principle on partially ordered complete metric spaces.

Staring from the results of these works, Bhaskar and Laksmikantham, in the article [**61**] published in 2006, extend this theory to partially ordered produced metric spaces $X \times X$ and introduce the concept of **coupled fixed point** for **mixed-monotone operators** type Picard, obtaining results regarding the existence, the existence and the uniqueness of the coincidence points for **mixed-g-monotone operators** which verify the contraction condition type **Matkowski-Rus**.

Berinde in the works [**23**], [**24**], obtaines more general results by considering some more weak contraction conditions, as for example:

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)].$$

As seen, this theory was presented in Chapter I. Since 2009 and until now, were published over 80 works that deal with the coupled fixed points theory and also with the coupled coincidence points theory for operators defined on different spaces and verify different contraction types.

The extention trend of the fixed point theory continues and in 2011 is presented a new extension in the works [**31**] **"tripled fixed point theorems for contractive type mappings in partially ordered metric spaces"** and [**32**] **"tripled coincidenc theorems for contractive type mappings in partially ordered metric spaces"** the authors Berinde-Borcut, respectivelly Borcut-Berinde, introduce the concepts **tripled fixed points** and **tripled coincidence points**. As seen in Chapters II and III, the results of this theory were obtained for monotone and mixed-monotone operators defined on partially ordered metric spaces, and the contraction conditions are the following:

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq\frac{k}{3}\left[d\left(x,u\right)+d\left(y,v\right)+d\left(z,w\right)\right],\text{ cu }k\in[0,1);$$

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,w\right),$$

where $j,k,l\in[0,1)$ with $j+k+l<1$, and for any $x\geq u,y\leq v,z\geq w$;

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq jd\left(g\left(x\right),g\left(u\right)\right)+kd\left(g\left(y\right),g\left(v\right)\right)+ld\left(g\left(z\right),g\left(w\right)\right);$$

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq\varphi\left(\max\left\{d(g(x),g(u));d(g(y),g(v));d(g(z),g(w))\right\}\right)$$

for any $x,y,z,u,v,winX$ with $g\left(x\right)\leq g\left(u\right),\ g\left(y\right)\geq g\left(v\right),\ g\left(z\right)\leq g\left(w\right)$, and φ is a comparison function.

We consider that the results obtained as regards the tripled fixed points are of a great importance within the fixed points theory because, beside the direct appliance in solving integral equations(paragraph 2.3), have generated the appearence of new articles on this subject [**1**], [**109**], [**15**], [**110**], [**9**] and the article [**31**] **Berinde, V.,Borcut, M.,** ***tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*****, Nonlinear Anal., 74, (2011) 4889-4897**; has **12 quotings from L. Ciric, E. Karapinar, B. Samet, M. Abbas, H. Aydi, K.P.R. Rao** and the research in the following directions can be continued:

1. Obtaining results as regards the existence and the uniqueness of the fixed points for mixed-monotone operators (monotone) defined on partially ordered metric spaces, in which the operator verifies another contraction type, as for example: **Rakotch, Kannan, Ćirić-Reich-Rus, Ćirić, Zanfirescu, Meir-Keeler, Istrăţescu, Rus-Kasahara-Rhoades and others**. Now, we will give as example the contraction condition type **Kannan** for fixed points, for coupled fixed points and for tripled points:

Suppose the operator $F:X\to X$ and $k\in[0,\frac{1}{2})$,

$$d(F(x),F(y))\leq k\left[d(x,F(x))+d(y,F(y))\right];$$

Suppose the operator $F:X^2\to X$ and $k\in[0,\frac{1}{4})$, then

$$d(F(x,y),F(u,v))\leq k\left[d(x,F(x,y))+d(y,F(y,x))+d(u,F(u,v))+d(v,F(v,u))\right];$$

Suppose the operator $F:X^3\to X$ and $k\in[0,\frac{1}{6})$ then

$$d(F(x,y,z),F(u,v,w))\leq k\left[d(x,F(x,y,z))+d(y,F(y,x,y))+d(z,F(z,y,x))\right.$$
$$\left.+d(u,F(u,v,w))+d(v,F(v,u,v))+d(w,F(w,v,u))\right].$$

In the same direction we can study the existence, the existence and the uniqueness of the tripled points for the operators that fulfill the weaker contaction conditions, as for example:

$$d\left(F(x,y,z),F(u,v,w)\right)+d\left(F(y,x,y),F(v,u,v)\right)+d\left(F(z,y,x),F(w,v,u)\right)$$

$$\leq k\left[d(x,u)+d(y,v)+d(z,w)\right].$$

2. Obtaining results as regards the existence and the uniqueness of the tripled points for mixed-monotone (monotonr) operators defined on another type of spaces:

3. Obtaining new results as regards the extention of the quadruple fixed points, extention introduced by Karapinar and, respectivelly, Karapinar-Berinde, in the works [**80**] and respectivelly [**81**], extention based on the tripled fixed points theory. e

4. Because there are operators which are not monotone nor mixed-monotone, considering the definition 2.1.1 it is justified to consider a tripled fixed points theory for operators that fulfill a hybrid monotony property, meaning: to be crescător on the first two components and descrescător on the third component.

In the [**41**], Berzig and Samet introduce the concet of m-mixed monotone and of fixed point of N order for an operator $F: X^N \to X$, where F is monotone crescător on the first m components, and on the next $N-m$ components is monotone descrescător. A research study can be a profound thoroughgoing study of what Berzig and Samet proppposed.

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