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RESUME OF PhD THESIS

STABILITY OF FIXED POINT ITERATIVE PROCEDURES

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Contents

Introduction

The fixed point theory is a very reaching domain of nonlinear analysis, with an expansive evolution in the last decades. There are many scientific papers in the literature based on this important researching area.

The basic result from metrical fixed point theory is the Contraction Principle of Picard-Banach-Caccioppoli [11] and it followed an important research on fixed point theory and applications of this theory to functional equations, differential equations, integral equations etc.

The problem of solving a nonlinear equation involves approximating fixed points of a corresponding contractive type mapping. There exists several methods for approximating fixed points: Picard iteration which is the most used for strict contractive type operators, Krasnoselskij, Mann and Ishikawa iterations etc.

In practical applications, it is important to establish if these methods are numerically stable or not. A fixed point iteration is numerically stable if small modifications due to approximation during computations, will produce small modifications on the approximate value of the fixed point computed by means of this method.

The concept of stability is fundamental in various mathematical domains, such as Differential Equations, Difference Equations, Dynamical Systems, Numerical Analysis etc. Our interest is for stability theory in Discrete Dynamical Systems.

In this context, one of the concepts of stability that we use in the paper is the one considered by Harder [44], Harder and Hicks [45], [46], who has been systematically studied this problem. Other stability results for several fixed point iteration procedures and for various classes of nonlinear operators were obtained by Berinde [21], [22], [23], Imoru and Olatinwo [49], Osilike [63], [64], Osilike and Udomene [65], Rhoades [78], [79] and many others.

The subject of this paper treats the problem of stability of fixed point, common fixed point, coincidence point and tripled fixed point iteration procedures, for certain class of mappings. The study material has been organized on six chapters, not including an introduction and a list of bibliographic resources, as follows:

The first chapter, **Preliminaries**, provides the terminology, basic concepts and notations from fixed point theory used in this paper. Most of the material in this chapter is taken from the monography named "Iterative Approximation of Fixed Points" of Professor Berinde [**22**]. In writing of this chapter, I also used the following bibliographical references [**1**], [**6**], [**48**], [**50**], [**52**], [**85**], [**86**].

The second chapter, **Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition**, presents the concept of stability of fixed point iteration procedures and surveys the most significant contributions to this area.

One of them was made by Berinde [**22**] who introduced a weaker and more natural notion of stability, called *weak stability*, by adopting approximate sequences instead of arbitrary sequences in the definition of stability. Following this concept, we continued to study the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings.

The author's original contributions in this chapter are: Definition 5.9, Theorem 4.4, Theorem 4.5, Examples 6.2-6.4, Example 6.5, Example 6.6, Definition 7.11 and Theorem 7.7.

Most of them were published in [**94**] (Timiş, I., *On the weak stability of fixed point iterative methods*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010), [**95**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. 37 (2) (2010), 106-114), [**96**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems*, International Journal of Computer Applications 37 (4) (2012), 9-13) and [**105**] (Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95).

In the third chapter, **Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined by implicit relations**, we study the stability of Picard iterative procedure and

also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation. This development has been initiated by Popa [**70**], [**71**], [**72**] and following this approach, a consistent part of the literature on fixed point, common fixed point and coincidence theorems, both for single valued and multi-valued mappings, in various ambient spaces have been accomplished.

For these new fixed point theorems did not exist corresponding stability results and Berinde [**14**], [**24**] filled this gap and established corresponding stability results for fixed point iterative procedures associated to contractive mappings defined by an implicit relation.

We continue the study of stability and the results obtained in this chapter are generalizations of fixed point theorems and stability theorems for Picard iteration existing in literature: see Berinde [**15**], [**19**], [**21**] [**22**], [**23**], [**25**], Chatterjea [**35**], Harder and Hicks [**45**], [**46**], Hardy and Rogers [**47**], Imoru and Olatinwo [**49**], Jungck [**51**], Kannan [**53**], Olatinwo [**59**], Osilike [**64**], [**63**], Ostrowski [**66**], Popa [**71**], Reich [**74**], Reich and Rus [**90**], Rhoades [**77**], [**78**], [**79**], Rus [**82**], [**83**], Zamfirescu [**106**] and most of their references.

The author's original contributions in this chapter are: Example 1.8, Theorem 1.9, Corollary 1.1, Corollary 1.2, Theorem 2.10, Examples 3.12-3.13, Example 3.15, Theorem 3.11, Corollary 3.3 and Corollary 3.4.

Most of them were published in [**97**] (Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567), [**99**] (Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011) and [**100**] (Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44).

The idea of the fourth chapter, **A new point of view on the stability of fixed point iterative procedures**, is due to Professor I. A. Rus [**81**], who unified the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis by new ones.

We consider these new notions in this chapter and study the stability of Picard iteration for mappings which satisfy certain contractive conditions. We also give some illustrative examples.

The author's original contributions in this chapter are: Theorem 1.12, Proposition 1.1, Corollary 1.5, Corollary 1.6, Corollary 1.7, Example 1.19, Corollary 1.8, Theorem 2.13, Corollary 2.9, Example 2.20, Theorem 2.14, Corollary 2.10, Examples 3.21 - 3.28, Definition 4.15, Definition 4.16, Proposition 4.2, Theorem 4.15, Theorem 4.16 and Theorem 5.17.

Some of them are included in [**92**] (Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012).

In the fifth chapter, **Stability of tripled fixed point iteration procedures**, following the results of Berinde and Borcut [**26**], [**32**] who introduced the concept of tripled fixed points, we introduce the notion of stability for tripled fixed point iterative procedures and also establish stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions. An illustrative example is also given.

The author's original contributions in this chapter are: Definition 2.19, Theorem 2.18, Corollary 2.11, Theorem 2.19, Theorem 2.20, Lemma 3.3, Definition 3.21, Theorem 3.21, Corollary 3.12, Theorem 3.22, Theorem 3.23, Example 4.29 and the contractive conditions (2.20)-(2.25), (3.28)-(3.33).

Most of them were published in [**102**] (Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7).

In the sixth chapter, **Conclusions**, we surveyed the original contributions from this thesis and we mentioned the possible research directions by following our results.

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I reserve for the end the most beautiful gratitude, to my husband Ilie, for his loyalty, for his support and for his love, with which he always surrounds me.

Preliminaries

The purpose of this chapter is to provide the terminology, basic concepts and notations from fixed point theory used in this paper.

Most of the material in this chapter is taken from the monography named "Iterative Approximation of Fixed Points" of Professor Berinde [**22**].

In writing of this chapter, I also used the following bibliographical references [**1**], [**6**], [**48**], [**50**], [**52**], [**85**], [**86**].

1. The background of metrical fixed point theory

Let X be a nonempty set and $T : X \to X$ be a selfmap. We say that $x \in X$ is a fixed point of T, if

$$T(x) = x,$$

and denote by F_T or $Fix(T)$ the set of all fixed points of T.

For any given $x \in X$, we define $T^n(x)$ inductively by

$$T^0(x) = x, \quad T^{n+1}(x) = T\left(T^n(x)\right),$$

and we call it the nth *iterate of x under T*. In order to simplify the notations, we will often use Tx instead of $T(x)$.

For any $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots$$

is called *the sequence of successive approximations with the initial value x_0*. It is also known as the *Picard iteration* starting at x_0.

For a given selfmap, the following properties obviously hold:

(1) $F_T \subset F_{T^n}$, for each $n \in \mathbb{N}^*$;
(2) $F_{T^n} = \{x\}$, for some $n \in \mathbb{N}^* \Rightarrow F_T = \{x\}$.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

(1) the existence and uniqueness of fixed points;

(2) the data dependence of fixed points;

(3) the construction of fixed points.

The ambient spaces X involved in fixed point theory cover a variety of spaces: lattice, metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions.

The following theorem is the classical method of successive approximations and is of fundamental importance in the metrical fixed point theory. It is called *contraction mapping theorem* or *Banach's theorem* or *theorem of Picard-Banach* or *theorem of Picard-Banach-Caccioppoli.*

Theorem 1.1. *(Contraction mapping principle) Let (X, d) be a complete metric space and $T : X \to X$ be a given contraction. Then T has an unique fixed point p, and*

$$T^n(x) \to p \ (\ as \ \ n \to \infty \ ), \ \ \forall x \in X.$$

There are various generalizations of the contraction mapping principle, roughly obtained in two ways:

(1) by weakening the contractive properties of the map and, possibly, by simultaneously giving to the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions;

(2) by extending the structure of the ambient space.

Several fixed point theorems have been also obtained by combining the two ways previously described or by adding supplementary conditions.

In order to prove several convergence theorems, we shall use various elementary results concerning recurrent inequalities, as the following lemmas:

Lemma 1.1. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and a constant h, $0 \le h < 1$, so that*

$$a_{n+1} \le ha_n + b_n, \ \ n \ge 0.$$

- *If $\lim_{n \to \infty} b_n = 0$, then $\lim_{n \to \infty} a_n = 0$.*

- *If $\sum_{n=0}^{\infty} b_n < \infty$, then $\sum_{n=0}^{\infty} a_n < \infty$.*

Lemma 1.2. *Let $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers. Then,*

$$\lim_{n \to \infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n \to \infty} \sum_{i=0}^{n} k^{n-i} \epsilon_i = 0, \quad k \in [0, 1) \,.$$

It appears that Jungck was the first who considered commutativity in connexion with common fixed point results, see [**51**].

Definition 1.1. *Let (X, d) be a metric space and S, $T : X \to X$ be two mappings. We say that S and T are commuting if*

$$STx = TSx, \quad \forall \, x \in X.$$

As a generalization of this notion, Sessa [**85**] defined S and T to be *weakly commuting* if

$$d(STx, TSx) \le d(Sx, Tx), \quad \forall \, x \in X.$$

There are several other concepts that weaken the notion of commuting mappings that were used for establishing common fixed point theorems. Here, we need the following concept, defined by Jungck [**52**].

Definition 1.2. *Let (X, d) be a metric space and S, $T : X \to X$ be two mappings. We say that S and T are compatible, as a generalization of weakly commuting, if*

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t, \quad t \in X.$$

Jungck [**52**] also showed that commuting implies weakly commuting which, in turn, implies compatibility property but the converse property is not true in general.

Definition 1.3. *A point $x \in X$ is called a coincidence point of a pair of self-maps S, T, if there exists a point $u \in X$, usually called a point of coincidence in X, such that $u = Sx = Tx$.*

Moreover, Jungck [**50**] defined S and T to be *weaky compatible* if they commute at their coincidence points, i.e., if

$$Sz = Tz \quad \Rightarrow \quad STz = TSz, \quad z \in X.$$

Jungck [**52**] established the inclusions between these notions, respectively that the *commuting property* implies *weakly commuting property* which, in turn, implies *compatibility property* that implies *weakly compatibility property* but the reverse is not generally true.

Secondly, Aamri and Moutawakil [**1**] introduced a notion which is independent of the notion of weakly compatibility.

Definition 1.4. *S and T mappings satisfy (E.A) property if there exists a sequence* $\{x_n\}_{n=0}^{\infty} \in X$ *such that*

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t, \quad for \ some \ \ t \in X.$$

2. Fixed point iteration procedures

Let (X, d) be a metric space, $D \subset X$ a closed subset of X (we often have $D = X$) and $T : D \to D$ a selfmap possessing at least one fixed point $p \in F_T$. For a given $x_0 \in X$ we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$(2.1) \qquad\qquad x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, ...$$

As we already mentioned, the sequence defined by (2.1) is known as the *sequence of successive approximations* or, simply, *Picard iteration.*

Picard iteration appears to have been introduced by Liouville [**56**] and used by Cauchy. It was developed systematically for the first time by Picard [**69**] in his classical and well-known proof of the existence and uniqueness of the solution of initial value problems for ordinary differential equations, dating back in 1890.

When the contractive conditions imposed on the map T are slightly weaker, then the Picard iteration need not converge to a fixed point of the operator T and some other iteration procedures must be considered.

CHAPTER 2

Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition

This chapter presents the concept of stability of fixed point iteration procedures and surveys the most significant contributions in this area.

The concept of stability is fundamental in various mathematical domains, such as Differential Equations, Difference Equations, Dynamical Systems, Numerical Analysis etc. Our interest is for stability theory in Discrete Dynamical Systems.

In this context, one of the concepts of stability that we use in the paper is the one considered by Harder [**44**], Harder and Hicks [**45**], [**46**], who has been systematically studied this problem.

The stability of Picard iterative procedure for a fixed point equation was first studied by Ostrowski [**66**] on metric spaces. This subject was formally developed by several authors.

One of the extensions was made by Berinde [**22**] who introduced a weaker and more natural notion of stability, called *weak stability*, by adopting approximate sequences instead of arbitrary sequences in the definition of stability. Following this concept, we continued to study the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings.

The author's original contributions in this chapter are: Definition 5.9, Theorem 4.4, Theorem 4.5, Examples 6.2-6.4, Example 6.5, Example 6.6, Definition 7.11 and Theorem 7.7.

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(2012), 9-13) and [**105**] (Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95).

1. Stability of fixed point iteration procedures

Intuitively, a fixed point iteration procedure is numerically stable if, "small" modifications in the initial data or in the data that are involved in the computation process will produce a "small" influence on the computed value of the fixed point.

Let (X, d) be a metric space and we define a fixed point iteration procedure by a general relation of the form

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, ...,$$

and considering that $f(T, x_n)$ does contain all parameters that define the fixed point iteration procedure, where $T : X \to X$ is an operator and $x_0 \in X$, with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ a sequence generated by a fixed point iteration procedure that ensure its convergence to a fixed point p of T.

In practical applications, when calculating $\{x_n\}_{n=0}^{\infty}$, we usually follow the steps:

(1) We choose the initial approximation $x_0 \in X$;
(2) We compute $x_1 = f(T, x_0)$ but, due to various errors that occur during the computations (rounding errors, numerical approximations of functions, derivatives or integrals etc.), we do not get the exact value of x_1, but a different one, say y_1, which is however close enough to x_1, i.e., $y_1 \approx x_1$.
(3) Consequently, when computing $x_2 = f(T, x_1)$, we will actually compute x_2 as $x_2 = f(T, y_1)$ and so, instead of the theoretical value x_2, we will obtain in fact another value, say y_2, again close enough but generally different of x_2, i.e., $y_2 \approx x_2$, ..., and so on.

In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$, defined by the given iterative method, we will practically obtain an approximate sequence $\{y_n\}_{n=0}^{\infty}$. We shall consider the given fixed point iteration method to be numerically **stable** if and only if, for y_n close enough (in some sense) to x_n at each stage, the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to a fixed point of T.

Following basically this idea, the next concept of stability was introduced.

Definition 1.5. [**44**] *Let (X, d) be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let assume that the sequence generated by the iteration procedure*

(1.2) $$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, ...,$$

converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, ...$$

We shall say that the fixed point iteration procedure (1.2) is T-stable or stable with respect to T if and only if

$$\lim_{n \to \infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n \to \infty} y_n = p.$$

2. Stability of common fixed point iterative procedures

The concept of stability of common fixed point iterative procedures for a pair of mappings (S, T) with a coincidence fixed point was introduced by Singh, Bhatnagar and Mishra [**87**].

Let X be an arbitrary nonempty set and (X, d) a metric space.

Let $S, T : X \to X$ be two mappings, such that $T(X) \subseteq S(X)$. For any $x_0 \in X$, consider the common fixed point iteration procedure

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, ...,$$

which is the iterative procedure introduced by Jungck [**51**].

The common fixed point iteration procedure becomes the Picard iterative procedure when $S = I$, the identity map on X.

Jungck [**51**] showed that the mappings S and T satisfying

(2.3) $$d(Tx, Ty) \leq kd(Sx, Sy), \quad 0 \leq k < 1, \quad \forall x, y \in X,$$

have a common fixed point in X, provided that S and T are commuting, $T(X) \subseteq S(X)$ and S is continuous.

The following significantly improved version of this result is generally called the Jungck contraction principle, obtained by Singh and Prasad [**88**].

Theorem 2.2. [**88**] *Let (X, d) be a metric space and let $S, T : X \to X$ satisfying (2.3). If $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a coincidence point.*

For any $x_0 \in X$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, and assume that $\{Sx_n\}_{n=0}^{\infty}$ converges to Sz for some z in X and $Sz = Tz = u$, respectively the point of coincidence of S and T.

If S and T commute just at z, then S and T have an unique common fixed point.

Definition 2.6. [**88**] *Let (X, d) be a metric space and let $S, T : X \to X$. Let z to be a coincidence point of T and S, that is, $Sz = Tz = u$.*

For any $x_0 \in X$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the general iterative procedure

$$(2.4) \qquad\qquad Sx_{n+1} = Tx_n, \quad n = 1, 2, ...,$$

and suppose that it converges to $u \in X$. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\epsilon_n = d(Sy_{n+1}, Ty_n)$, $n = 0, 1, 2, ...$.

Then the iterative procedure 2.4 is (S, T)-stable or stable with respect to (S, T) if and only if

$$\lim_{n \to \infty} \epsilon_n = 0 \quad \implies \quad \lim_{n \to \infty} Sy_n = u.$$

Some authors name (2.4) to be the Jungck common fixed point iteration procedure.

Definition 2.6 reduces to that of the stability of the fixed point iterative procedure due to Harder and Hicks [**45**], [**46**] when $S = I$, the identity map on X.

For several examples discussing the practical aspect and theoretical importance of the stability when S is the identity map on X in the above definition, see Berinde [**22**].

3. Several studies about stability

As we mentioned in section 1, the first stability result for fixed point iteration procedures has been obtained by Ostrowski[**66**].

Harder [**44**] introduced the concept of stability for general fixed point iteration procedures and made a systematical study by obtaining stability results that

extend Ostrowski's theorem to mappings satisfying more general contractive conditions for various fixed point iteration procedures.

Harder and Hicks [**46**] showed that the function iteration, for mappings T satisfying various contractive definitions is T-stable, as well as for several iteration schemes other that function iteration. Rhoades [**78**] extended some of the results of Harder and Hicks [**46**] to an independent contractive definition and also proved stability theorems for additional iteration procedures.

Moreover, Rhoades [**79**] continued the study of stability results by using a more general contractive definition than the ones studied by Harder and Hicks [**46**]: for $(E, \|\cdot\|)$ a normed linear space, T a selfmap of E, there exists a constant C, $0 \leq C < 1$ such that for each $x, y \in E$,

(3.5) $$\|Tx - Ty\| \leq CM(x, y),$$

where

$$M(x, y) := \max \left\{ \|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \right.$$
$$\left. \|x - Ty\|, \|y - Tx\| \right\},$$

and then proved several stability results which are generalizations and extensions of most of the results of Harder and Hicks [**46**] and Rhoades [**78**]. Osilike [**64**] continued the study of stability results of iteration procedures for mappings satisfying (3.5).

4. Stability results for common fixed point iteration procedures using certain classes of contractive nonself mappings

Let (X, d) be a metric space, $Y \subset X$ and $S, T : Y \to X$ two nonself mappings, satisfying the following contraction condition: $\exists q \in (0, 1)$ such that

(4.6) $$d(Tx, Ty) \leq qd(Sx, Sy), \quad \forall \ x, y \in Y.$$

Goebel [**43**] proved that S and T have a coincidence point in X (see Buică [**33**]) and Jungck [**50**] showed that the maps S and T satisfying (4.6) have an unique common fixed point in a complete space (X, d), provided that

(1) $T(X) \subseteq S(X)$;
(2) S is continuous;
(3) S and T commute.

The next theorem is an improved version of the Jungck's contraction principle [50], which has been obtained by Singh and Prasad [88].

Theorem 4.3. [88] *Let (X, d) be a metric space, Y a subset of X and let $S, T : Y \to X$ be two mappings satisfying (4.6).*

If $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X, then S and T have a coincidence point (that is, there exists $z \in Y$, such that $Sz = Tz$).

Moreover, for any $x_0 \in Y$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in Y, such that

(1) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$,

(2) $\{Sx_n\}_{n=0}^{\infty}$ converges to Sz for some coincidence point z in Y.

Further, if $Y = X$ and S and T commute (just) at z, then S and T have an unique common fixed point, that is, $Sz = Tz = z$.

Starting from the stability results of Singh and Prasad [88], we study the problem of stability of common fixed point iterative procedures for some classes of contractive type mappings.

As we have seen previously, the definition of (S, T)-stable iterative procedures used in [88] is based on the choice of an *arbitrary* sequence $\{Sy_n\}_{n=0}^{\infty}$. But, as shown in the paper [22], it is not natural to consider an arbitrary sequence in Definition 2.6, because in this way, we do not treat the problem of stability in its general context.

Our main result in this respect is given by the next theorem, which completes Theorem 4.3 by the result regarding the (S, T)-stability of the Jungck type iteration procedure.

Theorem 4.4. *(**Timiş**, [105]) Let (X, d) be a metric space, Y a subset of X and let $S, T : Y \to X$ be two mappings satisfying*

(4.7) $$d(Tx, Ty) \leq q d(Sx, Sy), \ \forall x, y \in Y, \ q \in [0, 1).$$

If $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X, then S and T have an unique coincidence point (that is, there exists $z \in Y$, such that $Sz = Tz = u$.).

Moreover, for any $x_0 \in Y$, there exists a sequence $\{Sx_n\}_{n=0}^{\infty} \in Y$ such that

(i) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$,

(ii) $\{Sx_n\}_{n=0}^{\infty}$ converges to u.

Let $\{Sy_n\}_{n=0}^{\infty} \subset Y$ be an approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$ and define

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \ n = 0, 1, 2, ...$$

Then,

(1) $d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + q^{n+1}d(Sx_0, Sy_0) + \sum_{r=0}^{n} q^{n-r}\epsilon_r$;

(2) $\lim_{n \to \infty} Sy_n = u$, if and only if $\lim_{n \to \infty} \epsilon_n = 0$, that is, the iterative procedure is (S, T)-stable.

Remark 4.1. *Particular cases of Theorem 4.4.*

(1) *If $Y = X$, then by Theorem 4.4, we obtain an improved result of stability for the Jungck's contraction principle, see Singh and Prasad [88].*

(2) *If f $Y = X$ and $S = I$ (the identity map on X), then by Theorem 4.4, then we obtain an improved result of stability for Banach's contraction mapping principle, see Ostrowski [66] and Harder and Hicks [46].*

Theorem 4.5. *($\mathbf{Timiş}$, [105]) Let (X, d) be a metric space, Y a subset of X and let $S, T : Y \to X$ be two mappings satisfying*

$$(4.8) \qquad d(Tx, Ty) \leq qd(Sx, Sy) + Ld(Sx, Tx), \ \forall x, y \in Y, \ q \in (0, 1), \ L \geq 0.$$

If $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X, then S and T have an unique coincidence point (that is, there exists $z \in Y$, such that $Tz = Sz = u$).

Moreover, for any $x_0 \in Y$, there exists a sequence $\{Sx_n\}_{n=0}^{\infty} \in Y$ such that

(i) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$,

(ii) $\{Sx_n\}_{n=0}^{\infty}$ converges to u.

Let $\{Sy_n\}_{n=0}^{\infty} \subset Y$ be an approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$ and define

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \ n = 0, 1, 2, ...$$

Then,

(1) $d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + q^{n+1}d(Sx_0, Sy_0) + L\sum_{r=0}^{n} q^{n-r}d(Sx_r, Tx_r) + \sum_{r=0}^{n} q^{n-r}\epsilon_r$;

(2) $\lim_{n \to \infty} Sy_n = u$ *if and only if* $\lim_{n \to \infty} \epsilon_n = 0$.

Remark 4.2. *Particular cases of Theorem 4.5.*

(1) *If $Y = X$, then by Theorem 4.5, we obtain an improved result of stability for the Jungck's contraction principle, see Singh and Prasad [88].*

(2) *If $Y = X$ and $S = I$ (the identity map on X), then by Theorem 4.5, we obtain an improved result of stability for Banach's contraction mapping principle, see Ostrowski [66].*

(3) *If $Y = X$ and $S = I$ (the identity map on X), then by Theorem 4.5, we obtain an improved result of stability for the Kannan's fixed point theorem [53], see Harder and Hicks [46].*

(4) If $Y = X$ and $S = I$ (the identity map on X), then by Theorem 4.5, we obtain an improved result of stability for the Zamfirescu's fixed point theorem, that is, Theorem 2 from Harder and Hicks [46].

(5) If $Y = X$ and $S = I$ (the identity map on X), then by Theorem 4.5, we obtain an improved result of stability for the Chatterjea's fixed point theorem [35].

5. Weak stability concept of fixed point iteration procedures and common fixed point iteration procedures

In this section, we review some existing results on the weak stability of fixed point iteration procedures and we transpose this concept to a pair of mappings with a coincidence point.

The concept of (almost) stability is slightly not very precise because of the sequence $\{y_n\}_{n=0}^{\infty}$ which is *arbitrary* taken. From a numerical point of view, $\{y_n\}_{n=0}^{\infty}$ must be an *approximate* sequence of $\{x_n\}_{n=0}^{\infty}$.

By adopting a concept of such kind of approximate sequences, Berinde [22] introduced a weaker and more natural concept of stability, called *weak stability*. So, any stable iteration will be also weakly stable but the reverse is not generally true.

Definition 5.7. [22] *Let (X, d) be a metric space and $\{x_n\}_{x=0}^{\infty} \subset X$ be a given sequence. We shall say that $\{y_n\}_{n=0}^{\infty} \in X$ is an approximate sequence of $\{x_n\}_{n=0}^{\infty}$ if, for any $k \in \mathbb{N}$, there exists $\eta = \eta(k)$ such that*

$$d(x_n, y_n) \leq \eta, \quad for \ all \ n \geq k.$$

Remark 5.3. *We can have approximate sequences of both convergent and divergent sequences.*

Definition 5.8. [22] *Let (X, d) be a metric space and $T : X \to X$ be a map. Let $\{x_n\}_{n=0}^{\infty}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0.$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. If for any approximate sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of $\{x_n\}_{n=0}^{\infty}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0 \quad implies \quad \lim_{n\to\infty} y_n = p,$$

then we shall say that the iteration procedure is weakly T-stable or weakly stable with respect to T.

All examples given by various authors that have studied the stability of the fixed point iteration procedures - examples intended to illustrate non stable fixed point iteration procedures - do not consider approximate sequences of $\{x_n\}_{n=0}^{\infty}$.

Berinde [**22**] presented in detail some of the aforementioned examples, in order to show how important and natural is to restrict the stability concept to approximate sequences $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$.

Example 5.1. [**22**]

Let $T : \mathbb{R} \to \mathbb{R}$ be given by $Tx = \frac{1}{2}x$, where $\mathbb{R}$ is endowed with the usual metric. T is an $\frac{1}{2}$-contraction, $F_T = \{0\}$.

The Ishikawa iteration $\{x_n\}_{n=1}^{\infty}$ is T-stable, hence it is almost T-stable and weakly T-stable, too.

However, Osilike [**62**] claimed that the Ishikawa iteration is not T-stable. To show this, it was used the sequence $\{y_n\}_{n=1}^{\infty}$ given by

$$y_n = \frac{n}{1+n}, \ n \geq 0.$$

But this is obviously nonsense, because $x_n \to 0$, as $n \to \infty$, the unique fixed point of T, while $y_n \to 1$, as $n \to \infty$, so, by construction, $\{y_n\}_{n=1}^{\infty}$ would have to be an approximate sequence of $\{x_n\}$.

Hence, using arbitrary sequences, the Ishikawa iteration is not T-stable.

In the following, we transpose the concept of (S, T)-stability used by Singh and Prasad [**88**] to (S, T)-weak stability in a metric space.

Definition 5.9. (*Timiş*, [**105**]) *Let (X, d) be a metric space and two mappings $S, T : X \to X$ be such as $T(X) \subseteq S(X)$ and let z is a coincidence point of S and T, that is, a point for which we have $Sz = Tz = u \in X$.*

For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the general iterative procedure

$$(5.9) \qquad\qquad Sx_{n+1} = f(T, x_n), \quad n = 1, 2, ...,$$

and assume that it converges to u.

If for any approximate sequence $\{Sy_n\}_{n=0}^{\infty} \subset X$ of $\{Sx_n\}_{n=0}^{\infty}$, we have that

$$\lim_{n\to\infty} d(Sy_{n+1}, f(T, y_n)) = 0 \quad implies \quad \lim_{n\to\infty} Sy_n = u,$$

then we shall say that (5.9) is weakly (S,T)-stable or weakly stable with respect to (S,T).

6. Examples of weak stable but not stable iterations

Harder and Hicks [**46**] presented some examples of mappings which satisfy various contractive conditions for which the corresponding iteration procedures are not stable.

In the following, we present some of these examples in order to study their weak stability.

We also present some examples of mappings with a coincidence point which satisfy certain contractive conditions in order to study their stability with respect to (S,T).

Example 6.2. *(Timiş, [**95**])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. T is continuous at each point of $[0,1]$ except at $\frac{1}{2}$ and T has an unique fixed point at $\frac{1}{2}$, see Harder and Hicks [**46**].

As shown in [**46**], T satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}, \quad \forall\, x, y \in X, \quad x \neq y.$$

Forward, concerning the stability, we obtained that the Picard iteration is not T-stable nor T-weakly stable.

Example 6.3. *(Timiş, [**95**])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. T is continuous at every point of $[0,1]$ except at $\frac{1}{2}$ and 0 is the only fixed point of T, see Harder and Hicks [**46**].

For each $x, y \in [0,1]$, $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

Forward, about the stability, we obtained that the Picard iteration is not T-stable but it is T-weakly stable.

Example 6.4. *(Timiş, [**95**])*

Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0 \\ \frac{1}{4}, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x > \frac{1}{2} \end{cases},$$

where $\mathbb{R}$ is endowed with the usual metric. T is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$. The only fixed point of T is $\frac{1}{4}$, see Harder and Hicks [**46**].

For each $x, y \in \mathbb{R}$, $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Forward, about the stability, we obtained that the Picard iteration is not T-stable but it is T-weakly stable.

Example 6.5. *(Timiş, [**96**])*

Let $S, T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$Sx = \begin{cases} \frac{1}{2} - x, & x \in \left[0, \frac{1}{2}\right] \\ x - \frac{1}{4}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. S and T are continuous at every point of $[0,1]$ except at $\frac{1}{2}$ which is their coincidence point, i.e., $T\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = 0 = u$ and $T([0,1]) = \left\{0, \frac{1}{2}\right\} \subseteq S([0,1]) = \left[0, \frac{1}{2}\right] \cup \left(\frac{1}{4}, \frac{3}{4}\right] = \left[0, \frac{3}{4}\right]$.

For each $x, y \in [0, 1]$, $x \neq y$, T and S satisfy the condition

$$d\left(Tx, Ty\right) < \max\left\{d\left(Sx, Ty\right), d\left(Sy, Tx\right)\right\}.$$

We obtained that the Picard iteration is not (S, T)-stable nor (S, T)-weakly stable.

Example 6.6. *(Timiş, [**96**])*

Let S, $T : [0, 1] \rightarrow [0, 1]$ be given by

$$Tx = \begin{cases} \frac{x+1}{2}, & x \in \left[0, \frac{1}{2}\right] \\[2mm] \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$Sx = \begin{cases} \frac{1}{2} - x, & x \in \left[0, \frac{1}{2}\right] \\[2mm] x - \frac{1}{4}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0, 1]$ is endowed with the usual metric. S and T have a two coincidence points, i.e., $T\left(0\right) = S\left(0\right) = T\left(\frac{3}{4}\right) = S\left(\frac{3}{4}\right) = \frac{1}{2} = u$ and $T\left([0, 1]\right) = \left[\frac{1}{2}, \frac{\frac{1}{2}+1}{2}\right] \cup \left\{\frac{1}{2}\right\} = \left[\frac{1}{2}, \frac{3}{4}\right] \subseteq S\left([0, 1]\right) = \left[0, \frac{1}{2}\right] \cup \left(\frac{1}{4}, \frac{3}{4}\right] = \left[0, \frac{3}{4}\right]$.

For each $x, y \in [0, 1]$, $x \neq y$, T and S satisfy the condition

$$d\left(Tx, Ty\right) < \max\left\{d\left(Sx, Ty\right), d\left(Sy, Tx\right)\right\}.$$

We obtained that the Picard iteration is not (S, T)-stable nor (S, T)-weakly stable.

7. Stability and weak stability of fixed point iterative procedures for multivalued mappings

By extending the contraction mapping principle form single-valued mappings to multivalued mappings, Nadler [**57**] proved that a multivalued contraction on a complete metric space has a fixed point. Ciric [**39**] extended this result for generalized multivalued contractions on metric spaces.

The concept of weak contraction from the case of single-valued mappings was extended to multi-valued mappings and then corresponding convergence theorems for the Picard iteration associated to a multi-valued weak contraction are obtained. M. Berinde and V. Berinde [**12**] extended, improved and unified a multitude of

classical results in the fixed point theory of single and multi-valued contractive mappings.

On the other hand, Singh and Chadha [**89**] extended Ostrowski's stability theorem (Theorem 7.6, in this paper) to multivalued contractions using Nadler's theorem and introduced the following definition of stability of iterative procedures for multivalued maps.

Definition 7.10. [**89**] *Let X be a metric space and $T : X \to P_{b,cl}(X)$. Let $x_0 \in X$ and $x_{n+1} \in Tx_n$ denote the Picard iterative procedure for T.*

Let $\{x_n\}_{n=0}^{\infty}$ be convergent to a fixed point u of T and $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence.

Set $\epsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots.$

The iterative procedure Tx_n is said to be T-stable provided that

$$\lim_{n \to \infty} \epsilon_n = 0 \quad implies \quad \lim_{n \to \infty} y_n = u.$$

The first result on the stability of Picard iterative procedure for multivalued mappings is due to Singh and Chadha [**89**] and it is stated as follows.

Theorem 7.6. [**89**] *Let X be a complete metric space and $T : X \to P_{b,cl}(X)$.*

Suppose there exists a positive number $q < 1$ such that T satisfies the condition

$$H_d(Tx, Ty) \leq qd(x, y), \quad \forall x, y \in X.$$

Let x_0 be an arbitrary point in X and assume that $\{x_n\}_{n=0}^{\infty}$ is a sequence which converges to a fixed point u of T.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X and set $\epsilon_n = H_d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots.$

If Tu is singleton then $\lim_{n \to \infty} y_n = u$ if and only if $\lim_{n \to \infty} \epsilon_n = 0$.

Afterwards, Czerwik, Dlutek and Singh [**41**] studied the stability of Picard iterative procedures for multivalued maps in b-metric spaces. Furthermore, Singh, Bhatnagar and Mishra [**86**] obtained a fixed point theorem for generalized multivalued contractions in b-metric spaces and further studied the stability of Picard iterative procedures for such maps.

In the following, we give a stability result for multivalued mappings satisfying an almost contraction condition.

Theorem 7.7. (***Timiş***, [**104**]) *Let (X, d) a complete metric space and $T : X \to P_{b,cl}(X)$ a mapping with $SFix(T) \neq \phi$, satisfying*

$$H_d(Tx, Ty) \leq q \cdot d(x, y) + L \cdot D(x, Tx),$$

for all $x, y \in X$, $q \in [0, 1)$ and $L \geq 0$.

Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} \in Tx_n$, for all $n \geq 0$ and assume that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^*, the unique strict fixed point of T.

Then, the Picard iteration is T-stable.

Remark 7.4. *Theorem 7.7 extends Theorem 7.6 of Singh and Chadha [89]. If we take $L = 0$ in Theorem 7.7, we get the stability result from Theorem 7.6.*

As argued in Section 2 of this Chapter, from a numerical point of view, the concept of weak stability is more natural than the one of usual stability considered in [41], [86], [89] etc., because of the *arbitrary* sequence taken. So, any stable iteration will be also weakly stable but the reverse is not generally true.

In the sequel, we give the transposition to multivalued mapping of Definition 5.8 of the weak stability with respect to T.

Definition 7.11. *(**Timiş**, [104]) Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$ be a multivalued mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0.$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a strict fixed point p of T. If for any approximate sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of $\{x_n\}_{n=0}^{\infty}$,

$$\lim_{n \to \infty} H_d(y_{n+1}, f(T, y_n)) = 0 \quad implies \quad \lim_{n \to \infty} y_n = p,$$

then we shall say that the iteration procedure $\{x_n\}_{n=0}^{\infty}$ is weakly T-stable or weakly stable with respect to T.

CHAPTER 3

Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined by implicit relations

Several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation. This development has been initiated by Popa [**70**], [**71**], [**72**] and following this approach, a consistent part of the literature on fixed point, common fixed point and coincidence theorems, both for single valued and multi-valued mappings, in various ambient spaces have been accomplished.

For these new fixed point theorems did not exist corresponding stability results and Berinde [**14**], [**24**] filled this gap and established corresponding stability results for fixed point iterative procedures associated to contractive mappings defined by an implicit relation.

We continue to study the stability of Picard iterative procedure and also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Since a metrical common fixed point theorem generally involves conditions of commutativity, a lot of researches in this domain are aimed at weakening these conditions. The evolution of weak commutativity of Sessa [**85**] and compatibility of Jungck [**52**] developed weak conditions in order to improve common fixed points theorems. We also give a general stability result for the common fixed point iteration procedure of Jungck-type in the class of weakly compatible mappings defined by means of an implicit contraction condition.

The results obtained in this chapter are generalizations of fixed point theorems and stability theorems for Picard iteration existing in literature: see Berinde [**15**], [**19**], [**22**], [**23**], [**25**], Chatterjea [**35**], Harder and Hicks [**45**], [**46**], Hardy and Rogers [**47**], Imoru and Olatinwo [**49**], Jungck [**51**], Kannan [**53**], Olatinwo [**59**], Osilike [**64**], [**63**], Ostrowski [**66**], Popa [**71**], Reich [**74**], Reich and Rus [**90**],

Rhoades [**77**], [**78**], [**79**], Rus [**82**], [**83**], Zamfirescu [**106**] and most of their references.

The author's original contributions in this chapter are: Example 1.8, Theorem 1.9, Corollary 1.1, Corollary 1.2, Theorem 2.10, Examples 3.12-3.13, Example 3.15, Theorem 3.11, Corollary 3.3 and Corollary 3.4.

Most of them were published in [**97**] (Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567), [**99**] (Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011) and [**100**] (Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44).

1. Stability of fixed point iterative procedure for contractive mappings satisfying implicit relations

Berinde [**24**] gave a general stability result of the Picard iteration for mappings satisfying an implicit relation with six parameters, using the set of all continuous real functions $F : \mathbb{R}_+^6 \to R_+$ introduced by Popa [**71**], [**72**], with the following conditions:

(F_{1a}) F is non-increasing in the fifth variable and $F(u, v, v, u, u + v, 0) \le 0$ for $u, v \ge 0 \implies \exists h \in [0, 1)$ such that $u \ge hv$;

(F_{1b}) F is non-increasing in the fourth variable and $F(u, v, 0, u + v, u, v) \le 0$ for $u, v \ge 0 \implies \exists h \in [0, 1)$ such that $u \ge hv$;

(F_{1c}) F is non-increasing in the third variable and $F(u, v, u + v, 0, v, u) \le 0$ for $u, v \ge 0 \implies \exists h \in [0, 1)$ such that $u \ge hv$;

(F_2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Theorem 1.8. [**24**] *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$*

$$F\left(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) \le 0.$$

If F satisfies (F_{1a}) and (F_2), then T has an unique fixed point.

If, additionally, F satisfies (F_{1b}), then Picard iteration is: a) T-stable; b) summable almost T-stable.

In the following, we study the stability of the Picard iterative procedure for mappings satisfying an implicit relation but we reduce the number of parameters to five.

Popa [**70**] introduced $\mathbb{F}$ to be the set of all continuous real functions $F : \mathbb{R}_+^5 \to \mathbb{R}$ with the following conditions:

(1) F is continuous in each coordinate variable,
(2) there exists $h \in [0, 1)$ such that, for all $u, v, w \geq 0$ satisfying
- (2a) $F(u, v, u, v, w) \leq 0$ or
- (2b) $F(u, v, v, u, w) \leq 0$,

we have that $u \leq h \max\{v, w\}$.

In the following, there are some examples of functions that satisfy some of the above conditions:

Example 1.7. [**22**] *Define* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *as*

(1) $F(t_1, ..., t_5) = t_1 - at_2, \quad a \in [0, 1)$;
(2) $F(t_1, ..., t_5) = a \max\{t_1, t_2, t_3, t_4, t_5\}, \quad a \in [0, 1)$;
(3) $F(t_1, ..., t_5) = a \max\left\{t_1, t_2, t_3, t_4, \frac{t_4 + t_5}{2}\right\}, \quad a \in [0, 1)$;
(4) $F(t_1, ..., t_5) = a(t_2 + t_3), \quad a \in \left[0, \frac{1}{2}\right)$;
(5) $F(t_1, ..., t_5) = at_1 + b(t_2 + t_3), \quad a, b \in \mathbb{R}_+, \quad a + 2b < 1$;
(6) $F(t_1, ..., t_5) = a \max\{t_2, t_3\}, \quad a \in (0, 1)$;
(7) $F(t_1, ..., t_5) = \left(\sum_{i=1}^5 a_i t_i^p\right)^{\frac{1}{p}}, \quad a_i \in \mathbb{R}_+, \quad \sum_{i=1}^5 a_i < 1, \quad p \geq 1$;
(8) $F(t_1, ..., t_5) = \max\{at_1, b(t_2 + t_4), c(t_3 + t_5)\}$, *where* $a \in [0, 1), \quad b, c \in \left[0, \frac{1}{2}\right]$.

Example 1.8. (**Timiş**, [**100**]) *Define* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *as*

$$F(t_1, ..., t_5) = t_1 - ct_2 - t_5, \quad c \in [0, \frac{1}{2}).$$

We establish the following general stability theorem for the Picard iteration procedure:

Theorem 1.9. (**Timiş**, [**100**]) *Let* (X, d) *be a complete metric space,* $T : X \to X$ *a map with* $Fix(X) \neq \emptyset$ *for which there exists* $F \in \mathbb{F}$ *such that for all* $x, y \in X$,

$$(1.10) \qquad F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

If F satisfyes (2a) then

(1) the fixed point p is unique in X;
(2) the Picard iteration is T-stable.

Corollary 1.1. *(**Timiş**, [**100**]) Let (X, d) be a complete metric space, $T : X \to X$ a map with $Fix(X) \neq \emptyset$ for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

If F satisfies (2a) then

   (1) the fixed point p is unique in X;
   *(2) the Picard iteration corresponding to the fixed point theorem obtained by Reich [**75**] and Rus [**83**] (see Taskovic [**90**]) is T-stable.*

Corollary 1.2. *(**Timiş**, [**100**]) Let (X, d) be a complete metric space, $T : X \to X$ a map with $Fix(X) \neq \emptyset$ for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

If F satisfies (2a) then

   (1) the fixed point p is unique in X;
   *(2) the Picard iteration corresponding to the fixed point theorem obtained by Bianchini [**28**] and Dugundij (1976) (see Rus [**83**]) is T-stable.*

Remark 1.5. *Some other important particular cases:*

   *(1) If F is given by Example 1.7 (1), then we obtain a stability result for Banach's contraction mapping principle, see Ostrowski [**66**].*
   *(2) If F is given by Example 1.7 (2), then we obtain a stability result for the Ciric's fixed point theorem [**37**], see Harder and Hicks [**46**].*
   *(3) If F is given by Example 1.7 (4), then we obtain a stability result for the Kannan's fixed point theorem [**53**], see Harder and Hicks [**46**].*
   *(4) If F is given by Example 1.7 (8), then we obtain a stability result for Zamfirescu's fixed point theorem, that is, Theorem 2 from Harder and Hicks [**46**].*
   *(5) If F is given by Example 1.8, then we obtain a stability result for Reich's fixed point theorem, that is, for Theorem 3 from Reich [**76**].*

Remark 1.6. *The contractive conditions obtained from (1.10) with F as in Examples 1-2 imply the contraction condition used by Rhoades in [**77**], [**78**], [**79**] and furthermore, they involve stability results for other well-known fixed point theorems.*

2. Stability of fixed point iterative procedure for common fixed points and coincidence points and contractive mappings satisfying implicit relations with six parameters

Popa [**71**], [**72**] also introduced $\mathbb{F}$ to be the set of all continuous real functions $F : \mathbb{R}_+^6 \to R_+$ with the following conditions:

(1) (a) F is non-increasing in the fifth variable and $F(u, v, v, u, u+v, 0) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;

(b) F is non-increasing in the fourth variable and $F(u, v, 0, u+v, u, v) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;

(c) F is non-increasing in the third variable and $F(u, v, u+v, 0, v, u) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;

(2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

The following examples of such functions appearing in Popa [**72**] correspond to well-known fixed point theorems and satisfy the above conditions.

Example 2.9. [**72**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}, \quad k \in (0, 1).$$

Example 2.10. [**72**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4), \quad b \in \left[0, \frac{1}{2}\right).$$

Example 2.11. [**72**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6), \quad c \in \left[0, \frac{1}{2}\right).$$

Imdad and Ali [**48**] proved a general common fixed point theorem for a pair of mappings using implicit functions due to Popa [**71**], [**72**].

In the following, using the results obtained in [**48**], we give a stability result for the common fixed point iterative procedure.

Theorem 2.10. *(Timiş, [**98**]) Let (X, d) be a complete metric space and $S, T : X \to X$ two mappings such that*

- *T and S satisfy (E.A) property;*
- *$\forall x, y \in X$, there exists $F \in \mathbb{F}$,*

(2.11) $F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) \leq 0,$

3. Stability of fixed point iterative procedure for common fixed points and coincidence points for contractive mappings satisfying implicit relations with five parameters

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- $S(X)$ *is a complete subspace of* X.

Then

(i) the pair (T, S) *has a point of coincidence;*

(ii) the pair (T, S) *has a unique common fixed point, as long as the pair* (T, S) *is also weakly compatible;*

(iii) if, additionally, F *satisfies (1b), then the associated iterative procedure is* (S, T)*-stable.*

Remark 2.7. *Particular cases:*

(1) In the case of F *given in Example 2.9, from Theorem 2.10 we obtain a stability result for the Ciric's fixed point theorem* [**38**].

(2) In the case of F *given in Example 2.10, from Theorem 2.10 we obtain a stability result for the Kannan's fixed point theorem* [**53**].

(3) In the case of F *given in Example 2.11, from Theorem 2.10 we obtain a stability result for the Chatterjea's fixed point theorem* [**35**].

Remark 2.8. *Theorem 2.10 gives a stability result for the common fixed point iteration procedure corresponding to Theorem 3.1 in* [**48**].

3. Stability of fixed point iterative procedure for common fixed points and coincidence points for contractive mappings satisfying implicit relations with five parameters

From the class of implicit functions due to Popa [**70**], [**71**], [**72**], now let $\mathbb{F}$ be the set of all continuous real functions $F : \mathbb{R}^5_+ \to \mathbb{R}$, satisfying the following conditions:

(1) F is continuous in each coordinate variable,

(2) there exists $h \in [0, 1)$ such that, $\forall u, v, w \geq 0$ satisfying

- (2a) $F(u, v, u, v, w) \leq 0$ or
- (2b) $F(u, v, v, u, w) \leq 0$

then we have that $u \leq h \max \{v, w\}$.

(3) $F(u, u, u, u, 0) > 0$, for all $u > 0$.

In the sequel, we present some examples of functions depending on five parameters, satisfying some of the conditions above.

Example 3.12. (*Timiş*, [97]) *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - kt_5,$$

where $k \in (0, 1)$, *satisfies (1),(2a),(2b) and (3), with* $h = k$.

Example 3.13. (*Timiş*, [97]) *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - at_2 - bt_5,$$

where $a, b \in (0, 1)$, *with* $a + 2b < 1$, *satisfies (1), (2a), (2b) and (3), with* $h = a$, *if* $\max\{v, w\} = v$ *and* $h = b$, *if* $\max\{v, w\} = w$.

Example 3.14. [70] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - a(t_3 + t_4),$$

where $a \in \left(0, \frac{1}{2}\right)$, *satisfies (1), (2a), (2b) and (3), with* $h = \frac{a}{1-a} \in (0, 1)$.

Example 3.15. (*Timiş*, [97]) *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5,$$

where $a, b, c, d \in [0, 1)$, *with* $a + b + c + 2d < 1$, *satisfies (1), (2a) with* $h = \frac{a+c}{1-b} \in [0, 1)$, *(2b) with* $h = \frac{a+b}{1-c} \in [0, 1)$, *and (3), where* $h = \frac{a+c}{1-b} \in [0, 1)$, *if* $\max\{v, w\} = v$ *and* $h = \frac{a+b}{1-c} \in [0, 1)$, *if* $\max\{v, w\} = w$.

Example 3.16. [68] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, t_5\right\},$$

where $a \in [0, 1)$, *satisfies (1), (2a), (2b) and (3), respectively when* $\max = t_2$ *or* $\max = t_5$, *then* $h = a$, *when* $\max = \frac{t_3+t_4}{2}$, *then* $h = \frac{\frac{a}{2}}{1-\frac{a}{2}}$.

Example 3.17. [70] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - c \max\{t_2, t_3, t_4, t_5\},$$

where $h = c \in [0, 1)$, *satisfies (1), (3), when* $\max = t_2$, $\max = t_4$ *or* $\max = t_5$ *is satisfied (2a) and when* $\max = t_3$ *is satisfied (2b).*

Example 3.18. [70] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1^2 - c \max\left\{t_2 t_3, t_2 t_4, t_3 t_4, t_5^2\right\},$$

where $c \in [0, 1)$, *satisfies (1), (2a) and (3), with* $h = c$.

Using the common fixed point theorem of Imdad and Ali [**48**], we give the following general stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings satisfying (E.A) property and defined by an implicit contraction condition.

Theorem 3.11. *(**Timiş**, [**97**]) Let (X, d) be a complete metric space and $S, T : X \to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

Assume there exists $F \in \mathbb{F}$ such that

(3.12)
$$F\left(d(Tx, Ty), d(Sx, Sy), d(Sx, Ty), d(Sy, Tx), \frac{d(Sx, Tx) + d(Sy, Ty)}{2} \right) \leq 0,$$

for all $x, y \in X$. Then

- *(1) if F satisfies (2b), then the pair (T, S) has a point of coincidence;*
- *(2) if F satisfies (3), the pair (T, S) has a unique common fixed point as long as the pair (T, S) is also weakly compatible;*
- *(3) if, additionally, F satisfies (2a), then the associated iterative procedure is (S, T)-stable.*

Remark 3.9. *Theorem 3.11 completes Theorem 3.1 in Imdad and Ali [**48**] with the information about the stability of the Jungck-type iterative procedure with respect to the mappings S and T, provided that the function F satisfies an additional condition.*

Corollary 3.3. *(**Timiş**, [**97**]) Let (X, d) be a complete metric space and $S, T : X \to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

Suppose there exists $F \in \mathbb{F}$ such that S and T satisfy (3.12), for all $x, y \in X$.

Then, the Jungck-type iterative procedure is (S, T)-stable.

Corollary 3.4. *(**Timiş**, [**97**]) Let (X, d) be a complete metric space and $S, T : X \to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

Suppose there exists $F \in \mathbb{F}$ such that S and T satisfy (3.12), for all $x, y \in X$.

Then, in the case of the contraction conditions of Zamfirescu's type, the associated common fixed point iterative procedure is (S, T)-stable.

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Remark 3.10. *Other particular cases.*

(1) *If F is given by Example 3.12 and $S = I$, the identity map on X, then we obtain a stability result for the Kannan's fixed point theorem, see* [**53**], *corresponding to a pair of mappings with a common fixed point;*

(2) *If F is given by Example 3.13 and $S = I$, the identity map on X, then we obtain a stability result for a fixed point theorem obtained by Reich (1971) and Rus (1971), see* [**90**], *corresponding to a pair of mappings with a common fixed point;*

(3) *If F is given by Example 3.14 and $S = I$, the identity map on X, then we obtain a stability result for the Chatterjea's fixed point theorem, see* [**35**], *corresponding to a pair of mappings with a common fixed point;*

(4) *If F is given by Example 3.15 and $S = I$, the identity map on X, then we obtain a stability result for the Hardy and Rogers's fixed point theorem, see* [**47**], *corresponding to a pair of mappings with a common fixed point;*

(5) *If F is given by Example 3.16 and $S = I$, the identity map on X, then we obtain a stability result for the Pathak and Verma's fixed point theorem, see* [**68**], *corresponding to a pair of mappings with a common fixed point in symmetric spaces;*

(6) *If F is given by Examples 3.17, 3.18 and $S = I$, the identity map on X, then we obtain stability results for the Popa's fixed point theorem, see* [**70**], *corresponding to two pairs of mappings on two metric spaces.*

Remark 3.11. *The contractive conditions obtained from (3.12) with F as in above examples also imply contractive conditions used by Rhoades in* [**77**], [**78**], [**79**], [**80**].

Conclusions:

Because of the inclusions between the commutativity definitions, the weakly compatible pair of mappings is the most general type from the mentioned notions and it includes the others. The above theorem use this kind of weakly compatible mappings and it follows that it holds also for compatible, commuting and weakly commuting pair of mappings.

In order to extend and generalize all the mentioned common fixed point theorems, it can be established corresponding stability results for fixed point iteration procedures associated to contractive mappings defined by a suitable implicit relation.

CHAPTER 4

A new point of view on the stability of fixed point iterative procedures

By taking account of the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis, Rus [**81**] unified these notions by new ones.

We consider these new notions in this chapter and study the stability of Picard iteration for mappings which satisfy certain contractive conditions. We also give some illustrative examples.

The author's original contributions in this chapter are: Theorem 1.12, Proposition 1.1, Corollary 1.5, Corollary 1.6, Corollary 1.7, Example 1.19, Corollary 1.8, Theorem 2.13, Corollary 2.9, Example 2.20, Theorem 2.14, Corollary 2.10, Examples 3.21 - 3.28, Definition 4.15, Definition 4.16, Proposition 4.2, Theorem 4.15, Theorem 4.16, Theorem 5.17.

Some of them are included in [**92**] (Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012).

1. New stability concept for Picard iterative procedures

Eirola, Nevanlinna and Pilyugin [**42**] introduced the notion of *limit shadowing property* and Rus [**81**] adopted it, in order to introduce a new concept of stability for fixed point iteration procedures which appears to be more general than the notion of stability introduced by Harder [**44**].

Definition 1.12. *(Rus,[**81**]) On the metric space (X, d), the operator $T : X \to X$ has stable Picard iterates at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that*

$$x \in X, \ d(x, x_0) < \delta(\epsilon) \ \Rightarrow \ d\left(T^n x, T^n x_0\right) < \epsilon, \ \ \forall n \in \mathbb{N}.$$

The operator T has stable Picard iterates on $Y \subset X$, if it has stable Picard iterates at all $x_0 \in Y$.

Definition 1.13. [**42**] *The operator T has the limit shadowing property with respect to Picard iteration, if*

$$y_n \in X, \quad n \in \mathbb{N}, \quad d(y_{n+1}, Ty_n) \to 0 \quad as \quad n \to \infty$$

imply that there exists $x_0 \in X$, such that

$$d(y_n, T^n x_0) \to 0 \quad as \quad n \to \infty.$$

Definition 1.14. [**81**] *Picard iteration is stable with respect to an operator T if it is convergent with respect to T and the operator T has the limit shadowing property with respect to this iterative procedure.*

Theorem 1.12. *(**Timiş**, [**93**]) Let (X, d) be a metric space and $T : X \to X$ be an a-contraction, i.e., T satisfies*

$$d(Tx, Ty) \leq ad(x, y), \quad \forall x, y \in X,$$

with $a \in [0, 1)$ fixed.

   Then, T has stable Picard iterates on X.

In the following, we study the relationship between the two stability definitions, the one of Harder [**44**] and the other one due to Rus [**81**].

Proposition 1.1. *(**Timiş**, [**93**])*

   Let (X, d) be a metric space and $T : X \to X$ be a mapping. Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...,$ converges to a fixed point p of T.

   Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus. (Definition 1.12)

Corollary 1.5. *(**Timiş**, [**93**]) Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying the contraction condition of Zamfirescu, i.e., there exists real numbers α, β, γ, satisfying $0 \leq \alpha < 1$, $0 \leq \beta, \gamma < \frac{1}{2}$, such that, for each $x, y \in X$, at least one of the following is true:*

   (1) $d(Tx, Ty) \leq \alpha d(x, y)$;
   (2) $d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty) \right]$;
   (3) $d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx) \right]$.

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.12).

Remark 1.12. *Corollary 1.5 gives a stability result corresponding to the fixed point theorem of Zamfirescu [**106**].*

Corollary 1.6. *(**Timiş**, [**93**]) Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying Kannan's contraction condition, i.e., there exists $a \in [0, 1)$ such that for all $x, y \in X$,*

$$d\left(Tx, Ty\right) \leq a\left[d(x, Tx) + d(y, Ty)\right].$$

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.12).

Remark 1.13. *Corollary 1.6 gives a stability result corresponding to the fixed point theorem of Kannan [**53**].*

Corollary 1.7. *(**Timiş**, [**93**]) Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying Chatterjea's contraction condition, i.e., there exists $a \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$,*

$$d\left(Tx, Ty\right) \leq a\left[d(x, Ty) + d(y, Tx)\right].$$

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.12).

Remark 1.14. *Corollary 1.7 gives a stability result corresponding to the fixed point theorem of Chatterjea [**35**].*

Remark 1.15. *The converse of Proposition 1.1 is not generally true, as shown by the following example.*

Example 1.19. *($Timiş$, [**93**])*

Let $T : [0, 1] \to [0, 1]$ be identity mapping on $[0, 1]$, that is, $Tx = x$, for each $x \in [0, 1]$, where $[0, 1]$ is endowed with the usual metric. Every point in $[0, 1]$ is a fixed point of T and T is nonexpansive, but not a contraction.

Harder [**46**] showed in this case that Picard iteration is not T-stable but we obtained that the Picard iteration is stable in the sense of Rus.

Corollary 1.8. *($Timiş$, [**93**])*

Let (X, d) be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let use assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point p of T.

If Picard iteration procedure is stable in the sense of Harder, then the fixed point is unique.

Remark 1.16. *Corollary 1.8 has been suggested by Professor I. A. Rus (private communication).*

2. Stability results of Picard iteration for mappings satisfying certain contractive conditions

According to above stability definitions of Rus [**81**], in the following we study the stability of Picard iterative procedure as well as the stability of Picard iterates at $x_0 \in X$, with respect to T.

A generalized contraction condition introduced by Berinde [**15**], named *almost contraction* condition has some surprising properties: it ensures the convergence of Picard iteration to a fixed point and under adequate conditions, an unique fixed point, but it does not require the sum of the coefficients on the right side of the contractive condition to be less than 1.

In a metric space (X, d), a self mapping $T : X \to X$ is called an *almost contraction* if there exists two constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d (Tx, Ty) \leq \delta d(x, y) + L d(y, Tx),$$

for any $x, y \in X$. Here, $\delta + L$ is not restricted to be less than 1.

Almost contractions have a very similar behavior to that of Banach contractions, which explains their name, except for the fact that the fixed point is generally not unique.

In order to ensure this uniqueness, Berinde [15] considered another condition, similar to the above one, namely

(2.13) $$d\left(Tx, Ty\right) \leq \delta_u d(x, y) + L_u d(x, Tx),$$

for any $x, y \in X$, where $\delta_u \in [0, 1)$ and $L_u \geq 0$ are constants.

Note that (2.13) has been used by Osilike [61], [63], Osilike and Udomene [65] in order to establish several stability results.

Berinde [16] also proved the existence of coincidence points and common fixed points for a large class of almost contractions in cone metric spaces.

Moreover, Berinde [13] proved the existence of coincidence points and common fixed points of noncommuting almost contractions in metric spaces and a method for approximating the coincidence points or the common fixed points is also constructed, for which both a priori and a posteriori error estimates are obtained.

Using this condition, we obtain the following stability result:

Theorem 2.13. *(Timiş, [93]) Let (X, d) be a metric space and $T : X \to X$ be a self mapping satisfying the contraction condition (2.13), i.e., for some $\delta_u \in [0, 1)$ and $L_u \geq 0$. For all $x, y \in X$, we have*

$$d\left(Tx, Ty\right) \leq \delta_u d(x, y) + L_u d(x, Tx).$$

Then, the associated Picard iteration is T-stable in the sense of Definition 1.14.

Remark 2.17. *For a metric space (X, d) and a self mapping T satisfying the almost contraction condition (2.13), the associated Picard iteration is T-stable in the sense of Rus, provided it is T-stable in the sense of Harder.*

Corollary 2.9. *(Timiş, [93]) Let (X, d) be a metric space and a mapping $T : X \to X$, satisfying Banach's contraction condition, i.e., there exists $a \in [0, 1)$ and for all $x, y \in X$, we have that T satisfy the contraction condition*

$$d\left(Tx, Ty\right) \leq a d(x, y).$$

Then, the associated Picard iteration is T-stable in the sense of Definition 1.14.

Example 2.20. *(Timiş, [93])*

Let $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, ...\right\}$ with the usual metric. Define $T : X \to X$ by $T(0) = \frac{1}{2}$, $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$, $n = 1, 2, 3, ...$

Babu, Sandhya and Kameswari [10] proved that T satisfies the almost contraction condition (2.13), with $\delta = \frac{1}{2}$ and $L = 1$, when $\delta + L = \frac{3}{4} > 1$.

Because T has no fixed points, Picard iteration is not stable in the sense of Harder but we obtained that the Picard iteration is stable in the sense of Rus.

Babu, Sandhya and Kameswari [10] found a different contractive condition that ensures the uniqueness of fixed points of almost contractions: if there exists $\delta \in (0, 1)$ and $L \geq 0$, such that for all $x, y \in X$,

$$(2.14) \quad d(Tx, Ty) \leq \delta d(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Using this condition, we obtain the following stability result:

Theorem 2.14. *(**Timiş**, [93]) Let (X, d) be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (2.14), i.e., there exists $\delta \in (0, 1)$ and $L \geq 0$, such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Then, the associated Picard iteration is T-stable in the sense of Harder.

Corollary 2.10. *(**Timiş**, [93]) Let (X, d) be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (2.14), i.e., there exists $\delta \in (0, 1)$ and $L \geq 0$, such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Then, the associated Picard iteration is T-stable in the sense of Rus, provided it is T-stable in the sense of Harder.

Conclusions:

1. A fixed point iteration procedure which is stable in the sense of Harder is also stable in the sense of Rus. But the reverse is not generally true, because Harder stability implies the uniqueness of fixed point, while the new one of Rus does not.

2. The stability of a fixed point iteration procedure in the sense of Rus may imply stability in the sense of Harder, if and only if the iterative procedure converges to the fixed point.

3. On the other hand, there are many examples of mappings that satisfy certain contractive conditions and for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

Open problem: Study the stability in the sense of Rus for general nonexpansive mappings as well as for general almost contractions (that do not satisfy a certain uniqueness condition).

3. Examples

In the following, we give some examples of mappings satisfying certain contractive conditions for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

Example 3.21. *(Timiş, [93])*

Let $T : [0, 2] \to [0, 2]$ be given by

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0, 1) \\ \\ 2, & x \in [1, 2], \end{cases}$$

where $[0, 2]$ is endowed with the usual metric. T has two fixed points, $Fix(T) = \{0, 2\}$.

Păcurar [67] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0, 1)$ and $L = 3 \geq 0$, such that, for any $x, y \in [0, 2]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + L d(y, Tx).$$

Note that $\delta + L = \frac{7}{2} > 1$.

Regarding the stability, we obtained that Picard iteration is not T-stable in sense of Harder but it is T-stable in sense of Rus.

Example 3.22. *(Timiş, [93])*

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{2}{3}x, & x \in \left[0, \frac{1}{2}\right) \\ \frac{2}{3}x + \frac{1}{3}, & x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where $[0, 1]$ is endowed with the usual metric.

T has two fixed points, $Fix(T) = \{0, 1\}$.

Păcurar [67] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{2}{3} \in [0, 1)$ and $L = 6 \geq 0$, such that, for any $x, y \in [0, 1]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Note that $\delta + L = 6 + \frac{2}{3} > 1$.

Concerning the stability, we obtained that Picard iteration is not T-stable in sense of Harder but it is T-stable in sense of Rus.

Example 3.23. *(Timiş, [93])*

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} x^2, & x \in \left[0, \frac{1}{4}\right) \\ 0, & x \in \left[\frac{1}{4}, 1\right], \end{cases}$$

where $[0, 1]$ is endowed with the usual metric. T has a fixed point at 0.

Păcurar [67] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0, 1)$ and $L = \frac{1}{3} \geq 0$, such that, for any $x, y \in [0, 1]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Note that in this case $\delta + L = \frac{5}{6} < 11$.

We obtained that Picard iteration is T-stable in sense of Harder and it is also T-stable in sense of Rus.

Example 3.24. *(Timiş, [93])*

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{2}{3}, & x \in [0, 1) \\ 0, & x = 1, \end{cases}$$

where $[0, 1]$ is endowed with the usual metric.

T has one fixed point at $\frac{2}{3}$, $Fix(T) = \left\{\frac{2}{3}\right\}$.

Păcurar [**67**] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{2}{3} \in [0,1)$ and $L \geq \delta \geq 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \leq \delta d(x,y) + Ld(y, Tx).$$

Note that in this case $\delta + L \geq \frac{4}{3} > 1$.

Concerning the stability, we obtained that Picard iteration is T-stable in sense of Harder and hence it is also T-stable in sense of Rus.

Example 3.25. *(Timiş, [**93**])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{2}, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric.

T has one fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$.

Păcurar [**67**] showed that T is an almost contraction, i.e., there exists two constants $\delta_u = \frac{1}{2} \in [0,1)$ and $L_u = 1 \geq 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \leq \delta_u d(x,y) + L_u d(x, Tx).$$

Note that in this case $\delta + L = \frac{3}{2} > 1$.

About the stability, we obtained that Picard iteration is T-stable in sense of Harder and it is also T-stable in sense of Rus.

Example 3.26. *(Timiş, [**93**])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. T is continuous at each point of $[0,1]$ except at $\frac{1}{2}$.

T has an unique fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$.

We already showed in Example 6.2 that for each $x, y \in [0, 1]$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max \{d(x, Tx), d(y, Ty)\},$$

and also we showed that the associated Picard iteration is not T-stable in the sense of Harder.

Moreover, here we obtained that the Picard iteration is stable in the sense of Rus.

Example 3.27. *(Timiş, [93])*

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0, 1]$ is endowed with the usual metric. T is continuous at every point of $[0, 1]$ except at $\frac{1}{2}$.

T has an unique fixed point at 0, $Fix(T) = \{0\}$.

We already showed in Example 6.3 that for each $x, y \in [0, 1]$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max \{d(x, Ty), d(y, Tx)\},$$

and also showed that the associated Picard iteration is not T-stable in the sense of Harder.

Moreover, here we obtained that the Picard iteration is stable in the sense of Rus.

Example 3.28. *(Timiş, [93])*

Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0 \\ \frac{1}{4}, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x > \frac{1}{2} \end{cases},$$

where $\mathbb{R}$ is endowed with the usual metric. T is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$.

The only fixed point of T is $\frac{1}{4}$, $Fix(T) = \left\{\frac{1}{4}\right\}$.

We already showed in Example 6.4 that for each $x, y \in \mathbb{R}$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

and also showed that the associated Picard iteration is not T-stable in the sense of Harder.

Moreover, here we obtained that the Picard iteration is stable in the sense of Rus.

4. New stability concepts of fixed point iteration for common fixed points and contractive type mappings

By adapting Definition 1.13 of limit shadowing property of Eirola, Nevanlinna and Pilyugin [42] to common fixed points, we introduce the following:

Definition 4.15. *(Timiş, [92]) Let (X, d) be a metric space and $S, T : X \to X$ be two mappings such that $T(X) \subseteq S(X)$. Let u be a common fixed point of S and T, that is, $Tu = Su = u$.*

For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by the Jungck type iterative procedure

$$(4.15) \qquad\qquad Sx_{n+1} = Tx_n, \ n = 0, 1, 2, ...,$$

and assume that it converges to u.

Then, we say that the mappings T and S have the limit shadowing property with respect to Jungck type iteration procedure, if

$$Sy_n \in X, \ \ n \in \mathbb{N}, \ \ d(Sy_{n+1}, Ty_n) \to 0 \ \ as \ \ n \to \infty$$

imply that there exists $x_0 \in X$, such that

$$d(Sy_n, T^n x_0) \to 0 \ \ as \ \ n \to \infty.$$

Remark 4.18. *If $S = I$, the identity map on X, then, by Definition 4.15, we get Definition 1.13 of the limit shadowing property introduced by Eirola, Nevanlinna and Pilyugin [42].*

The notion of stability introduced by Rus [81] in Definition 1.14 will be transposed to common fixed points, as follows:

Definition 4.16. *(Timiş, [92]) Let (X, d) be a metric space and $S, T : X \to X$ be two mappings such that $T(X) \subseteq S(X)$. Let u be a common fixed point of S and T, that is, $Tu = Su = u$.*

For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by the Jungck type iterative procedure $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, and assume that it converges to u.

Then, the Jungck type iteration procedure is stable with respect to the mappings T and S if it is convergent with respect to T and S and the mappings T and S have the limit shadowing property with respect to this iterative procedure.

In the following, we study the relationship between the stability concept introduced by Singh and Prasad [88] in Definition 2.6 which is given for a pair of mappings (S, T) with a coincidence point and our new stability concept introduced by Definition 4.16.

Proposition 4.2. *(Timiş, [92])*

Let (X, d) be a metric space and let $S, T : X \to X$, where $T(X) \subseteq S(X)$ and the mappings S and T have a common fixed point, that is, $Su = Tu = u$.

For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be defined by (4.15) and assume that it converges to $u \in X$.

Suppose that the Jungck type iteration procedure is stable in the sense of Singh and Prasad [88], by Definition 2.6.

Then, the Jungck type iteration procedure is also stable in the sense of Definition 4.16..

Remark 4.19. *If If $S = I$, the identity map on X, Proposition 4.2 reduces to Proposition 1.1.*

In the following, we give some stability results for the iteration procedure defined by (4.15), with respect to two mappings which satisfy various contractive conditions.

Theorem 4.15. *(Timiş, [92]) Let (X, d) be a complete metric space and $S, T : X \to X$ be two mappings, satisfying*

(4.16) $d(Tx, Ty) \leq a d(Sx, Sy),$

for each $x, y \in X$ and some constant $a \in [0, 1)$.

S and T have an unique common fixed point u, with $Tu = Su = u$, if

i) $T(X) \subseteq S(X)$;

ii) S is continuous;

iii) S and T commute.

For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by (4.15) which is assumed to converge to u.

Then, the Jungck type iteration procedure is stable with respect to the mappings T and S, in the sense of Definition 4.16.

Remark 4.20. *If $S = I$, the identity map on X, the stability result in the case of Jungck type iteration procedure in the sense of Rus, i.e., Theorem 4.15, reduces to the stability result of Picard iteration procedure, i.e., Theorem 2.13.*

Theorem 4.16. *(**Timiş**, [**92**]) Let (X, d) be a metric space and $S, T : X \to X$ be two mappings. Suppose there exists $h \in [0, 1)$ such that, for every $x, y \in X$,*

$$(4.17) \qquad d(Tx, Ty) \leq h \max \{d(Sx, Ty), d(Sy, Tx)\}.$$

S and T have an unique common fixed point u, with $Tu = Su = u$, if
i) $T(X) \subseteq S(X)$;
ii) S is continuous;
iii) S and T commute.
For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by (4.17) which is assumed to converge to u.

Then, iteration procedure defined by (4.17) is stable with respect to the mappings T and S, in the sense of Definition 4.16.

5. New stability of Picard iteration for mappings defined by implicit relations

We recommence the set of all continuous real functions $\mathbb{F}$, introduced by Popa [**71**], [**72**] and used in Chapter 3, Section 2, i.e., $F : \mathbb{R}_+^6 \to R_+$ for which we consider the following conditions:

(1) (a) F is non-increasing in the fifth variable and $F(u, v, v, u, u+v, 0) \leq 0$ for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

(b) F is non-increasing in the fourth variable and $F(u, v, 0, u+v, u, v) \leq 0$ for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

(c) F is non-increasing in the third variable and $F(u, v, u+v, 0, v, u) \leq 0$ for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

(2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

.

For the complete metric space (X, d) and $T : X \to X$ a self mapping for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,

$$(5.18) \qquad F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0,$$

Berinde [**24**] proved that if F satisfies (1a) and (2), then

- T has an unique fixed point x^* in X;
- The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ converges to x^*, for any $x_0 \in X$.

In the following, using the above assumptions, we study the stability of Picard iteration in the sense of Definition 1.14.

Theorem 5.17. (**Timiş**, [**91**]) *Let (X, d) be a complete metric space and $T : X \to X$ a self mapping for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$, F satisfies (5.18), i.e.*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

If F satisfies (1a), (1b) and (2), then the Picard iteration is T-stable in the sense of Definition 1.14.

CHAPTER 5

Stability of tripled fixed point iteration procedures

In this chapter we introduce the concept of stability for tripled fixed point iterative procedures and also establish some stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions. An illustrative example is also given.

The author's original contributions in this chapter are: Definition 2.19, Theorem 2.18, Corollary 2.11, Theorem 2.19, Theorem 2.20, Lemma 3.3, Definition 3.21, Theorem 3.21, Corollary 3.12, Theorem 3.22, Theorem 3.23, Example 4.29 and the contractive conditions (2.20)-(2.25), (3.28)-(3.33).

Most of them were published in [**102**] (Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7).

1. Tripled fixed point iterative procedures

Banach-Caccioppoli-Picard Principle has been generalized by enriching the metric space structure with a partial order. The first result of this kind for monotone mappings in ordered metric spaces was obtained by Ran and Reurings [**73**].

Following the same approach, Bhaskar and Laksmikantham [**27**] obtained some coupled fixed point results for mixed-monotone operators of Picard type, obtaining results involving the existence, the existence and the uniqueness of the coincidence points for mixed-monotone operators $T : X^2 \to X$ in the presence of a contraction type condition, in a partially ordered metric space.

This concept of coupled fixed points in partially ordered metric spaces and cone metric spaces have been studied by several authors, including Abbas, Ali Khan and Radenovic [**2**], Berinde [**17**], [**18**], [**20**], Choudhury and Kundu [**36**], Ciric and Lakshmikantham [**40**], Karapinar [**54**], Lakshmikantham and Ciric [**55**], Olatinwo [**58**], Sabetghadam, Masiha and Sanatpour [**84**].

Recently, Berinde and Borcut [26], [32] obtained extensions to the concept of tripled fixed points and tripled coincidence fixed points and also obtained tripled fixed points theorems and tripled coincidence fixed points theorems for contractive type mappings in partially ordered metric spaces.

The research on tripled fixed point was continued by Abbas, Aydi and Karapinar [3], Aydi and Karapinar [8], Aydi, Karapinar and Vetro [9], Amini-Harandi [7], Borcut [29], [30], [31], Charoensawan [34], Rao and Kishore [73].

By adapting the concept of stability from fixed point iterative procedures, Olatinwo [60] studied the stability of the coupled fixed point iterative procedures using some contractive conditions for which the existence of a unique coupled fixed point has been established by Sabetghadam, Masiha and Sanatpour [84].

In the following, we introduce the concept of stability for tripled fixed point iterative procedures and establish stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions by extension from coupled fixed points to tripled fixed points of contractive conditions employed by Olatinwo [60].

2. Stability of tripled fixed point iteration procedures for monotone mappings

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Borcut [31] endowed the product space X^3 with the following partial order:

$$(x, y, z), \ (u, v, w) \in X^3, \ (u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, \ y \leq v, \ z \geq w.$$

Definition 2.17. [31] *Let $(X, \leq)$ be a partially ordered set and $T : X^3 \to X$ a mapping. We say that T has the monotone property if $T(x, y, z)$ is monotone nondecreasing in x, y and z, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, \ x_1 \leq x_2 \Rightarrow T(x_1, y, z) \leq T(x_2, y, z),$$

$$y_1, y_2 \in X, \ y_1 \leq y_2 \Rightarrow T(x, y_1, , z) \leq T(x, y_2, z),$$

$$z_1, z_2 \in X, \ z_1 \leq z_2 \Rightarrow T(x, y, z_1) \leq T(x, y, z_2).$$

Definition 2.18. [31] *An element $(x, y, z) \in X^3$ is called tripled fixed point of $T : X^3 \to X$, if $T(x, y, z) = x$, $T(y, x, z) = y$, $T(z, y, x) = z$.*

A mapping $T : X^3 \to X$ is said to be a (k, μ, ρ)-contraction, if and only if there exists the constants $k \geq 0$, $\mu \geq 0$, $\rho \geq 0$, $k + \mu + \rho < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$(2.19) \qquad d(T(x, y, z), T(u, v, w)) \leq kd(x, u) + \mu d(y, v) + \rho d(z, w).$$

In relation to (2.19), we introduce some new contractive conditions.

Let (X, d) be a metric space. For a mapping $T : X^3 \to X$, suppose there exists $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, with $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$(2.20) \qquad (i) \ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), x\right) + b_1 d\left(T(u, v, w), u\right);$$

$$(2.21) \qquad d\left(T(y, x, z), T(v, u, w)\right) \leq a_2 d\left(T(y, x, z), y\right) + b_2 d\left(T(v, u, w), v\right);$$

$$(2.22) \qquad d\left(T(w, y, x), T(z, v, u)\right) \leq a_3 d\left(T(z, y, x), z\right) + b_3 d\left(T(w, v, u), w\right);$$

$$(2.23) \qquad (ii) \ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), u\right) + b_1 d\left(T(u, v, w), x\right);$$

$$(2.24) \qquad d\left(T(y, x, z), T(v, u, w)\right) \leq a_2 d\left(T(y, x, z), v\right) + b_2 d\left(T(v, u, w), y\right);$$

$$(2.25) \qquad d\left(T(w, y, x), T(z, v, u)\right) \leq a_3 d\left(T(z, y, x), w\right) + b_3 d\left(T(w, v, u), z\right).$$

For $(x_0, y_0, z_0) \in X^3$, the sequence $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ defined by

$$(2.26) \qquad x_{n+1} = T(x_n, y_n, z_n), \ y_{n+1} = T(y_n, x_n, z_n), \ z_{n+1} = T(z_n, y_n, x_n),$$

with $n = 0, 1, 2, ...$, is said to be a *tripled fixed point iterative procedure.*

We give the following definition of stability with respect to T, in metric spaces, relative to tripled fixed points iterative procedures:

Definition 2.19. *(Timiş, [102]) Let (X, d) be a complete metric space and a maping $T : X^3 \to X$, with*

$$Fix_t(T) = \left\{(x^*, y^*, z^*) \in X^3 \ | \ T(x^*, y^*, z^*) = x^*, \ T(y^*, x^*, z^*) = y^*, \right.$$

$T(z^*, y^*, x^*) = z^*\}$, *the set of tripled fixed points of T.*

Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the sequence generated by the iterative procedure defined by (2.26), where $(x_0, y_0, z_0) \in X^3$ is the initial value, which converges to a tripled fixed point (x^*, y^*, z^*) of T.

Let $\{(u_n, v_n, w_n)\}_{n=0}^{\infty} \subset X^3$ an arbitrary sequence and set

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \quad \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, w_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right), \quad n = 0, 1, 2, ...$$

Then, the tripled fixed point iterative procedure defined by (2.26) is T-stable or stable with respect to T, if and only if

$$\lim_{n \to \infty} (\epsilon_n, \delta_n, \gamma_n) = 0_{R^3} \ \text{implies that} \ \lim_{n \to \infty} (u_n, v_n, w_n) = (x^*, y^*, z^*).$$

Theorem 2.18. (*Timiş*, [102]) *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the monotone property on X and satisfying (2.19).

If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \leq T(y_0, x_0, z_0) \ \ and \ \ z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \ \ and \ \ z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.26).

Then, the tripled fixed point iterative procedure is T-stable.

Remark 2.21. *Theorem 2.18 completes the existence theorem of tripled fixed points of Borcut [31] with the stability result for the tripled fixed point iterative procedures, using monotone operators.*

Corollary 2.11. (*Timiş*, [102]) *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the monotone property on X.

Assume that there exists $\kappa \in [0, 1)$, such that for each $x, y, z, u, v, w \in X$, T satisfies the following contraction condition:

$$d(T(x, y, z), T(u, v, w)) \leq \frac{\kappa}{3} \left\{ d(x, u) + d(y, v) + d(z, w) \right\}.$$

If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \ \ and \ \ z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \ \ and \ \ z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.26).

Then, the tripled fixed point iterative procedure is T-stable.

Remark 2.22. *Corollary 2.11 completes the existence theorem of tripled fixed points of Borcut [**31**] with the stability result for the tripled fixed point iterative procedures, using monotone operators.*

Theorem 2.19. *(**Timiş**, [**102**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the monotone property on X and satisfying (2.20), (2.21) and (2.22).

If there exists x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.26).

Then, the tripled fixed point iterative procedure is T-stable.

Theorem 2.20. *(**Timiş**, [**102**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the monotone property on X and satisfying (2.23), (2.24) and (2.25).

If there exists x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.26).

Then, the tripled fixed point iterative procedure is T-stable.

3. Stability of tripled fixed point iteration procedures for mixed monotone mappings

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Berinde and Borcut [**26**] endowed the product space X^3 with the following partial order:

$$(x, y, z), \ (u, v, w) \in X^3, \ (u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, \ y \leq v, \ z \geq w.$$

Definition 3.20. [**26**] *Let $(X, \leq)$ be a partially ordered set and $T : X^3 \to X$ a mapping. We say that T has the mixed monotone property if $T(x, y, z)$ is monotone nondecreasing in x, monotone nonincreasing in y and monotone nondecreasing in z, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, \ x_1 \leq x_2 \Rightarrow T(x_1, y, z) \leq T(x_2, y, z),$$

$$y_1, y_2 \in X, \ y_1 \leq y_2 \Rightarrow T(x, y_1, , z) \geq T(x, y_2, z),$$

$$z_1, z_2 \in X, \ z_1 \leq z_2 \Rightarrow T(x, y, z_1) \leq T(x, y, z_2).$$

A mapping $T : X^3 \to X$ is said to be a (k, μ, ρ)-contraction, if and only if there exists three constants $k \geq 0$, $\mu \geq 0$, $\rho \geq 0$, $k + \mu + \rho < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$(3.27) \qquad d(T(x, y, z), T(u, v, w)) \leq k d(x, u) + \mu d(y, v) + \rho d(z, w).$$

In relation to (3.27), we introduce some new contractive conditions:

Let (X, d) be a metric space. For a mapping $T : X^3 \to X$, there exists $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, with $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$, such that $\forall x, y, z, u, v, w \in X$, we introduce the following definitions of contractive conditions:

$$(3.28) \qquad (i) \ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), x\right) + b_1 d\left(T(u, v, w), u\right);$$

$$(3.29) \qquad d\left(T(y, x, y), T(v, u, v)\right) \leq a_2 d\left(T(y, x, y), y\right) + b_2 d\left(T(v, u, v), v\right);$$

$$(3.30) \qquad d\left(T(w, y, x), T(z, v, u)\right) \leq a_3 d\left(T(z, y, x), z\right) + b_3 d\left(T(w, v, u), w\right);$$

$$(3.31) \qquad (ii) \ d\left(T(x,y,z), T(u,v,w)\right) \le a_1 d\left(T(x,y,z), u\right) + b_1 d\left(T(u,v,w), x\right);$$

$$(3.32) \qquad d\left(T(y,x,y), T(v,u,v)\right) \le a_2 d\left(T(y,x,y), v\right) + b_2 d\left(T(v,u,v), y\right);$$

$$(3.33) \qquad d\left(T(w,y,x), T(z,v,u)\right) \le a_3 d\left(T(z,y,x), w\right) + b_3 d\left(T(w,v,u), z\right).$$

In the case of two matrices $A, B \in M_{(m,n)}(\mathbb{R})$, we say that $A \le B$, if $a_{ij} \le b_{ij}$, for all $i = \overline{1,m}$, $j = \overline{1,n}$.

In order to prove our main stability result, we give the next result which extends Lemma 1.1 to vector sequences, where inequalities between vectors means inequality on its elements:

Lemma 3.3. (Timiş, [101]) *Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers and consider a matrix $A \in M_{3,3}(\mathbb{R})$ with nonnegative elements, so that*

$$(3.34) \qquad \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} \le A \cdot \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} + \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}, \ n \ge 0,$$

with

(i) $\lim_{n\to\infty} A^n = O_3$;
(ii) $\sum_{k=0}^{\infty} \epsilon_k < \infty$, $\sum_{k=0}^{\infty} \delta_k < \infty$ and $\sum_{k=0}^{\infty} \gamma_k < \infty$.

Then, $\lim_{n\to\infty} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

Let (X, d) be a metric space and $T : X^3 \to X$ a mapping. For $(x_0, y_0, z_0) \in X^3$, the sequence $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ defined by

$$(3.35) \qquad x_{n+1} = T(x_n, y_n, z_n), \ y_{n+1} = T(y_n, x_n, y_n), \ z_{n+1} = T(z_n, y_n, x_n),$$

with $n = 0, 1, 2, ...$, is said to be a *tripled fixed point iterative procedure.*

We give the following definition of stability with respect to T, in metric spaces, relative to tripled fixed points iterative procedures:

Definition 3.21. (Timiş, [101]) *Let (X, d) be a complete metric space and a maping $T : X^3 \to X$, with*

$$Fix_t(T) = \left\{ (x^*, y^*, z^*) \in X^3 \ | \ T(x^*, y^*, z^*) = x^*, \ T(y^*, x^*, y^*) = y^*, \right.$$

$T(z^*, y^*, x^*) = z^*\}$, *the set of tripled fixed points of T.*

Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the sequence generated by the iterative procedure defined by (3.35), where $(x_0, y_0, z_0) \in X^3$ is the initial value, which is supposed to converge to a tripled fixed point (x^*, y^*, z^*) of T.

Let $\{(u_n, v_n, w_n)\}_{n=0}^{\infty} \subset X^3$ an arbitrary sequence and set

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \quad \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right), \quad n = 0, 1, 2, ...$$

Then, the tripled fixed point iterative procedure defined by (3.35) is T-stable or stable with respect to T, if and only if

$$\lim_{n \to \infty} (\epsilon_n, \delta_n, \gamma_n) = 0_{\mathbb{R}^3} \text{ implies that } \lim_{n \to \infty} (u_n, v_n, w_n) = (x^*, y^*, z^*).$$

Theorem 3.21. (**Timiş**, [**101**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.

Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on X and satisfying (3.27).

If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.35).

Then, the tripled fixed point iterative procedure is T-stable or stable with respect to T.

Remark 3.23. Theorem 3.21 completes the existence theorem of tripled fixed points of Berinde and Borcut [**26**] with the stability result for the tripled fixed point iterative procedures, using mixed-monotone operators.

Corollary 3.12. (**Timiş**, [**101**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.

Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on X.

There exists $\kappa \in [0,1)$, such that for each $x, y, z, u, v, w \in X$, T satisfies the following contraction condition:

$$(3.36) \qquad d(T(x,y,z), T(u,v,w)) \leq \frac{\kappa}{3} \left\{ d(x,u) + d(y,v) + d(z,w) \right\}.$$

If there exists x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x,y,z), $(x_1, y_1, z_1) \in X^3$, there exists $(u,v,w) \in X^3$ which is comparable to (x,y,z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.35).

Then, the tripled fixed point iterative procedure is T-stable or stable with respect to T.

Remark 3.24. *Corollary 3.12 completes the existence theorem of tripled fixed points of Berinde and Borcut [26] with the stability result for the tripled fixed point iterative procedures, using mixed-monotone operators.*

Theorem 3.22. *(Timiş, [101]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on X and satisfying (3.28), (3.29) and (3.30).

If there exists x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x,y,z), $(x_1, y_1, z_1) \in X^3$, there exists $(u,v,w) \in X^3$ which is comparable to (x,y,z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.35).

Then, the tripled fixed point iterative procedure is T-stable or stable with respect to T.

Theorem 3.23. *(**Timiş**, [**101**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space.*

Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on X and satisfying (3.31), (3.32) and (3.33).

If there exists x_0, y_0, $z_0 \in X$ such that

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad and \quad z_0 \leq T(z_0, y_0, x_0),$$

then there exists $x^*, y^*, z^* \in X$ such that

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad and \quad z^* = T(z^*, y^*, x^*).$$

Assume that for every (x, y, z), $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to (x, y, z) and (x_1, y_1, z_1).

For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.35).

Then, the tripled fixed point iterative procedure is T-stable or stable with respect to T.

4. Illustrative example

Example 4.29. *(**Timiş**, [**101**])*

Let (X, d) be a complete metric space, where $X = \mathbb{R}$, $d(x, y) = |x - y|$ and a continuous and mixed monotone mapping $T : \mathbb{R}^3 \to \mathbb{R}$, with $T(x, y, z) = \frac{2x - 2y + 2z + 1}{12}$.

Berinde and Borcut [**26**] proved the existence and the uniqueness of the tripled fixed point of T, respectively $(x^*, y^*, z^*) = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$, using $(x_0, y_0, z_0) = \left(\frac{1}{20}, \frac{1}{5}, \frac{1}{20}\right)$.

For $k = \frac{1}{2}$, T satisfies the contraction condition (3.36), i.e.,

$$d(T(x, y, z), T(u, v, w)) \leq \frac{\kappa}{3} \left[d(x, u) + d(y, v) + d(z, w)\right],$$

for each $x, y, z, u, v, w \in X$, with $x \geq u$, $y \leq v$ and $z \geq w$.

Applying Corollary 3.12, we obtained the stability of the tripled fixed point iteration procedure defined by (2.26) with respect to T.

CHAPTER 6

Conclusions

Fixed point theory has an important role in the nonlinear analysis domain, with an expansive evolution in the last decades and with many concrete results.

Following the basic result from the metrical fixed point theory, respectively the Contraction Principle of Picard-Banach-Caccioppoli [11], an important part of the scientific literature appeared, with applications to functional equations, differential equations, integral equations etc.

In order to solve a nonlinear equation, we appeal to approximating fixed points of a corresponding contractive type mappings. From the existing methods for approximating fixed points, we studied the Picard iteration and the Jungck type iteration procedure.

Establishing the stability of these methods is very important in practical applications, because a fixed point iteration which is numerically stable will produce small modifications on the approximate value of the fixed point during the computational process.

The concept of stability of a fixed point iteration procedure has been systematically studied by Harder [44], Harder and Hicks [45], [46], and since then, many other stability results for several fixed point iteration procedures and for various classes of nonlinear operators were obtained.

In this paper, we treat the problem of stability of fixed point, common fixed point, coincidence point and tripled fixed point iteration procedures, for certain class of mappings.

In the chapter named **Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition**, we present the concept of stability of fixed point iteration procedures and we survey the most significant contributions to this area.

Berinde [**22**] introduced a natural concept of stability, called *weak stability*, and we transposed this notion to the case of two mappings S and T with a coincidence point, named (S, T)-*weak stability*.

We established weak stability results for common fixed points iteration procedures, on the metric space (X, d), with $Y \subset X$ and $S, T : Y \to X$ two nonself mappings with a coincidence point, satisfying the following contraction condition:

(i) $d(Tx, Ty) \leq qd(Sx, Sy)$, for all $x, y \in Y$ and $q \in (0, 1)$;

(ii) $d(Tx, Ty) \leq qd(Sx, Sy) + Ld(Sx, Tx)$, for all $x, y \in Y$, $q \in (0, 1)$ and $L \geq 0$.

Because some fixed point iteration procedures are not weakly stable and because the stability can be obtained in the meaning of a new concept, we developed a weaker notion, named w^2-*stability*.

Therefore, we gave some stability results on a complete metric space (X, d) and using a mapping $T : X \to X$ satisfying the following contractive conditions:

(1) $d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}$;

(2) $d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y)\}$;

(3) $d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y), d(x, Ty), d(y, Tx)\}$;

(4) $d(Tx, Ty) < \max\left\{d(x, Tx), d(y, Ty), d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$;

for all $x, y \in X$ and $x \neq y$.

Moreover, we gave stability results on a complete metric space (X, d), using two mappings $S, T : X \to X$ with a coincidence point and satisfying the following contractive conditions:

(1) $d(Tx, Ty) < \max\{d(Sx, Tx), d(Sy, Ty)\}$;

(2) $d(Tx, Ty) < \max\{d(Sx, Ty), d(Sy, Tx)\}$;

(3) $d(Tx, Ty) < \max\{d(Sx, Tx), d(Sy, Ty), d(Sx, Sy)\}$;

(4) $d(Tx, Ty) < \max\{d(Sx, Tx), d(Sy, Ty), d(Sx, Sy), d(Sx, Ty), d(Sy, Tx)\}$;

(5) $d(Tx, Ty) < \max\left\{d(Sx, Tx), d(Sy, Ty), d(Sx, Sy), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}$;

for all $x, y \in X$ and $x \neq y$.

We also presented some examples of weak stable, w²-stable but nor stable iterations with respect to T and with respect to (S, T).

Our research can be extended by using other iterative methods, e.g. Ishikawa, Mann, or another contractive conditions.

On the chapter entitled **Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined by implicit relations**, we continued to study the stability of Picard iterative procedure and also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Using the set of all continuous real functions $\mathbb{F}$ introduced by Popa [**70**], $F : \mathbb{R}_+^5 \to \mathbb{R}$, with the following conditions:

(1) F is continuous in each coordinate variable,
(2) there exists $h \in [0, 1)$ such that, for all $u, v, w \geq 0$ satisfying
  - $F(u, v, u, v, w) \leq 0$ or
  - $F(u, v, v, u, w) \leq 0$,

  we have that $u \leq h \max \{v, w\}$,

we established a general stability result for the Picard iteration procedure, on the complete metric space (X, d), for a mapping $T : X \to X$, with $Fix(X) \neq \emptyset$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F \left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2} \right) \leq 0.$$

We also gave a stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings satisfying (E.A) property and defined by an implicit contraction condition on the complete metric space (X, d), $S, T : X \to X$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F \left(d(Tx, Ty), d(Sx, Sy), d(Sx, Ty), d(Sy, Tx), \frac{d(Sx, Tx) + d(Sy, Ty)}{2} \right) \leq 0.$$

On the other hand, using the set of all continuous real functions $\mathbb{F}$ introduced by Popa [**71**], [**72**], $F : \mathbb{R}_+^6 \to R_+$, with the following conditions:

(1) (a) F is non-increasing in the fifth variable and $F(u, v, v, u, u+v, 0) \leq 0$
        for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

    (b) F is non-increasing in the fourth variable and $F(u, v, 0, u+v, u, v) \leq 0$
        for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

    (c) F is non-increasing in the third variable and $F(u, v, u+v, 0, v, u) \leq 0$
        for $u, v \geq 0 \implies \exists h \in [0, 1)$ such that $u \geq hv$;

(2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$,

we also established a stability result for common fixed point iterative procedures, on the complete metric space (X, d), for two mappings $S, T : X \to X$, with $Fix(X) \neq \emptyset$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F\left(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\right) \leq 0.$$

Our research can be extended by using another iterative methods, e.g. Ishikawa, Mann, another contractive conditions, or by modifying the number of the parameters.

The idea of the chapter **A new point of view on the stability of fixed point iterative procedures** was due to Professor I. A. Rus [**81**], who unified the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis by new ones.

By considering these new notions, we gave some stability result for Picard iteration procedure for mappings which satisfy certain contractive conditions.

We studied the relationship between the two stability definitions, the one of Harder [**44**] and the other one due to Rus [**81**].

We gave stability results on the metric space (X, d), for self mappings $T : X \to X$ satisfying the following contraction conditions:

(1) $d(Tx, Ty) \leq \delta_u d(x, y) + L_u d(x, Tx)$, $\delta_u \in [0, 1)$, $L_u \geq 0$;
(2) $d(Tx, Ty) \leq \delta d(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, $\delta \in (0, 1)$, $L \geq 0$;

for all $x, y \in X$.

We also presented some examples of mappings satisfying certain contractive conditions for which the associated Picard iteration was not stable in the sense of Harder but it was actually stable in the sense of Rus.

On the other hand, we transposed the notion of stability introduced by Rus [**81**] to common fixed points and we studied the relationship between the stability concept introduced by Singh and Prasad [**88**] given for a pair of mappings (S, T) with a coincidence point and our new stability concept.

We gave some stability results for the Jungck-type iteration procedure, with respect to two mappings which satisfy the following contractive conditions:

(1) $d(Tx, Ty) \leq ad(Sx, Sy)$, $a \in [0, 1)$;

(2) $d(Tx, Ty) \leq h \max \{d(Sx, Ty), d(Sy, Tx)\}$, $h \in [0, 1)$;

for each $x, y \in X$.

Our research can be extended by using another iterative methods, e.g. Ishikawa, Mann, or another contractive conditions.

An open problem is the study of the stability in the sense of Rus for general nonexpansive mappings as well as for general almost contractions, that do not satisfy a certain uniqueness condition.

In the chapter **Stability of tripled fixed point iteration procedures**, we introduced the concept of stability for tripled fixed point iterative procedures and also established stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions by extension from coupled fixed points to tripled fixed points of contractive conditions employed by Olatinwo [**60**].

We established stability results for the tripled fixed point iteration procedure, on the metric space (X, d), for mappings $T : X^3 \to X$, in the case of the monotone property and also in the case of the mixed monotone property of T, satisfying the following contraction conditions:

(1) for $k \geq 0$, $\mu \geq 0$, $\rho \geq 0$, $k + \mu + \rho < 1$,

$$d(T(x, y, z), T(u, v, w)) \leq kd(x, u) + \mu d(y, v) + \rho d(z, w);$$

(2) for $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$,

$$d(T(x, y, z), T(u, v, w)) \leq a_1 d(T(x, y, z), x) + b_1 d(T(u, v, w), u);$$

$$d(T(y, x, z), T(v, u, w)) \leq a_2 d(T(y, x, z), y) + b_2 d(T(v, u, w), v);$$

$$d(T(w, y, x), T(z, v, u)) \leq a_3 d(T(z, y, x), z) + b_3 d(T(w, v, u), w);$$

(3) for $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$,

$$d(T(x, y, z), T(u, v, w)) \leq a_1 d(T(x, y, z), u) + b_1 d(T(u, v, w), x);$$

$$d\left(T(y,x,z),T(v,u,w)\right) \le a_2 d\left(T(y,x,z),v\right) + b_2 d\left(T(v,u,w),y\right);$$

$$d\left(T(w,y,x),T(z,v,u)\right) \le a_3 d\left(T(z,y,x),w\right) + b_3 d\left(T(w,v,u),z\right);$$

$\forall x,y,z,u,v,w \in X.$

Moreover, we have illustrated these results with an example.

Our research can be extended by using some other contractive conditions, for mappings satisfying various properties.

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[91] **Timiş, I.**, *New stability of Picard iteration for mappings defined by implicit relations* (in preparation)

[92] **Timiş, I.**, *New stability results of Picard iteration for common fixed points and contractive type mappings* (in preparation)

[93] **Timiş, I.**, *New stability results of Picard iteration for contractive type mappings* (submitted)

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[100] **Timiş, I.**, *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. **19** (2012), no. 4, 37-44

[101] **Timiş, I.**, *Stability of tripled fixed point iteration procedures for mixed monotone mappings* (submitted)

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1. Addend: Published and Communicated Research Papers

This thesis is developed on the basis of the following published and communicated papers:

I. List of published research papers:

1. Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. 37 (2) (2010), 106-114

2. Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems*, International Journal of Computer Applications 37 (4) (2012), 9-13

3. Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567

4. Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44

5. Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7

6. Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95

II. List of communicated research papers:

1. Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012

2. Timiş, I., *On the weak stability of fixed point iterative methods*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010

3. Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011

III. List of submitted research papers:

1. Timiş, I., *New stability results of Picard iteration for contractive type mappings*

2. Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings satisfying an implicit relation*

3. Timiş, I., *Stability of tripled fixed point iteration procedures for mixed monotone mappings*