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A FIXED POINT CHARACTERIZATION OF THE POINTS OF DISCONTINUITY OF A DERIVATIVE

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§1. Preliminaries

Let $[a,b]$ be a compact interval in $\mathbb{R}$ with $a<b$. It is wellknown that a continuous function $f:[a,b]\to[a,b]$ has a fixed point in $[a,b]$ and that the continuity of f is not necessary for the validity of this assertion.

In order to obtain an extension of this result, the continuity condition of f may be replaced by a weaker condition, as in [1] where the following is given:

THEOREM 1.1. If $f:[a,b]\to[a,b]$ is a Derboux function in the first class of Baire, then f has a fixed point in $[a,b]$.

On the other hand, if a real valued function f defined on an interval in $\mathbb{R}$ is a derivative, then f is a Darboux function. Starting from this fact, the aim of this note is to emphasize the relation between the primitivity and the fixed point property for a class of discontinuous functions.

To this end we need some definitions and theorems which appear in the paper quoted above, but we present them for the convenience of the reader.

DEFINITION 1.1. A real valued function f defined on an interval I in $\mathbb{R}$ is said to be a Darboux function if whenever a and b are in I, $a<b$, and y is any number between f(a) and f(b), there is a $x\in[a,b]$, such that $f(x) = y$.

DEFINITION 1.2. A real valued function f defined on an interval I in $\mathbb{R}$ is said to be in the first class of Baire if there exists a sequence of continuous functions $f_n:I\to\mathbb{R}$ such as $\lim_{n\to\infty} f_n(x)=f(x)$ for each $x\in I$.

THEOREM 1.2. If the set of points of discontinuity of a real valued function on an interval in $\mathbb{R}$ is at most countable, then that function is in the first class of Baire (see [1]).

§2. Discontinuous derivatives

Let be I and J two intervals in $\mathbb{R}$, $J\subseteq I$, such that

$$x\in J,\ x\neq 0 \Rightarrow \frac{1}{x}\in J$$

Now consider the following three functions

$$p,r : I\to\mathbb{R}, \text{ and } q : I\to J$$

and denote

$$s(x) = r(x).q^2(x), \text{ for each } x\in I.$$

THEOREM 2.1. Suppose that

i) The set $A = \{x\in I \,/\, q(x) = 0\}$ is at most countable;

ii) p,q and r are differentiable on I;

iii) p is bounded on I;

iv) $\lim_{x \to a} s(x)/(x-a) = 0$, for every $a \in A$;

v) $\lim_{x \to a} s'(x) = 0$, for every $a \in A$.

Then the function $f: I \to \mathbb{R}$, defined by

$$f(x)=\begin{cases} s(x).q'(x).p'(\frac{1}{q(x)}), & \text{if } x \in I \setminus A \\ 0, & \text{if } x \in A, \end{cases} \qquad (1)$$

admits a primitive (is a derivative) on I.

Proof. From the conditions iii) and v) it follows that the function $g: I \to \mathbb{R}$, given by

$$g(x)=\begin{cases} s'(x).p(\frac{1}{q(x)}), & \text{if } x \in I \setminus A \\ 0, & \text{if } x \in A \end{cases} \qquad (2)$$

is continuous. Hence g is a derivative. Let be G a primitive of g on I. We shall prove that the function

$$F(x)=\begin{cases} G(x) - s(x).\ p(\frac{1}{q(x)}), & \text{if } x \in I \setminus A \\ G(x), & \text{if } x \in A, \end{cases} \qquad (3)$$

is a primitive of f on I. Indeed, from the condition ii) it results that F is differentiable on $I \setminus A$ and, by a simple calculation, we obtain

$$F'(x) = f(x), \text{ for each } x \in I \setminus A.$$

When $a \in A$, we have

$$F'(a)= \lim_{x \to a} \frac{F(x)-F(a)}{x-a} = \lim_{x \to a} \frac{G(x)-G(a)}{x-a} - \lim_{x \to a} \frac{s(x)p(\frac{1}{q(x)})}{x-a}$$

According to conditions iii) and iv), it follows that

$$F'(a)= \lim_{x \to a} \frac{G(x)-G(a)}{x-a} = G'(a) = 0 = f(a),$$

wich completes the proof.

Using the theorem 2.1 one can prove the following result:

THEOREM 2.2. If the conditions i) – v) from the theorem 2.1 are satisfied, then the function $f : I \to R$ given by

$$f(x)=\begin{cases} r(x).q'(x).p'\left(\frac{1}{q(x)}\right), & x\in I\setminus A \\ \lambda, & x\in A, \end{cases} \qquad (4)$$

admits a primitive if and only if $\lambda = 0$.

Proof. The sufficiency results from the theorem 2.1. In order to prove the necessity we take into consideration the fact that any primitive F_1 of f has the form:

$$F_1 = F + c, \quad c = \text{const.}$$

where F is given by (3). Then, for each $a \in A$, we have

$$F_1'(a) = F'(a) = 0.$$

Because, from (4)

$$F_1'(a) = f(a) = \lambda,$$

it results that $\lambda = 0$, which finishes the proof.

It is possible to weak the conditions in theorems 2.1 and 2.2. We have

THEOREM 2.3. The conclusion of the theorems 2.1 and 2.2 is preserved if we replace the conditions iii), iv) and v) by the following two

iii)' $\lim\limits_{x\to a} \dfrac{s(x)p\left(\frac{1}{q(x)}\right)}{x-a} = 0$, for each $a \in A$;

iv)' $\lim\limits_{x\to a} s'(x)p\left(\frac{1}{q(x)}\right) = 0$, for each $a \in A$.

REMARKS. 1. The proof of theorem 2.1 shows that r may be defined only on $I\setminus A$. In this case, the condition ii) relative to r must be accordingly modified;

2. In the conditions of the theorem 2.1, the function f given by (1) is a derivative, hence it is a Darboux function. Because the set of points of discontinuity of f is a

subset of A, which is at most countable, it follows from theorem 1.2 that f is a Darboux function in the first class of Baire.

But the range of f, f(I), is generally not included in I, hence theorem 1.1 does not apply.

3. For a Darboux function, the points of discontinuity are necessarly of the second kind, except, among them, the points of infinite discontinuity, [2].

Hence, if there exists no at least one of the one-sided limits of the function f given by (1) at the point $a \in A$, then f is discontinuous at the point a.

§3. Characterizations of the points which are fixed points

In this section we present, using the theorems 2.2 and 2.3, the relationship which exists between the primitivity, by a hand, and the points of discontinuity which are fixed points, on the other hand, for a discontinuous function of the form (5).

THEOREM 3.1. In the conditions of the theorem 2.2, a point $x_0 \in A$ is a fixed point of the function $f: I \to R$, given by

$$f(x) = \begin{cases} r(x)q'(x)p'\left(\frac{1}{q(x)}\right) + x_0, & x \in I \setminus A \\ \lambda, & x \in A \end{cases} \quad (5)$$

if and only if f admits a primitive (that is $\lambda = x_0$)

On the base of the theorem 2.3 a similar result can be proved. It is contained in

THEOREM 3.2. In the conditions of the theorem 2.3, a point $x_0 \in A$ is a fixed point of the function f given by (5),

<u>if and only if f admits a primitive (which is equivalent whit the fact that $\lambda = x_0$).</u>

<u>EXAMPLE.</u> The function $f: [0, +\infty) \to R$, defined by

$$f(x)=\begin{cases} \ln x.\sin \frac{1}{x}, & x > 0 \\ \lambda , & x = 0 \end{cases}$$

is discontinuous at the point $x = 0$.

The conditions of the theorem 2.2 are satisfied, hence $x = 0$ is a fixed point of f if and only if f is a derivatives, i.e. $\lambda = 0$.

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