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THE STABILITY OF FIXED POINTS FOR A CLASS OF φ - CONTRACTIONS

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In the paper [1] has been studied the comparison functions $\varphi : R_+ \to R_+$ which fulfil the following condition

(c) There exist the numbers k_0 and α, $0 < \alpha < 1$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} a_k$, such that

$$\varphi^{k+1}(r) \le \alpha\left[\varphi^k(r)+a_k\right] \text{ , for each } k \ge k_0,\ (\forall)\ r \in R_+ .$$

The aim of the present paper is to show the relationship between the strict comparison functions and the (c)-comparison functions and to prove some results, established for the strict φ - contractions in [4] , which remains valid for the φ- contractions with φ (c) - comparison function.

We refer to [4] for the definition and basic properties of comparison functions.

DEFINITION 1 ([4]). A function $\varphi : R_+ \to R_+$ is called comparison function if satisfies the following conditions

(i) φ is monotone increasing ;

(ii) $(\varphi^n(t))_{n \in N}$ converges to o, for all $t > 0$.

A comparison function φ is called strict comparison function if, in addition, φ satisfies the conditions

(iii) is continuous ;

(iv) $t - \varphi(t) \to +\infty$ as $t \to \infty$.

DEFINITION 2.

A function $\varphi : R_+ \to R_+$ is called (c) - comparison function if φ is monotone increasing and satisfies the condition (c).

The following lemma is given in [1]

LEMMA 1.

If $\varphi : R_+ \to R_+$ is a(c) - comparison function then φ is a comparison function.

REMARK 1.

The condition (c) is, in fact, a necesary and sufficient condition for the convergence of the series of decreasing terms

$$\sum_{k=1}^{\infty} \varphi^k(t), \quad \text{for all} \quad t \in R_+ \tag{1}$$

This result is proved in [1] using a generalization of the ratio (or D'Alembert's) test for the series of positive terms.

For convenience we denote by F, F_s, F_c and F_{cs} , respectively, the set of all comparison functions, of all strict comparison functions, of all (c) - comparison functions and of all subadditive (c) - comparison functions.

EXAMPLE 1.

If $\varphi : R_+ \to R_+$, $\varphi(t) = \frac{t}{t+1}$, $t \in R_+$, we have $\varphi \in F$ and $\varphi \in F_s$ (see [3] , example 3.1.2.) but $\varphi \notin F_c$, because the series

$$\sum_{k=1}^{\infty} \varphi^k(1) = \sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{diverges .}$$

EXAMPLE 2.

For the function $\varphi : R_+ \to R_+$, $\varphi(t) = 0$ if $t \in [0,1)$ and $\varphi(t) = \frac{1}{2} t$, if $t \geq 1$, we have $\varphi \in F_c$, and $\varphi \notin F_s$, because φ is not continuous.

EXAMPLE 3.

Let $\varphi : R_+ \to R_+$, $\varphi(t) = at$, $t \in R_+$ and $0 < a < 1$, be a function. Then $\varphi \in F_s$ and $\varphi \in F_c$.

From this considerations we obtain

LEMMA 2.

i) $F_s \subset F$ and $F_c \subset F$;

ii) $F_s \cap F_c \neq \emptyset$;

iii) $F_s \setminus F_c \neq \emptyset$ and $F_c \setminus F_s \neq \emptyset$.

DEFINITION 3.

A function $\varphi: \mathbb{R} \to \mathbb{R}$ is called subadditive if

$$\varphi(t_1+t_2) \leq \varphi(t_1) + \varphi(t_2), \ (\forall)\ t_1,t_2 \in \mathbb{R} .$$

EXAMPLE 4.

For the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{1}{2} t^2 + \frac{3}{10} t$, if $t \in [0,1]$ and $\varphi(t) = \frac{1}{2} t$, if $t > 1$, we have $\varphi \in F_s$ and $\varphi \notin F_{cs}$, because

$$\frac{1}{2} = \varphi(1) > 2\varphi(\tfrac{1}{2}) = \frac{2}{5} .$$

EXAMPLE 5.

For the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = -\frac{2}{5} t^2 + \frac{9}{10} t$, if $t \in [0,1]$ and $\varphi(t) = \frac{2}{3}$, if $t \geq 1$, we have $\varphi \in F_{cs}$ and $\varphi \notin F_s$.

Indeed, from the inequality

$$\varphi^k(t) < (\tfrac{9}{10})^k t, \quad n \in \mathbb{N} \text{ and } t \in [0,1)$$

it results that the series (2) converges uniformly, hence, it is pointwise convergent, which implies that $\varphi \in F_c$ and $\varphi \in F_{cs}$, but $\varphi \notin F_s$ because φ is not continuous.

REMARK 4.

The comparison function from example 1 is subadditive, $\varphi \in F_s$ but $\varphi \notin F_{cs}$, because $\varphi \notin F_c$.

EXAMPLE 6. ([3]), Example 3.1.3.)

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function given by $\varphi(t) = \frac{1}{2}t$, for $t \in [0,1)$ and $\varphi(t) = t - \frac{1}{2}$, if $t > 1$. Then

1) $\varphi \in F$ and $\varphi \in F_c$ but $\varphi \notin F_{cs}$ (φ is not subadditive) ;

2) $\varphi \notin F_s$, because $t - \varphi(t) = \frac{1}{2}$, for $t > 1$.

From the preceding examples and remarks it follows

LEMMA 3.

i) $F_s \cap F_{cs} \neq \emptyset$;

ii) $F_s \setminus F_{cs} \neq \emptyset$ and $F_{cs} \setminus F_s \neq \emptyset$.

Other results concerning some bundles between the comparison functions and the rates of convergence (a (c) - comparison function is a rate of convergence) can be found in the recent paper [2] .

To prove the main result of this paper we need some definitions and lemmas.

DEFINITION 4. ([4])

Let (X,d) be a matric space. A mapping $f : X \to X$ is called (strict) φ- contraction if and only if $\varphi \in F$ (respectively $\varphi \in F_s$) and

$$d(f(x),f(y)) \leq \varphi(d(x,y)), \quad (\forall)\ x,y \in X . \tag{2}$$

LEMMA 4.

If $\varphi \in F$ then φ is continuous at 0 .

Proof.

From (i) and (ii) it results $\varphi(t) < t$, $(\forall)$ $t > 0$ and $\varphi(0) = 0$ (see [4] , chapter III).

From $\varphi(t) \geq 0$, we obtain $\lim\limits_{t \to 0} \varphi(t) \geq 0$,

and from $\varphi(t) < t$, $\lim\limits_{t \to 0} \varphi(t) \leq 0$, hence $\lim\limits_{t \to 0} \varphi(t) = \varphi(0)$,

as claimed. _

LEMMA 5.

If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c) - comparison function, and

$$s(t) = \sum_{k=1}^{\infty} \varphi^k(t) , \quad t \in \mathbb{R}_+$$

then s is continuous at zero.

Proof.

Let $a > 0$ a fixed number. From the monotonocity of φ , we obtain

$$\varphi^{k}(t) \leq \varphi^{k}(a), \quad (\forall) \quad t \in [0,a]$$

hence the series (1) converges uniformly on $[0,a]$.

Since φ is continuous in zero, it is obvious that s is continuous in zero and the proof is completed.

The continuous dependence of the fixed point of the strict φ - contractions is given by

THEOREM 1. ([4]).

Let (X,d) be a complete matric space and $f,g : X \rightarrow X$ two mappings. We suppose

(u) f is a strict φ - contraction,

(v) $x_g^* \in F_g$,

(w) there exist $\eta > 0$ so that

$d(f(x), g(x)) \leq \eta$, for all $x \in X$.

Then

$$d(x_g^*, x_f^*) \leq t_\eta$$

where $F_f = \{x_f^*\}$ and $t_\eta = \sup\{ t \,/\, t - \varphi(t) \leq \eta \}$.

In the proof of Theorem 3 we shall use the following statement of [1] , which is a generalization of the Theorem 3.2.1. from [3] .

THEOREM 2.

Let (X,d) be a complete matric space, $f : X \rightarrow X$ a mapping satisfying (2) with $\varphi \in F_c$.

Then

1) $F_f = \{x^*\}$;

2) The sequence of succesive approximations, $(x_n)_{n \in N}$, $x_n = f(x_{n-1})$, $n \geq 1$, converges to x^* , for every $x_0 \in X$;

3) We have

$$d(x_n, x^*) \leq s(d(x_0,x_1)) - S_{n-1}(d(x_0,x_1)) ,$$

where $s(t)$, $S_{n-1}(t)$ denote the sum, respectivaly, the partial sum of rank $n-1$, of the series (1) ;

4) If $\varphi \in F_{cs}$ and $g : X \to X$ is a mapping satisfying (w), then

$$d(y_n, x) \leq \eta + s(\eta) + s(d(x_0, x_1)) - S_{n-1}(d(x_0, x_1)) ,$$

where $y_n = g^n(x_0)$.

We are now able to prove

THEOREM 3.

Let (X,d) be a complete metric space and $f, g : X \to X$ two mappings. If f is a φ- contraction with $\varphi \in F_{cs}$ and the assumptions (v), (w) are satisfied,

then

$$d(x_f^*, x_g^*) \leq \eta + s(\eta) ,$$

where $F_f = \{x_f^*\}$ and $s(\eta)$ is the sum of the series $\sum_{k=1}^{\infty} \varphi^k(\eta)$.

Proof.

We apply theorem 2, with $x_0 = x_g^*$, to obtain

$$d(x_f^*, x_g^*) \leq \eta + s(\eta) + s(r_0) - S_{n-1}(r_0) ,$$

where $r_0 = d(x_g^*, f(x_g^*))$.

Now it is enough to take $n \to \infty$ and the proof, is completed.

REMARK 5.

1) From the continuity of s we have

$$\lim_{\eta \to 0} (\eta + s(\eta)) = 0 ;$$

2) If $\varphi(t) = at$, $0 < a < 1$, than (see [4] , example 7.1.1.)

$$\varphi(t) = \eta + s(\eta) = \frac{\eta}{1 - a} ;$$

3) If $\varphi(t) = \frac{t}{1 + t}$, then (see [4] , exemple 7.1.2) the Theorem 1 give an estimation with

$$t_\eta = \frac{1}{2}(\eta + \sqrt{\eta^2 + 4}) ,$$

but theorem 3 fails, because $\varphi \notin F_c$, and consequently , $\varphi \notin$

F_{cs} .

4) If we take φ as in example 5, then Theorem 3 applies, but Theorem 1 not.

An analogous to Theorem 7.2.1. from [4] holds.

THEOREM 4.

Let (X,d) be a complete metric space and $f_n, f : X \to X$, $n \in N$, such as :

1) f_n is a φ - contraction with $\varphi \in F_{cs}$, for all $n \in N$;

2) $(f_n)_{n \in N}$ converges pointwise to f .

Then f is a φ - contractions with $\varphi \in F_{cs}$ and $(x_n)_{n \in N}$ converges to x^*,

where $F_{f_n} = \{x_n^*\}$ and $F_f = \{x^*\}$.

Proof.

Using 1) and 2), from the inequality

$$d(f(x),f(y)) \leq d(f(x),f_n(x)) + d(f_n(y),f_n(y)) + d(f_n(y),f(y)),$$

we obtain, latting $n \to \infty$, that f is a φ - contraction, with $\varphi \in F_{cs}$.

Since $\varphi \in F_{cs}$, from

$d(x_n^*,x^*) = d(f_n(x_n^*),f(x^*)) \leq \varphi(d(x_n^*,x^*)) + d(f_n(x^*), f(x^*))$, it follows

$d(x_n^*,x^*) \leq \varphi^{k+1}(d(x_n^*,x^*)) + \sum_{i=0}^{k} \varphi^i(d(f_n(x^*),f(x^*)))$, $k \geq 0$.

which, together with (ii), yields, letting $k \to \infty$

$$d(x_n^*,x^*) \leq r_0 + s(r_0) ,$$

where $r_0 = d(f_n(x^*) , f(x^*))$.

From (2) we now obtain $\lim_{n \to \infty} r_0 = 0$ and, from lemma 5 ,

$\lim_{n \to \infty} s(r_0) = 0$,

hence

$$\lim_{n \to \infty} d(x_n^*, x^*) = 0$$

REMARK 6.

The theorem 7.1.2. and 7.2.3. [4] remain also valid if we replace the condition $\varphi \in F_s$ with the condition $\varphi \in F_{cs}$.

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