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A FIXED POINT THEOREM OF MAIA TYPE IN K - METRIC SPACES

by

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1. Introduction

Banach's fixed point principle in metric spaces was generalized in [11] to generalized metric spaces as follow (see also [13])

THEOREM A. (A.I. Perov's theorem)

Let (X,d) be a complete generalized metric spaces with $d : X \times X \longrightarrow R^r$ and let $T : X \longrightarrow X$ be a mapping such that

(A 1) $\quad d(Tx,Ty) \leq A\, d(x,y), \quad \forall\, x,y \in X$,

where $A \in \mathcal{M}_{r,r}(R_+)$ and

(A 2) $\quad A^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Then T has a unique fixed point x^* and $T^n x_0 \longrightarrow x^*$ as $n \longrightarrow \infty$ for each $x_0 \in X$ (here T^n stands for the n-th iterate of T).

Moreover,
$$d(T^n x_0, x^*) \leq A^n (I-A)^{-1} d(x_0, Tx_0) .$$

Also, the same fixed point principle was generalized in [8] to sets endowed with two metrics :

THEOREM B. (Maia's theorem)

Let X be a nonempty set, d and d' two metrics on X and $T : X \longrightarrow X$ a mapping.

Suppose that

(B 1) $d'(x,y) \le d(x,y)$, for $x,y \in X$;

(B 2) (X,d') is a complete metric space;

(B 3) $T : (X,d') \longrightarrow (X,d)$ is continuous;

(B 4) $d(Tx,Ty) \le a d(x,y)$, $(x,y \in X)$, for a certain $a \in [0,1)$.

Then T has a unique fixed point x^* and

$T^n x_0 \xrightarrow{d'} x^*$ as $n \longrightarrow \infty$, for any $x_0 \in X$.

This theorem remains true if condition (B 4) is replaced by

(B 5) $d'(T^k x, T^k y) \le a\, d(x,y)$, $(x,y \in X)$,

for a certain $k \in N$ and $a > 0$ (see [15] , Remark 2.3.1., and [13]).

In [18] and [19] , B. Rzepecki using the idea of [13] gives some another fixed point theorems of Maia type.

Then, taking into consideration a set endowed with two generalized metrics, in [3] is obtained the following result

<u>THEOREM C</u>

Let X be a set endowed with the generalized metrics

$$d,\ d' : X \times X \longrightarrow R^p .$$

If the conditions are fulfilled

(C 1) $d'(x,y) \le d(x,y)$, for $x,y \in X$;

(C 2) (X,d') is a generalized complete metric space;

(C 3) $T : (X,d') \longrightarrow (X,d')$ is continuous;

(C 4) There exist a matrix convergent toward 0

$$A \in \mathcal{M}_{p,p}(R) ,$$

such that

$$d(Tx,Ty) \le A\, d(x,y) , \quad x,y \in X ,$$

then T has a unique fixed point x^*.

We have the estimation

$$d(T^n x_0, x^*) \le A^n (I-A)^{-1} d(x_0, Tx_0) .$$

The basic problem of the fixed point theory for theorem C is solved in [10] .

The purpose of this paper is to extend theorem C to K - metric spaces, using a result from [4] which generalize theorem A.

2. Generalized φ - contractions

Referring to generalized φ - contractions we shall follow, both in terminology and notation, the paper [17] .

We need some definitions and results from [17] , [4] , [6] .

DEFINITION 2.1.

Let $(Y, \|\cdot\|)$ be a real Banach space. A set $K \subset Y$ is called a cone if

(i) K is closed

(ii) $x, y \in K$ implies $ax + by \in K$ for all $a, b \in R_+$;

(iii) $K \cap (-K) = \{\theta_Y\}$.

Remark.

The cone K induces a reflexive, transitive and antisimetrical order relation in Y , by

$$x \leq y \iff y - x \in K ,$$

related to the linear structure by the properties

$$u \leq v \quad \text{implies} \quad u + z \leq v + z , \ \forall z \in Y$$

and

$$u \leq v \quad \text{implies} \quad tu \leq tv , \ \forall t \in R_+ \quad .$$

The space Y with this order relation is called an ordered Banach space, while K is termed as its positive cone.

DEFINITION 2.2.

Let K be the positive cone in the ordered Banach space $(Y, \|\cdot\|)$. We say that the norm of Y is monotone if $x, y \in Y$,

$$\theta_Y \leq x \leq y \ \text{implies} \ \|x\| \leq \|y\| .$$

Remark.

If the norm of $(Y, \|\cdot\|)$ is monoton then K is a normal cone.

Throughout this paper K will be the positive cone in a real ordered Banach space $(Y, \|\cdot\|)$ with monotone norm.

DEFINITION 2.3.

Let X be a nonempty set. A mapping $d : X \times X \longrightarrow K$ is said to be a K-metric on X if

(i) $d(x,y) \geq \theta_Y$ and $d(x,y) = \theta_Y \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$, $\forall\, x,y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(y,z)$, $\forall\, x,y,z \in X$.

The obtained entity: the nonempty set X with a K-metric d is called K-metric space, denoted as usually by (X,d).

DEFINITION 2.4.

A mapping $\varphi : K \longrightarrow K$ is a comparison function if

(a) φ is monotone increasing;

(b) $(\varphi^n(t))_{n \in N}$ converges to θ_Y , for all $t \in K$.

DEFINITION 2.5.

Let (X,d) be a K-metric space and $\varphi : K \longrightarrow K$ a comparison function. A mapping $T : X \longrightarrow X$ is an abstract φ - contraction iff

$$d(Tx,Ty) \leq \varphi(d(x,y)), \text{ for all } x,y \in X .$$

DEFINITION 2.6.

A mapping $\varphi : K \longrightarrow K$ is called (c)-comparison function if is monotone increasing and fulfils the following convergence condition

(c) There exist two numbers k_0, α , $0 < \alpha < 1$, and a convergent series with nonnegative real terms $\sum_{k=1}^{\infty} a_k$, such that

$$\|\varphi^{k+1}(t)\| \leq \alpha(\|\varphi^k(t)\| + a_k), \text{ for } k \geq k_0, (\forall) t \in K .$$

REMARKS.

1) If φ is a (c)-comparison function then (see [2]) the series $\sum_{k=1}^{\infty} \|\varphi^k(t)\|$ converges for all $t \in K$ and, consequently,

the series
$$\sum_{k=1}^{\infty} \varphi^{k}(t) \qquad (1)$$

converges in T ;

2) Every (e)-comparison function is a comparison function;

3) For every comparison function we have
$$\varphi(t) < t \text{ , for all } t \in E, \ t > \theta_E .$$

hence every φ - contraction on a K-metric space is a continuous mapping.

3. A theorem of Maia type in K-metric spaces

In the paper [4] we have proved a theorem of Perov type in K-metric spaces with K normal cone. In the following theorem we consider two generalized metrics on the K-metric space X.

THEOREM 3.1.
Let d and d' be two K-metrics on X and $T : X \longrightarrow X$ a mapping.

Suppose that

(i) (X,d') is a complete K-metric space;

(ii) $d'(x,y) \le d(x,y)$, $(\forall) x,y \in X$;

(iii) $T : (X,d') \longrightarrow (X,d')$ is continuous;

(iv) $T : (X,d) \longrightarrow (X,d)$ is an abstract φ-contraction with a (e)-comparison function.

Then

1) T has a unique fixed point x^* ;

2) The sequence $(x_n)_{n \in N}$, $x_n = Tx_{n-1}$, $n \ge 1$, converges to x^* , for every $x_0 \in X$;

3) We have $d(x_n,x^*) \le s(d(x_0,x_1)) - s_{n-1}(d(x_0,x_1))$, where $s(t)$, $s_{n-1}(t)$ denote the sum and the partial sum of rank $n-1$, respectively, of the series (1);

4) If, in addition, φ is subadditive and there exist $\eta \in E$, $\eta \neq \theta_E$, and a mapping $U : X \longrightarrow X$ so that

$$d(Tx,Ux) \leq \eta \text{ , for all } x \in X \text{ ,}$$

then

$$d(y_n, x^*) \leq \eta + s(\eta) + s(d(x_0,x_1)) - S_{n-1}(d(x_0,x_1)) \text{ ,}$$

where $y_n = U^n(x_0)$.

<u>Proof.</u>

By (iv) we obtain

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0,x_1))$$

which yields

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} \varphi^k(d(x_0,x_1)) \text{ .}$$

Since φ is a (c)-comparison function, the series

$$\sum_{k=1}^{\infty} \varphi^k(d(x_0,x_1))$$

converges and, consequently, (x_n) is a Cauchy sequence in (X,d) .

By (ii), (x_n) is a Cauchy sequence in (X,d') and by (i), it converges.

Let $x^* = \lim_{n\to\infty} x_n$. Then, by (iii), we obtain

$$x^* = Tx^* \text{ ,}$$

and the fixed point is unique. Indeed, let x_1^*, x_2^* be two fixed points of T, $x_1 \neq x_2$.

We have

$$d(x_1^*, x_2^*) = d(T^n x_1^*, T^n x_2^*) \leq \varphi^n(d(x_1^*, x_2^*)) \text{ .}$$

Now, using the fact that the norm is monotone and

$$\varphi^n(d(x_1^*, x_2^*)) \longrightarrow \theta_E \text{ , as } n \longrightarrow \infty, \text{ it results}$$

$$d(x_1^*, x_2^*) = \theta_E \text{ ,}$$

i.e. the uniqueness of the fixed point of T.

Thus 1) and 2) are proved.

For 3) and 4) we use theorem 1 from [4].

REMARK.

This theorem remains true if condition (ii) is replaced by

(ii)' there exists a real number a, $a > 0$, such that

$$d'(Tx,Ty) \le a\, d(x,y), \quad \text{for all } x,y \in X,$$

see [17].

For the applications of a theorem of Maia type see [20].

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(iii) $F(\lambda,x,y) = F(1-\lambda,y,x)$ for any $x,y \in X$ and $\lambda \in [0,1]$.

(iv) $F(\lambda,x,y) = F(\lambda,x,z)$, $x,y,z \in X$, $\lambda \in [0,1]$ implies that $y = z$.

Definition 4. Let (X,F) be a semi-convex structure. A subset Y of X is called F semi-starshaped if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in [0,1]$, we have $F(\lambda,x,p) \in Y$.

Definition 5. Let (X,F) be a convex structure. A subset Y of X is called:

a) F - starshaped if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in [0,1]$ we have $F(\lambda,x,p) \in Y$.

b) F-convex if for any $u,v \in Y$ and $\lambda \in [0,1]$ we have $F(\lambda,u,v) \in Y$.

Petrusel / 4 / made a remark that for $F(\lambda,u,v) = \lambda u + (1-\lambda)v$, we obtain the known notions of starshaped and convexity from linear spaces. Petrusel also noted with an example that a set can be a F-semi-convex structure without being a convex structure.

2. Main Result

We prove the following theorem:

Theorem 1. Let X be a Banach space with a semi-convex structure F, where the mapping $F:[0,1] \times X \times X \to X$ satisfies the following conditions.

(i) F is ϕ-contractive relative to the second argument, i.e., there exists a mapping $\phi : [0,1[\to [0,1[$ so that,

$$\| F(\lambda, x, p) - F(\lambda, y, p) \| \leq \phi(\lambda) \| x - y \|,$$

for any $x, y, p \in X$ and $\lambda \in [0,1]$.

(ii) F is continuous relative to the first argument.

Let $T: X \longrightarrow X$ be a mapping satisfying the following:

(I) $Tx_0 = x_0$, $x_0 \in X$.

(II) $T(\partial Y) \subseteq Y$, where Y is a subset of X.

(III) The set D, the set of best Y-approximants to x_0 is closed, F-starshaped, and, $T(D)$ is compact.

(IV) T is non-expansive on D and

$$\| Tx - Tx_0 \| \leq \| x - x_0 \| \quad \text{for all } x \in D.$$

Then T has a fixed point in D closest to x_0.

<u>Proof</u>. We claim that $T: D \longrightarrow D$.

Let $y \in D$. Then, by (IV),

$$\| Ty - x_0 \| \leq \| Ty - Tx_0 \| \leq \| y - x_0 \| .$$

By Lemma 1 of Hicks and Humphries / 3 / we get $y \in \partial Y$. Since y is nearest to x_0 and $T(\partial Y) \subset Y$, so Ty is also nearest to x_0, i.e. $Ty \in D$.

We choose $p \in Y$ so that for any $u \in Y$ and $\lambda \in [0,1]$, we get $F(\lambda, u, p) \in Y$ and hence $\in D$.

Let $k_n = 1 - 1/n$ be a sequence of reals such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. We define $T_n: D \longrightarrow D$ by $T_n(x) = F(k_n, T(x), p)$ for all $x \in D$. This is well defined since T is nonexpansive mapping on D and D is F-starshaped.

Now

$$\| T_n(x_1) - T_n(x_2) \| = \| F(k_n, T(x_1), p) - F(k_n, T(x_2), p) \| \leq$$
$$\leq \phi(k_n) \| T(x_1) - Tx_2 \| \leq \phi(k_n) \| x_1 - x_2 \| .$$

so that T_n is a contractive map and has a unique fixed point, say, y_n for each n, i.e. $T_n y_n = y_n$, $n = 1,2,\ldots$.

Since $T(D)$ is compact, therefore, $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ so that $y_{n_i} \longrightarrow y^*$ for $n_i \longrightarrow \infty$.

We claim that $Ty^* = y^*$ in D. Since T is continuous, so $T(y_{n_i}) \longrightarrow T(y^*)$, when $n_i \longrightarrow \infty$.

Again;

$$y_{n_i} = T_{n_i}(y_{n_i}) = F(k_{n_i}, T(y_{n_i}), p).$$

When

$$n_i \longrightarrow \infty, \text{ we have } y_{n_i} \longrightarrow y^* \text{ and}$$

$$F(k_{n_i}, T(y_{n_i}), p) \longrightarrow F(1, T(y^*), p) = T(y^*).$$

Thus,

$$T(y^*) = y^*.$$

<u>Remark</u>: For $T: X \longrightarrow X$ instead of our condition (II) of Theorem 1 and X-starshaped, i.e., $F(\lambda, u, v) = \lambda u + (1-\lambda)v$, we obtain theorem of Singh / 5 / which is a further extension of the result of Brosowski / 1 / who considered X-convex.

Using the same working technique we may also obtain the following theorem:

<u>Theorem 2.</u> Let X be a Banach space, with a semi-convex structure $F: [0,1] \times X \times X \longrightarrow X$, which satisfies the conditions:

(i) F is ϕ-contractive relative to the second argument.

(ii) F is continuous relative to the first argument.

Let $T: X \longrightarrow X$ be a mapping satisfying the following:

(I) $Tx_0 = x_0 \in X$

(II) $T(\partial Y) \subset Y$, where Y is a weakly compact and * F-semi-starshaped subset of X.

(III) T is non-expansive and weakly continuous map on $\partial Y \cup x_0$. Then T has a fixed point in Y closest to x_0.

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