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ON THE PROBLEM OF DARBOUX - IONESCU USING A GENERALIZED LIPSCHITZ CONDITION

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1. Introduction. In applications of the contraction mapping principle to concrete problems associated with the fixed points of a given operator an usual way is to find a space where the operator in question is contractive, i.e. to construct a norm, equivalent to the norms of the spaces into consideration, with respect to which the operator in question is contractive.

Results concerned with the construction of some Bielecki adequate norms are given, for example, in [7], [8], [10].

In this note we shall give some "existence and uniqueness" theorems for the problem of Darboux-Ionescu [8], using an alternative of the above mentioned approach, i.e., the generalized contraction mapping principle [2], when generalized norms (with values in the positive cone of a real Banach space), rather than equivalent Bielecki norms are considered.

2. A fixed point theorem in K-metric spaces. In this paper we consider, as in [2], [3], a generalized norm, i.e., a norm which takes values in an abstract cone K and comparison maps $\varphi : K \to K$, which enjoys certain properties in common with the map $t \to \alpha t$ where $0 < \alpha < 1$, but is not necessarily linear.

We need some definitions and results from [1]-[3], [9], [4],

[5].

Let $(E,\|\cdot\|)$ be a real Banach space.

A set $K \subset E$ is called a **cone** if

(i) K is closed

(ii) $x,y \in K$ implies $ax + by \in K$ for all $a,b \in \mathbb{R}_+$;

(iii) $K \cap (-K) = \{\theta\}$, where θ is the null element of E.

The cone K induces a reflexive, transitive and antisimetrical order relation in E, by

$$x \leq y \text{ if and only if } y - x \in K,$$

related to the linear structure by the properties

$$u \leq v \text{ implies } u + z \leq v + z, \text{ for each } z \in E$$

and

$$u \leq v \text{ implies } tu \leq tv, \text{ for each } t \in \mathbb{R}_+,$$

that is "$\leq$" is a linear order relation.

The space E endowed with this order relation is called an **ordered Banach space**, while K is termed as its **positive cone.**

We say that the norm of E is **monotone** if $x,y \in E$,

$$\theta_E \leq x \leq y, \text{ implies } \|x\| \leq \|y\|.$$

The cone K is **normal** if there exists $\delta > 0$ such that, from $x,y \geq \theta$, $x,y \in E$ and $\|x\| = \|y\| = 1$, it results $\|x + y\| \geq \delta$.

Recall that if the norm of $(E,\|\cdot\|)$ is monotone, then K is a normal cone [4].

Throughout this paper K will be the positive cone in a real ordered Banach space $(E,\|\cdot\|)$ with monotone norm.

DEFINITION 2.1. *A mapping $\varphi : K \to K$ is a* ***comparison function*** *if*

(i) *φ is monotone increasing;*

(ii) *$(\varphi^n(t))_{n\in\mathbb{N}}$ converges to θ, for all $t \in K$.*

Example 2.1. Let $E = \mathbb{R}$, the real axis, with the usual norm. In this case $K = \mathbb{R}_+$ and a typical comparison function is $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\varphi(t) = at, \quad 0 < a < 1, \quad t \in \mathbb{R}_+.$$

DEFINITION 2.2. *A mapping $\varphi : K \to K$ is called* (c)*-comparison function if φ is monotone increasing and fulfils the following convergence condition*

(c) *There exist two numbers k_0, α, $0 < \alpha < 1$, and a convergent series with nonnegative real terms $\sum_{k=1}^{\infty} a_k$, such that*

$$\|\varphi^{k+1}(t)\| \le \alpha(\|\varphi^k(t)\| + a_k), \text{ for } k \ge k_0, \ (\forall) t \in K.$$

Remark. Every (c) - comparison function is a comparison function since, if φ is a (c) - comparison function, then (see [1], [2]) the series $\sum_{k=1}^{\infty} \|\varphi^k(t)\|$ converges for all $t \in K$ and, consequently, the series

$$\sum_{k=1}^{\infty} \varphi^k(t) \tag{1}$$

converges in E, hence condition (ii) in Definition 2.1 is satisfied.

The function φ from example 2.1 is a (c) - comparison function.

DEFINITION 2.3. *Let X be a nonempty set. A mapping $d : X \times X \to K$ is said to be a K-metric on X if*

(i) $d(x,y) \ge \theta$ *and* $d(x,y) = \theta \leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$, *for all* $x,y \in X$;

(iii) $d(x,y) \le d(x,z) + d(y,z)$, *for all* $x,y,z \in X$.

The obtained entity: the nonempty set X with a K-metric d is called K - **metric space**, denoted as usually by (X,d).

DEFINITION 2.4. *Let (X,d) be a K-metric space and $\varphi : K \to K$ a comparison function. A mapping $f : X \to X$ is called abstract φ - contraction if there exist a comparison function $\varphi : K \to K$ such that*

$$d(f(x),f(y)) \le \varphi(d(x,y)), \text{ for all } x,y \in X. \qquad (2)$$

Example 2.2. Let $K = \mathbb{R}_+$ and let φ be the comparison function from example 2.1. Then a $\mathbb{R}_+$ - metric space is an usual metric space, while condition (2) becomes the well - known contraction condition

$$d(f(x),f(y)) \le a\cdot d(x,y), \quad 0 < a < 1, \quad x,y \in X.$$

Remark. In a K-metric space the concepts as K-fundamental sequence, K-convergent sequence and complete K-metric space are defined in a similar manner to the usual metric spaces.

As shown by example 2.2, the following result [2] is a generalization of the contraction mapping principle.

THEOREM 2.1. *Let (X,d) be a complete K-metric space, where K is a normal cone, and $f:X \to X$ an abstract φ-contraction, with φ (c)-comparison function.*

Then

(1) $F_f = \{x^*\}$, *where* $F_f = \{ x \in X | f(x) = x \}$;

(2) *The sequence* $(x_n)_{n\in\mathbb{N}}$, $x_n = f(x_{n-1})$, $n \ge 1$, *converges to* x^*, *for every* $x_0 \in X$;

(3) *We have*

$d(x_n,x^*) \le s(d(x_0,x_1)) - s_{n-1}(d(x_0,x_1))$,

where $s(t)$, $s_{n-1}(t)$ *denote the sum and the partial sum of rank n-1, respectively, of the series (1);*

(4) *If, in addition,* φ *is subadditive and there exist* $\eta \in K$, $\eta \neq \theta$, *and a mapping* $g : X \to X$, *so that*

$$d(f(x), g(x)) \le \eta, \text{ for all } x \in X,$$

then

$$d(y_n, x^*) \le \eta + s(\eta) + s(d(x_0, x_1)) - S_{n-1}(d(x_0, x_1)),$$

where

$$y_n = g^n(x_0).$$

Remark. $X = \mathbb{R}_+$ and φ is as in example 2.1, from theorem 2.1 we obtain the Banach fixed point theorem [7], [9].

3. Differential equations with deviating argument. Let us first consider the following problem of Darboux [8]

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x, y, u(h(x,y))), \tag{3}$$

$$\begin{cases} u(x,0) = 0, & x \in [0,a] \\ u(0,y) = 0, & y \in [0,b] \end{cases}, \tag{4}$$

where $f \in C([0,a] \times [0,b] \times \mathbb{R})$, $h \in C([0,a] \times [0,b],$ $[0,a] \times [0,b])$.

Recall that by a solution of the problem (3)+(4) we mean a function $u: [0,a] \times [0,b] \to \mathbb{R}$ which is continuous together with its partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial x \partial y}$, and which satisfies (3)+(4).

The problem (3)+(4) is equivalent to the following integral equation of Volterra type ([7], [8])

$$u(x,y) = \int_0^x \int_0^y f(s, t, u(h(s,t)))\, ds\, dt. \tag{5}$$

Let $D = [0,a] \times [0,b]$ and let X be the space $C(D)$, endowed with the usual norm.

We denote by K the cone of the positive functions from X and we define a mapping

$$\|\cdot\|_* : X \to K,$$

by

$$\|u(x,y)\|_* = |u(x,y)|, \quad (x,y) \in D.$$

It is obvious that $\|\cdot\|_*$ is a generalized norm on X, that is

(i) $\|u\|_* \geq \theta$, for each $u \in X$, where θ is the null function, and $\|u\|_* = \theta$ if and only if $u = \theta$;

(ii) $\|\lambda u\|_* = |\lambda| \cdot \|u\|_*$, for each $u \in X$ and any $\lambda \in \mathbb{R}$;

(iii) $\|u + v\|_* \leq \|u\|_* + \|v\|_*$, for each $u,v \in X$.

By other hand, K is the positive cone of X endowed with the Cebîşev's norm, which is monotone. Hence, K is a normal cone.

The partial order induced by K in X is given by, $u,v \in X$,

$$u \leq v \Leftrightarrow u(x,y) \leq v(x,y), \ \forall (x,y) \in D.$$

Let us consider the mapping

$$T : X \to X,$$

defined by

$$Tu(x,y) = \int_0^x \int_0^y f(s,t,u(h(s,t)))dsdt, \quad (x,y) \in D. \tag{6}$$

Every solution of equation (5), hence every solution of the problem (3)+(4) too, is a fixed point of T and vice versa.

Thus we obtain the main result of this paper given by

THEOREM 3.1. *Assume that*

(i) $f \in C(D \times \mathbb{R})$ and $h \in C(D,D)$;

(ii) *There exists a function* $g: D \times \mathbb{R}_+ \to \mathbb{R}_+$, *integrable with respect to the first argument and monotone with respect to the second argument (i.e.,* $g(\cdot,\cdot,u)$ *is integrable on* D *for fixed* $u \in \mathbb{R}_+$, *and*

$$u \leq v \Rightarrow g(x,y,u) \leq g(x,y,v), \ \forall (x,y) \in D) \tag{7}$$

such that

$$|f(x,y,u) - f(x,y,v)| \le g(x,y,|u-v|), \text{ for each } (x,y) \in D, \quad u,v \in \mathbb{R} \tag{8}$$

(iii) *For the fu ction $\varphi : K \to K$, defined by*

$$\varphi u(x,y) = \int_0^x \int_0^y g(s,t,u(h(s,t)))\,dsdt, \tag{9}$$

there exist a number $\alpha \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=0}^{\infty} a_k$, such that, begining from a fixed rank the following inequality holds

$$|\varphi^{k+1}(r)| \le \alpha\,(|\varphi^k(r)| + a_k), \text{ for every } r \in K, \tag{10}$$

Then the problem (3)+(4) has a unique solution u^*, which may be obtained by the succesive approximation method, starting from an arbitrary element $u_0 \in X$.

The sequence of succesive approximations $(u_p)_{p\in\mathbb{N}}$, is given by

$$u_p(x,y) = \int_0^x \int_0^y f(s,t,u_{p-1}(h(s,t)))\,dsdt,$$

and we have the following estimation

$$|u_p - u^*|_* \le s(|u_0-u_1|_*) - S_{p-1}(|u_0-u_1|_*).$$

Proof. From condition (7), we deduce that φ is monotone increasing and from (iii) it results that φ is a (c)-comparison function.

We have

$$|Tu(x,y) - Tv(x,y)| = \left|\int_0^x \int_0^y [f(s,t,u(h(s,t))) - f(s,t,v(h(s,t)))]\,dsdt\right| \le \int_0^x \int_0^y |f(s,t,u(h(s,t))) - f(s,t,u(h(s,t)))|\,dsdt.$$

Using (8) we then obtain

$$|Tu(x,y) - Tv(x,y)| \le \int_0^x \int_0^y g(s,t,|u(h(s,t))) - v(h(s,t))|)\,ds\,dt,$$

for each $(x,y) \in D$,

that is

$$|Tu - Tv|_* \le \varphi(|u - v|_*), \quad u,v \in X,$$

hence T is an abstract φ-contraction.

Now, theorem 2.1 completes the proof.

Remarks.

1) For $g(x,y,u) = L\cdot u$, where $L > 0$ is constant, (8) is just the condition (i) from Theorem 16.3.1 [7];

2) In a similar manner one can treat the following problem of Darboux - Ionescu [8]

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = F(x,y,u(g(x,y),h(x,y))), \ (x,y) \in D, \tag{11}$$

$$\begin{cases} u(x,0) = \alpha(x), & x \in [0,a] \\ u(0,y) = \beta(y), & y \in [0,b] \end{cases} \tag{12}$$

where

$$\alpha \in C^1[0,a], \quad \beta \in C^1[0,b], \quad (g,h) \in C(D,D)$$

and

$$\alpha(0) = \beta(0).$$

We obtain

THEOREM 3.2. *Assume that the following conditions are fulfiled*

(i) $F \in C(D \times \mathbb{R})$, $\alpha \in C^1[0,a]$, $\beta \in C^1[0,b]$ *and* $\alpha(0) = \beta(0)$;

(ii) $g \in C(D,[0,a])$, $h \in C(D,[0,b])$;

(iii) *There exists a function* $\psi : D \times \mathbb{R}_+ \to \mathbb{R}_+$ *integrable on* D *with respect to the first argument and monotone* *increasing with respect to the argument, i.e.,*

$$u \le v \Rightarrow \psi(x,y,u) \le \psi(x,y,v), \text{ for each } (x,y) \in D, \; u,v \in \mathbf{R}, \text{ such that}$$

$$|F(x,y,u) - F(x,y,v)| \le \psi(x,y,|u-v|), \text{ for each } (x,y) \in D \text{ and } u,v \in \mathbf{R}; \tag{13}$$

(iv) *The function* $\varphi : K \to K$, *defined by*

$$u(x,y) = \int_0^x \int_0^y \psi(s,t,u(g(s,t),h(s,t)))\, ds\, dt, \quad (x,y) \in D,$$

satisfies the convergence condition (8) *from theorem* 3.1.

Then, the problem of Darboux - Ionescu has a unique solution u^*.

Proof. The problem (11)+(12) is equivalent to the following integral equation of Volterra type

$$u(x,y) = \alpha(x) + \beta(y) - u_0 + \int_0^x \int_0^y F(x,t,u(g(s,t),h(s,t)))\, ds\, dt.$$

Remark. For $\psi(x,y,u) = L(x,y)\cdot u$, (13) is the Lipschitz condition from theorem 3.1 [7].

The generalized Lipschitz conditions used in Theorem 3.1 and Theorem 3.2 are termed "Perron condition" in [5].

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