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GENERALIZED CONTRACTIONS IN QUASIMETRIC SPACES

by

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Fixed point theorems for generalized contractions on metric, K-metric or uniform spaces, as well as for generalized contractions on σ-complete vector lattices, have been given in the papers [2]-[4], [6], [8]-[9], [10]-[11].

The aim of the present paper is to extend a result of BAHTIN,I.A. [1], given for usual contractions in quasimetric spaces, to a class of φ-contractions, with φ a comparison function.

A **quasimetric space** is a nonempty set X endowed with a **quasimetric**, i.e. a function $d : X \times X \to \mathbb{R}_+$, satisfying the following conditions:

d1) $d(x,y) = 0$ if and only if $x = y$;

d2) $d(x,y) = d(y,x)$, $\forall\ x,y \in X$;

d3) $d(x,z) \le a\,[d(x,y) + d(y,z)]$, $\forall\ x,y,z \in X$,

where $a \ge 1$ is a given real number, see [1].

Obviously, when $a = 1$ we obtain the usual notion of **metric (space)**.

Example 1.[1] The space $l_p (0 < p < 1)$,

$$l_p = \left\{ (x_n) \subset \mathbb{R} / \sum_{n=1}^{\infty} |x_n|^p < \right\},$$

together the function $d: l_p \times l_p \to \mathbb{R}$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = (x_n)$, $y = (y_n) \in l_p$, is a quasimetric space. Indeed, by an elementary calculation we obtain

$$d(x,z) \leq 2^{\frac{1}{p}} [d(x,y) + d(y,z)],$$

hence $a = 2^{\frac{1}{p}} > 1$ in this case.

Example 2.[1] The space $L_p(0 < p < 1)$ of all real functions $x(t)$, $t \in [0,1]$, such that

$$\int_0^1 |x(t)|^p dt < \infty,$$

becomes a quasimetric space if we take

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}, \text{ for each } x,y \in L_p.$$

The constant a is the same as in the previous example

$$a = 2^{\frac{1}{p}}.$$

In order to obtain our main result we need some definitions and results from [2]-[4] and [11], which we summarise here.

DEFINITION 1. ([8]) A mapping $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called *comparison function* if

(i) φ is monotone increasing;

(ii) $\varphi^n(t) \to 0$, as $n \to \infty$, for each $t \in \mathbb{R}_+$.

Example 3. The mapping $\varphi(t) = \alpha t$, $t \in \mathbb{R}_+$, where $0 \leq \alpha < 1$, is a comparison function.

Let (X,d) be, for instance, a metric space and $f: X \to X$ a mapping. An usual way to obtain generalizations of the contraction mapping principle is to replace the classical condition

$$d(f(x),f(y)) \leq \alpha \cdot d(x,y), \ \forall\, x,y \in X, \tag{1}$$

by a generalized contraction condition

$$d(f(x),f(y)) \leq \varphi(d(x,y)), \ \forall\, x,y \in X, \tag{2}$$

where φ is a certain comparison function, see, for example RUS,A.I. [10], [11].

Generally, such a fixed point theorem asserts, under suitable conditions, the existence or the existence and unicity of a fixed point of f but it is not able to say anything about the approximation of this fixed point.

A class of generalized contractions for which a fixed point theorem furnishes an estimation of the convergence of the sequence of successive approximations is studied in our papers [2]-[4], [8]-[9].

DEFINITION 2. ([2], [3]) A mapping $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies:

(i) φ is monotone increasing (isotone);

(ii) There exist a convergent series of positive terms $\sum_{n=0}^{\infty} v_n$ and a real number α, $0 \leq \alpha < 1$, such that

$$\varphi^{k+1}(t) \leq \alpha\, \varphi^k(t) + v_k, \text{ for each } t \in K \text{ and } n \geq N \text{ (fixed)} \tag{3}$$

is called (c) - *comparison function* (φ^k stands for the k iterate of φ).

Remark. 1) Using a generalization of the ratio test [5], [7], it results that if φ is a (c)-comparison function then the series

$$\sum_{k=0}^{\infty} \varphi^k(t) \tag{4}$$

converges for each $t \in \mathbb{R}_+$, hence

$$\varphi^k(t) \to 0, \text{ as } k \to \infty,$$

i.e. any (c)-comparison function is a comparison function.

2) If we denote by $s(t)$ the sum of the series (4), then, see [2]-[4], s is monotone increasing and continuous at zero.

Example 4. The function given in example 3 is a (c)-comparison function but, generally, a comparison function is not a (c)-comparison function, see [2]-[3].

The following theorem extends theorem 1 from [1].

THEOREM 1. *Let (X,d) be a complete quasimetric space and $f: X \to X$ a φ-*

contraction, i.e. *a mapping which satisfies* (2).

Then f has a unique fixed point if and only if there exists $x_0 \in X$, *such that the sequence* $(x_n)_{n\in\mathbb{N}}$ *of successive approximations,*

$$x_n = f(x_{n-1}),\ n \in \mathbb{N},$$

is bounded.

Proof. The sufficiency is obvious: if f has a unique fixed point, say x^*, then for $x_0 = x^*$, the sequence (x_n) is bounded, being constant.

The necessity. We suppose that (x_n) is bounded for a certain $x_0 \in X$. This means there exists a constant $c > 0$ and an element $y \in X$ such that

$$d(x_n, y) \le c, \quad \text{for each} \quad n \in \mathbb{N},$$

and then, for $n,m \in \mathbb{N}$ we have

$$d(x_n, x_m) \le a[d(x_n, y) + d(y, x_m)] \le 2ac.$$

Therefore

$$d(x_n, x_{n+p}) = d(f(x_{n-1}), f(x_{n+p-1})) \le \varphi(d(x_{n-1}, x_{n+p-1})) \le$$
$$\le \dots \le \varphi^{n-1}(d(x_1, x_{p+1})) \le \varphi^{n-1}(2ac),\ n,p \in \mathbb{N},$$

which shows (x_n) is fundamental, hence (x_n) is convergent, because (X,d) is a complete quasimetric space.

Let $x^* = \lim_{n\to\infty} x_n$. Then

$$\lim_{n\to\infty} d(x_n, x^*) = 0$$

and from

$$0 \le d(f(x^*), x^*) \le a[d(f(x^*), f(x_n)) + d(f(x_n), x^*)] \le$$
$$\le a[\varphi(d(x^*, x_n)) + d(x_{n+1}, x^*)],$$

we deduce

$$d(f(x^*), x^*) = 0,$$

since φ is continuous at zero. Hence $x^* \in F_f$.

The unicity is an immediate consequence of the contraction condition (2). The proof is now complete.

Remark. For φ as in example 3, from theorem 1 we obtain theorem 1[1].

Now let $a \geq 1$ be a given real number and let $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function for which there exists a convergent series of positive terms $\sum_{n=0}^{\infty} v_n$ and a real number α, $0 \leq \alpha < 1$ such that

$$a^{k+1}\varphi^{k+1}(t) \leq \alpha \cdot a^k\varphi^k(t) + v_k, \text{ for each } t \in \mathbb{R}_+ \text{ and each } k \geq N(\text{fixed}) \tag{5}$$

Then, in view of the generalized ratio test [5], the series

$$\sum_{k=0}^{\infty} a^k\varphi^k(t), \tag{6}$$

converges for each $t \in \mathbb{R}_+$ and its sum, denoted by $s_a(t)$, is monotone increasing and continuous at zero.

Obviously, when $a = 1$ (i.e. d is actually a metric on X) such a comparison function is a (c)-comparison function.

The main result of this paper is the following theorem.

THEOREM 2. *Let (X,d) be a complete quasimetric space, $f: X \to X$ a φ-contraction, with φ satisfying the condition* (5).

If $x_0 \in X$ is such that the sequence (x_n),

$$x_n = f(x_{n-1}), \quad n \in \mathbb{N}^*,$$

is bounded and $F_f = \{x^*\}$, then we have

$$d(x_n, x^*) \leq a \cdot s_a(d(x_n, x_{n+1})), \quad n \geq 0, \tag{7}$$

where $s_a(t)$ is the sum of the series (6).

Proof. From the contraction condition (2) we deduce, for $n \geq 1$,

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \varphi(d(x_{n-1}, x_n)),$$

and for $n \geq 2$,

$$d(x_{n-1}, x_n) \leq \varphi(d(x_{n-2}, x_{n-1})),$$

which yields, using the monotocity of φ,

$$d(x_n, x_{n+1}) \leq \varphi^2(d(x_{n-2}, x_{n-1})), \text{ for } n \geq 2.$$

By induction, we then obtain

$$f(x_{n+k-1}, x_{n+k}) \leq \varphi^{k-1}(d(x_n, x_{n+1})), \text{ for each } k. \tag{8}$$

By on other hand, the axiom d3) gives, for each $p \in \mathbb{N}$,

$$d(x_n, x_{n+p}) \leq a[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})] \leq$$

$$\leq ad(x_n, x_{n+1}) + a^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+p})] \leq$$

$$\leq \ldots \leq ad(x_n, x_{n+1}) + a^2 d(x_{n+1}, x_{n+2}) + \ldots + a^p d(x_{n+p-1}, x_{n+p}),$$

hence, from (8) we obtain

$$d(x_n, x_{n+p}) \leq a\sum_{k=0}^{p-1} a^k \varphi^k(d(x_n, x_{n+1})). \tag{9}$$

Now if we take $p \to \infty$ in (9), we obtain the desired estimation (7).

The proof is complete.

Remarks. 1) If $a = 1$, theorem 2 is just the generalized contraction principle given in [2]-[4];

2) When $\varphi(t)$ is as in example 3, the condition (5) is satisfied if $\alpha \in [0;1[$ is such that

$$\alpha a < 1.$$

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