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# ERROR ESTIMATES FOR A CLASS OF (δ,φ) - CONTRACTIONS

by

**Vasile BERINDE**

The purpose of this paper is to establish an estimates for a class of generalized contractions.

Fixed point theorems for generalized contractions are given in [1] - [5], [10] - [13]. It is well known that in a fixed point theorem is very important to have an error estimates, that is a method for the approximation of the fixed point.

Because the fixed point for (δ - φ)-contractions in [12], [13] does not give an error estimation of the fixed point, we are led to consider a class of (δ,φ)-contractions, as in [5], for which such an estimation may be obtained. To this end we need some notations, definitions and lemmas from [12], [13], [5].

Let $(X,d)$ be a metric space and $f: X \to X$ a mapping.

We denote as usually

$$F_f := \{x \in X \mid f(x) = x\},$$

$$O(x;f) := \{x, f(x), \dots, f^n(x), \dots\},$$

$$O(x,y;f) := O(x;f) \cup O(y;f),$$

$$\delta(A) = \sup\{d(a,b) \mid a, b \in A\},\ A \subseteq X$$

and

$$I_b(f) = \{A \subseteq X \mid A \neq \emptyset, f(A) \subseteq A, \delta(A) < +\infty\}.$$

DEFINITION 1. (BERINDE [1]). A function $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ is called *(c)-comparison function* if the following two conditions are satisfied

($c_1$) $\varphi$ is monotone increasing.

($c_2$) There exist two numbers $k_0$, $\alpha$, $0 < \alpha < 1$, and a convergent series with nonnegative terms $\sum_{k=1}^{\infty} a_k$, such as

$$\varphi^{k+1}(t) \le \alpha\varphi^k(t) + a_k, \text{ for each } t \in \mathbf{R}_+ \text{ and } k \ge k_0.$$

LEMMA 1 (BERINDE [1], [5]). *If $\varphi$ is a (c)-comparison function then*

($c_3$) *$\varphi(t) < t$, for each $t > 0$;*

($c_4$) *$\varphi$ is continuous in 0;*

($c_5$) *The series*

$$\sum_{k=0}^{\infty} \varphi^k(t) \tag{1}$$

*converges for each $t \in \mathbf{R}_+$.*

($c_6$) *The sum of the series (1), $s(t)$, is monotone increasing and continuous in 0.*

($c_7$) *$(\varphi^n(t))_{n\in N}$ converges to 0, as $n \to \infty$, for each $t \in \mathbf{R}_+$.*

*Remark.* A function $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ satisfying ($c_1$) and ($c_7$) is called comparison function.

*Example* 1. If $a \in (0,1)$, then $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) = at$, $t \in \mathbf{R}_+$ is a (c)-comparison function, hence $\varphi$ is a comparison function too. But $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) = \dfrac{t}{1+t}$, $t \in \mathbf{R}_+$, is a comparison function which is not a (c)-comparison function.

DEFINITION 2. (RUS [12]). Let $(X,d)$ be a metric space.

A mapping $f: X \to X$ is a $(\delta,\varphi)$ - *contraction* if there exists a comparison function $\varphi$ such that:

$$\delta(f(A)) \le \varphi(\delta(A)), \tag{2}$$

for all $A \in I_b(f)$.

*Example* 2. If f is a φ-contraction (see RUS [12]), i.e. a mapping which satisfies instead of (2) the following condition

$$d(f(x),f(y)) \le \varphi(d(x,y)),\ \forall x,y \in X,$$

where φ is a comparison function, then $f$ is a (δ-φ)-contraction.

*Example* 3. If $f$ is a φ-contraction with φ a 5-dimensional comparison function (see BERINDE [5]), then $f$ is a (δ-φ)-contraction.

*Example* 4. Let $f: [0,1] \to [0,1]$ be a function defined by $f(x) = x + \frac{1}{3}$, for $x \in \left[0, \frac{1}{3}\right]$ and $f(x) = \frac{1}{2}x - \frac{1}{6}$, for $x \in \left(\frac{1}{3}, 1\right]$. Then

a) $F_f = \varnothing$;

b) $I_\delta(f) = \left\{[0,b] / b \ge \frac{2}{3}\right\}$;

c) For $b > \frac{2}{3}$, any set $A = [0,b]$ satisfies (2), with $\varphi(t) = \frac{2}{3}t$, but $f$ is not a (δ,φ)-contraction, because for $A = \left[0, \frac{2}{3}\right]$, (2) is not satisfied.

DEFINITION 3. Let $(X,d)$ be a metric space and $f: X \to X$ a mapping. An element $x \in X$ is called *regular for* $f$ if the set $O(x;f)$ is bounded. Two elements $x$ and $y$ of $X$ are called *asymptotic under* $f$ if the sequence $(d(f^n(x),f^n(y)))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$.

LEMMA 2. (RUS [13]). *Let $(X,d)$ be a metric space and $f: X \to X$ a (δ,φ)-contraction. If $x$ and $y$ are regular elements for $f$ then $x$ and $y$ are asymptotic under $f$.*

The following characterization of the (δ,φ)-contraction will be useful in the sequel.

LEMMA 3. (RUS [13]). *Let $(X,d)$ be a metric space, φ a given comparison function and $f: X \to X$ a mapping. Then the following statements are equivalent*

(i) *$f$ is a (δ,φ)-contraction;*

(ii) $d(O(f(x),f(y);f)) \le \varphi(\delta(O(x,y;f)))$, *for all regular elements $x$ and $y$ of $X$;*

(iii) $d(f(x),f(y)) \le \varphi(\delta(O(x,y;f)))$, *for all regular elements $x$ and $y$ of $X$.*

LEMMA 4. (RUS [13]). *Let $(X,d)$ be a complete metric space and $f: X \to X$ a (δ,φ)-*

*contraction.*

(i) *If* $F_f \neq \varnothing$, *then* $F_f = \{x^*\}$;

(ii) *If* $x^* \in F_f$ *and* $x$ *is a regular element for* $f$, *then* $(f^n(x))_{n\in\mathbb{N}}$ *converges to* $x^*$.

The main result of this paper is given by

THEOREM 1. *Let* $(X,d)$ *be a complete metric space and* $f: X \to X$ *a* $(\delta,\varphi)$*-contraction with* $\varphi$ *a (c)-comparison function.*

*If there exists a regular element* $x \in X$ *for* $f$ *then*

a) $F_f = \{x^*\}$;

b) *If* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of the succesive approximations with* $x_0 \in X$ *a regular element for* $f$, *then*

$$x_n \to x^*;$$

c) $d(x_n, x^*) \leq s(\delta(0(x_n, x_{n+1})))$ (3)

*where* $s(t)$ *is the sum of the series* (1).

*Proof.* From Theorem 1 [12] and example 2 it results a), b). In order to prove c), we use Lemma 3 and condition (2). We deduce

$$\delta(0(f(x), f(y); f)) \leq \varphi(\delta(0(x, y; f))), \tag{4}$$

for all regular elements $x, y \in X$.

Let $x_0$ be a regular element and $(x_n)_{n\in\mathbb{N}}$ the sequence of succesive approximations,

$$x_n = f^n(x_0),\ n \geq 1. \tag{5}$$

Obviously, $x_n$ is a regular element for $f$ too, for each $n \geq 1$ and then, from (4) we deduce

$$\delta(0(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \varphi(\delta(0(f^n(x_0), f^{n+1}(x_0); f))),$$

that is

$$\delta(0(x_{n+1}, x_{n+2}; f)) \leq \varphi(\delta(0(x_n, x_{n+1}; f)).$$

By induction we obtain

$$\delta(O(x_{n+k}, x_{n+k+1}; f) \leq \varphi^k(\delta(x_n, x_{n+1}; f)).$$

Since

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p}),$$

it results

$$d(x_n, x_{n+p}) \leq r + \varphi(r) + \dots + \varphi^{n+p-1}(r), \tag{6}$$

where

$$r = \delta(O(x_n, x_{n+1}; f)).$$

Now we take $p \to \infty$ in (6) and we obtain just the desired estimation (3).

*Remarks.*

1) Theorem 1 in the present paper completes Theorem 5.2.2. [12] by the estimation (3);

2) It is possible to obtain a more general result if consider two metrics $d$ and $\rho$ defined on $X$. In this case we denote by $\delta_d(A)$ and $\delta_\rho(A)$ the diameter of the set $A$ with respect to the metric $d$ and $\rho$ respectively.

The following theorem corrects the statement of Theorem 1 in MUREŞAN [10] and completes it by the estimation (7).

THEOREM 2. *Let $X$ be a nonempty set endowed with two metrics $d$ and $\rho$ and $f: X \to X$ a mapping satisfying the following conditions*

1) *There exists a constant $c > 0$ such that*

$$d(f(x), f(y)) \leq c\rho(x, y), \text{ for all } x, y \in X;$$

2) *$(X,d)$ is a complete metric space;*

3) *$f: (X,d) \to (X,d)$ is continuous;*

4) *$f: (X,\rho) \to (X,\rho)$ is a $(\delta,\varphi)$-contraction with $\varphi$ a (c)-comparison function;*

5) *There exists a regular element $x \in X$ for $f: (X,\rho) \to (X,\rho)$*

*Then:*

(i) $F_f = \{x^*\}$,

(ii) *The sequence* $(f^n(x_0))_{n\in N}$ *converges to* $x^*$ *for any regular element* $x_0 \in X$;

(iii) *For* $x_0 \in X$ *a regular element for f and* $x_n = f^n(x_0)$, *we have*

$$d(x_n, x^*) \le cs(\delta_f(0(x_n, x_{n+1}; f))) \qquad (7)$$

*Proof.* In a similar manner to theorem 1 we obtain, using (1) and (4),

$$d(x_n, x_{n+p}) \le c[\varphi^{n-1}(r) + \varphi^n(r) + \ldots + \varphi^{n+p-2}(r)], \qquad (8)$$

where

$$r = \delta_f(0(x_0, x_1; f))$$

Since $\varphi$ is a (c)-comparison function, it results $(x_n)$ is a Cauchy sequence in the complete metric space $(X, d)$. Hence $(x_n)$ is convergent. Let $x^*$ be its limit. From condition (3) we obtain $x^* \in F_f$ and then $F_f = \{x^*\}$, in view of Lemma 4. In order to obtain (7) it suffices to take $p \to \infty$ in (8). The proof is now complete.

*Remark.* If $\varphi$ is not a (c)-comparison function then, from (8) does not result that $(x_n)$ is a Cauchy sequence. For example, if $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ is given by $\varphi(t) = \frac{t}{1+t}$, $t \in \mathbf{R}_+$, then

$$\varphi^{n-1}(1) + \varphi^n(1) + \ldots + \varphi^{n+p-2}(1) \to \infty, \text{ as } p \to \infty$$

## REFERENCES


1. Berinde, V., *Error estimates in the approximation of the fixed point for a class of φ-contractions*. Studia Univ. Cluj-Napoca, XXXV(1990), fasc.2, 86-89
2. Berinde, V., *The stability of fixed points for a class of φ-contractions*. Seminar on Fixed Point Theory, 1990, 3, 13-20
3. Berinde, V., *A fixed point theorem of Maia type in K-metric spaces*. Seminar on Fixed

Point Theory, 1991, 3. 7-14.

4. Berinde, V., *Abstract φ-contractions which are Picard mappings*, Mathematica, Tome 34(57), No. 2, 1992, 107-111.

5. Berinde, V., *Generalized contractions and applications* (Romanian), Ph.D.Thesis. Univ. Cluj-Napoca, 1993.

6. Čirić, L.B., *A generalization of Banach's contraction principle*. Proc. Amer. Math. Soc., 45(1974), 267-273.

7. Čirić, L.B., *A note on fixed point mappings with contracting orbital diameters*, Publ. l'Institut Math., 27(1980), 31-32.

8. Fuchssteiner, B., *Iterations and fix points*, Pacific J.Math., 68(1977), 73-80.

9. Hegedus, M., Kasahara, S., *A contraction principle in metric spaces*, Math. Sem. Notes, 7(1979), 597-603.

10. Mureşan, A., *Some fixed point theorems of Maia type*, Seminar on Fixed Point Theory, 1988, 3, 35-42.

11. Rus, I.A., *Some metrical fixed point theorems*, Studia Univ. Babeş-Bolyai, Mathematica, 1, (1979), 73-77.

12. Rus, I.A., *Generalized contractions*, Seminar on Fixed Point Theory, 1983, 3, 1-130.

13. Rus, I.A., *Generalized φ-contractions*, Mathematica, Tome 24(47), No. 1-2, 1982, 175-178.