STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXV, 4, 1990

ON THE SOLUTIONS OF A FUNCTIONAL EQUATION USING PICARD MAPPINGS

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Received: January 18, 1990
AMS subject classification: 40J05

REZUMAT. — Asupra soluţiilor unei ecuaţii funcţionale folosind aplicaţii Picard. În lucrare sînt prezentate soluţii ale ecuaţiei funcţionale (1), în cazul în care f este o aplicaţie Picard. Acestea sînt exprimate prin intermediul unei serii care poate fi convergentă dar şi divergentă dar sumabilă printr-o metodă de sumare dată (Césaro, Abel, Toeplitz).

0. **Introduction.** This work is concerned with the solutions g of the functional equation (1). The solutions are given in the form of convergent series or divergent series but summable by some method of summability.

1. **Preliminaries.** Let (X, d) be a complete metric space, Y a real Banach space and let us consider the following functional equation:

$$g(f(x)) + g(x) = F(x), \text{ for all } x \in X \tag{1}$$

where $f: X \to X$, $F: X \to Y$ are given mappings and $g: X \to Y$ is an unknown mapping.

When f, F and g are functions of a single real variable $x \in [a, b]$, the solutions of equation (1) have been studied by many authors [1], [3], [5]. Thus, the work [3] of Kuczma extends some results due to Hardy, respectively Steinhaus, [1] extends [3], [5] extends [1] etc. (for more details see the first section in [5] and also [4], [3]).

The present work may be regarded as an extension of a part of the results given in the papers quoted above, especially in [1]. One can treat similarly other results in [1] and [4], e.g. theorem 3.1 and theorem 4.1 from [5].

In the papers [1], [3], [5] f is assumed to satisfy the following conditions: f is a continuous and strictly increasing function on $[a, b]$; $f(a) = a$, $f(b) = b$ and $f(x) > x$, for all $x \in (a, b)$.

Under thase assumptions it can be readly verified [2] that the sequence of the iterates of f, $(f^n(x))_{n \geq 0}$, defined, as usually, by $f^{(0)}(x) = x$ and $f^n(x) = f(f^{n-1}(x))$, for $n \geq 1$ and for all $x \in [a, b]$, converges to b, i.e.

$$\lim_{n \to \infty} f^n(x) = b, \text{ for all } x \in (a, b] \tag{2}$$

Let us observe that only the condition (2) is essentially in the proof of theorem I from [1], and, consequently, we may consider a slower condition on f, i.e., f is a Picard mapping [6].

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This weakness is effective, as is shown by

Example 1.1. The function $f: [0, 1] \to [0, 1]$, $f(0) = 0$,

$$f(x) = 6x^2 - \frac{7}{2}x + 1, \text{ for } x \in \left(0, \frac{1}{2}\right) \text{ and } f(x) = 1, \text{ for}$$

$x \in \left[\frac{1}{2}, 1\right]$ is not continuous, also not strictly increasing on $[0, 1]$, but it can be easily verified that

$$\lim_{n\to\infty} f^n(x) = 1, \text{ for all } x \in (0, 1].$$

The aim of this note is to show that the theorem I in [1] remains valid under these weak conditions.

2. **Picard mappings.** For the definitions, examples, properties and other results concerning Picard mappings we refer to [6].

As usually, we denote by F_f the set of all fixed points of a mapping f.

DEFINITION 2.1. Let (X, d) be a metric space. A mapping $f: X \to X$ is said to be a *Picard mapping* if there exists $b \in X$ such that

$$F_f = \{b\}$$

and

$$(f^n(x))_{n\in\mathbb{N}} \text{ converges to } b, \text{ for all } x \in X.$$

Example 2.1 Let (X, d) be a complete metric space. A contraction mapping $f: X \to X$ is a Picard mapping.

Example 2.2. Let (X, d) be a compact metric space. A contractive mapping $f: X \to X$ is a Picard mapping.

Example 2.3. [3] Let $f: [a, b] \to [a, b]$ be a continuous and strictly increasing function on $[a, b]$, $f(b) = b$ and $f(x) > x$, for all $x \in [a, b)$. Then f is a Picard mapping.

DEFINITION 2.2. A function $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ is a *comparison function* if satisfies the conditions

i) φ is monotone increasing.

ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0, for all $t \geq 0$.

DEFINITION 2.3. Let (X, d) be a metric space. A mapping $f: X \to X$ is a *φ-contraction* if φ is a comparison function and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

Now let us recall (see [6], p. 33–34) some results which give sufficient conditions for the mapping f be a Picard mapping.

THEOREM 2.1. *Let (X, d) be a complete metric space and $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ a function which satisfies condition* i) *and the following two conditions*

iii) $\varphi(t) < t$, *for all* $t > 0$;

iv) φ *is right continuous.*

If $f: X \to X$ is a mapping such that

$d(f(x), f(y)) \leqslant \varphi(d(x, y))$, *for all* $x, y \in X$, *then f is a Picard mapping.*

THEOREM 2.2. *Let (X, d) be a complete metric space and $f: X \to X$ a φ-contraction. Then f is a Picard mapping.*

THEOREM 2.3. *Let (X, d) be a complete metric space and $\varphi: \mathbf{R}_+ \to [0, 1)$ a monotone decreasing function. If $f: X \to X$ is such that*

$$d(f(x)), f(y)) \leqslant \varphi(d(x, y)) \cdot d(x, y), \text{ for all } x, y \in X,$$

then f is a Picard mapping.

THEOREM 2.4. *Let (X, d) be a complete metric space and $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ a function such that $\varphi(r) > 0$, for $r > 0$.*

If $f: X \to X$ is a mapping such that

$$d(f(x), f(y)) \leqslant d(x, y) - \varphi(d(x, y)), \text{ for all } x, y \in X,$$

then f is a Picard mapping.

3. Sequences and series. In this section, we recall some definitions and properties concerning sequences in a metric space, series in a real Banach space and we also introduce some concepts of summability of divergent series in Banach spaces, analogous to those for number series in [2].

For simplicity, in the sequel we shall assume, without any special mention, that (X, d) is a complete metric space and Y is a real Banach space.

DEFINITION 3.1. A series $\sum_{n=0}^{\infty} a_n$ in Y is said to be *convergent* with the sum S, and we write $S = \sum_{n=0}^{\infty} a_n$ if the sequence of partial sums $(S_n)_{n \in \mathbf{N}}$ converges to S (in norm).

A series of operators $\sum_{n=0}^{\infty} f_n$, $f_n: X \to Y$, is *convergent* on X to f if, for all $x \in X$, the series

$$\sum_{n=0}^{\infty} f_n(x) \text{ converges to } f(x) \in Y.$$

DEFINITION 3.2. Let $\sum_{n=0}^{\infty} x_n$ be a series in Y. We denote, as usually, by $(S_n)_{n \in \mathbf{N}}$ the sequence of partial sums, and let us consider the sequence $(S_n^{(k)})_{k \in \mathbf{N}}$, defined by $S_n^{(0)} = S_n$ and, for $k \geqslant 1$,

$$S_n^{(k)} = S_0^{(k-1)} + S_1^{(k-1)} + \ldots + S_n^{(k-1)}, \quad (n = 0, 1, 2, \ldots)$$

If, for some k, the sequence $(C_n^{(k)})_{n\in\mathbf{N}}$

$$C_n^{(k)} = \frac{1}{\binom{n+k}{k}} S_n^{(k)},\ n \geq 0,$$

$\left(\text{where } \binom{n+k}{k} \text{ is the binomial coefficient}\right)$ converges to S, we say that the the series $\sum_{n=0}^{\infty} x_n$ is *Césaro-summable* or C_k — *summable* with the sum S.

DEFINITION 3.3. Let $(S_n)_{n\in\mathbf{N}}$ be the sequence of partial sums of the series $\sum_{n=0}^{\infty} x_n$ in Y. Let us consider an infinite matrix $T = (a_{kn})$ of real numbers and let us construct the following series

$$\sum_{n=0}^{\infty} a_{kn} S_n,\ (k = 0,\ 1,\ 2,\ \ldots). \tag{3}$$

We assume that the seies (3) is convergent, with the sum S_k', $k = 0,\ 1,\ 2, \ldots$ If the sequence $(S_k')_{k\in\mathbf{N}}$ converges to S, then the series $\sum_{n=0}^{\infty} x_n$ is said to be *Toeplitz-summable* or *T-summable* with the sum S.

DEFINITION 3.4. A series $\sum_{n=0}^{\infty} x_n$ in Y is *summable by Abel's method* of summability, or *A-summable*, with the sum S, if the series $\sum_{n=0}^{\infty} t^n x_n$ converges for all $t \in [-r,\ r]$, $r \geq 1$, and there exists

$$\lim_{t\to 1-0} \left(\sum_{n=0}^{\infty} t^n x_n\right) = S.$$

(If $h : \mathbf{R} \times Y \to Y$, the limit

$$\lim_{t\to 1-0} h(t,\ x) = l_x$$

must be understood as

$$\lim_{t\to 1-0} \| h(t,\ x) - l_x \| = 0).$$

Since theorems 1 and 2 [2] pp. 404—405 are important for this work we quote them here, adapted to Banach spaces.

THEOREM 3.1. *Let $(z_n)_{n\in\mathbf{N}}$ be a sequence, $z_n \in Y$, for every $n = 0,\ 1,\ 2,\ \ldots,$ and $\lim_{n\to\infty} z_n = 0 \in Y$. If $T = (a_{kn})$ is an infinite regular matrix whose elements $a_{kn} \in \mathbf{R}$ satisfy the conditions* t1) *For each* $n \geqslant 0$, $a_{kn} \to 0$ *as* $k \to \infty$,

t2) *There exists a constant* C *such that*

$$|a_{k_0}| + |a_{k_1}| + \ldots + |a_{kn}| < C, \ (k \geqslant 0, \ n \geqslant 0),$$

then the series $\sum_{n=0}^{\infty} a_{kn} z_n$ *converges for all* $k \geqslant 0$.

Moreover, the sequence $(z'_k)_{k \in \mathbf{N}}$

$$z'_k = \sum_{n=0}^{\infty} a_{kn} z_n, \ k = 0, 1, 2, \ldots \tag{4}$$

is convergent in Y *and* $\lim_{k \to \infty} z'_k = 0$.

THEOREM 3.2. *If* $T = (a_{kn})$ *is a regular matrix whose elements* $a_{kn} \in \mathbf{R}$ *satisfy the conditions* t1), t2) *from theorem* 3.1 *and the following*

t3) $$\lim_{k \to \infty} \left(\sum_{n=0}^{\infty} a_{kn} \right) = 1,$$

then, for any sequence $(z_n)_{n \in \mathbf{N}}$ *converging to* z *in* Y, *as* $n \to \infty$, *the sequence* $(z'_k)_{k \in \mathbf{N}}$, *given by* (4), *is convergent and* $\lim_{k \to \infty} z'_k = z$.

4. **Solutions of the functional equation.** Now we return to equation (1). Let $f: X \to X$ be a Picard mapping and $F_f = \{b\}$. As in [1], [3], [5] we consider the following series

$$\frac{1}{2} F(b) + \sum_{n=0}^{\infty} (-1)^n \{F[f^n(x)] - F(b)\}. \tag{5}$$

The aim of this section is to give sufficient conditions for the existence of a solution of equation (1), by using the convergence or the summability of the series (5).

The main result is stated in

THEOREM 4.1. *Let* (X, d) *be a complete metric space,* Y *a real Banach space,* $F: Y \to Y$ *a given mapping and* $f: X \to X$ *a given Picard mapping.*

a) *If the series* (5) *converges on* X, *its sum* $g(x)$ *is a solution of equation* (1).

b) *If the series* (5) *is* T — *summable with the sum* $g(x)$, *where* $T = (a_{kn})$ *is a regular matrix transformation whose elements* $a_{kn} \in \mathbf{R}$ *satisfy the conditions* t1)—t3), *then* $g(x)$ *is a solution of the equation* (1) *if* F *is continuous in* b.

c) *If the series* (5) *is* C_k — *summable with the sum* $g(x)$, *then* $g(x)$ *is a solution of* (1).

d) *If the series* (5) *is* A — *summable with the sum* $g(x)$ *then* $g(x)$ *is a solution of* (1).

Proof.

a) Substituting $g(x)$ given by the series (5) in (1), we obtain that the first statement holds.

b) Putting

$$S_n(x) = \frac{1}{2} F(b) + \sum_{i=0}^{n} (-1)^i [F(f^i(x)) - F(b)], \; n = 0, 1, 2, \ldots$$

and

$$S'_k(x) = \sum_{n=0}^{\infty} a_{kn} S_n(x), \; (k = 0, 1, 2, \ldots)$$

it results from the T — summability of the series (5) that

$$g(x) = \lim_{k\to\infty} S'_k(x).$$

On the other hand, we have

$$S_n[f(x)] = F(x) - S_n(x) - (-1)^{n+1}[F(f^{n+1}(x)) - F(b)]$$

and consequently

$$S'_k[f(x)] = \sum_{n=0}^{\infty} a_{kn} F(x) - S'_k(x) - \sum_{n=0}^{\infty} (-1)^{n+1} a_{kn} [F(f^{n+1}(x)) - F(b)]. \quad (6)$$

From t1), t2) and the continuity of F in b, it follows that

$$\lim_{n\to\infty} \{(-1)^{n+1}[F(f^{n+1}(x)) - F(b)]\} = 0$$

hence, applying theorem 3.1 we obtain

$$\lim_{k\to\infty} \left(\sum_{n=0}^{\infty} (-1)^{n+1} a_{kn} [F(f^{n+1}(x)) - F(b)]\right) = 0$$

It is also obvious, using theorem 3.2 that

$$\lim_{k\to\infty} \left(\sum_{n=0}^{\infty} a_{kn} F(x)\right) = F(x), \text{ for all } x \in X$$

and therefore, taking $k \to \infty$, (6) becomes

$$g[f(x)] = F(x) - g(x).$$

c) The proof is similar with that of the third part of theorem 1 [1] and the unnecessary details will be omitted.

If we denote by $S_n(x)$, $S_n(f(x))$ the partial sum of the series (5), respectively of the series obtained from (5) replacing x by $f(x)$, it results by an elementary calculation

$$C_n^{(k)}(x) = \frac{n}{n+k} C_{n-1}^{(k)}(x) + \frac{1}{\binom{n+k}{k}} S_n^{(k-1)}(x),$$

and

$$C_n^{(k)}(x) + C_n^{(k)}(f(x)) = F(x) - \frac{k}{2(n+k)} F(b) + \frac{1}{\binom{n+k}{k}} S_n^{(k-1)}(f(x)). \qquad (7)$$

Now, in (7) we take $n \to \infty$ and follows, from the C_k — summability of the series (5), i.e.

$$\lim_{n\to\infty} C_n^{(k)}(x) = g(x);$$

that

$$g(x) + g[f(x)] = F(x).$$

d) This follows from theorem 1 [1], d).

The proof of theorem is now complete.

Finally, let us observe that all results in this paper remain valid if X and Y are complex rather than real spaces.

The author would like to thank Professor I. A. Rus for helpful suggestions in preparing this article.

REFERENCES

1. M. Bajraktarevic, *Sur une solution de l'équation fonctionelle* $\varphi(x) + \varphi[f(x)] = F(x)$, Glasnik Mat-Fiz i Astr, **15** (1960), 2.
2. K. Knopp, *Theorie und Anwendung der Unendlichen Reihan*, Springer Verlag, 1964.
3. M. Kuczma, *On the functional equation* $\varphi(x) + \varphi[f(x)] = F(x)$, Ann. Polon. Math., **6** (1959) 281–287.
4. M. Kuczma, *Functional equations in a single variable*, Warsawa, 1968.
5. M. Malenica, *On the solutions of the functional equation* $\varphi f(x) + \varphi(x) = F(x)$ *using* θ*-summability*, Glašnik Mat. vol. 18, **38** (1983), 305–315.
6. I. A. Rus, *Generalized Contractions*, Seminar on Fixed Point Theory, Preprint, nr. 3, 1983, Univ. "Babeş-Bolyai", Cluj-Napoca.