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ϕ-MONOTONE AND ϕ-CONTRACTIVE OPERATORS IN HILBERT SPACES

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REZUMAT. - Operatori ϕ-monotoni şi ϕ-contractivi în spaţii Hilbert. Lucrarea introduce noţiunea de ϕ-monotonie, care generalizează conceptul de tare-monotonie, şi arată că familia operatorilor ϕ-monotoni este echivalentă, intr-un anume sens, cu cea a operatorilor ϕ-contractivi.

Introduction. The aim of this paper is to establish a relation between a class of monotone operators, by a hand, and a class of contractive mappings, on the other hand, using a generalization of the contraction mapping principle due to KRASNOSELSKI and STECENKO [6]. As shown by CEA [3], which applies this results in optimization theory, if the operator G satisfies certain monotonic conditions, then $T_\gamma = I - \gamma G$, for a certain $\gamma > 0$, is a contraction. In [5] DINCĂ argued that CEA's conditions are only sufficient and, consequently furnishes necessary and sufficient conditions, obtaining the following generalization of CEA's result: G is strongly monotone and Lipschitz operator if and only if there exists $\gamma > 0$ such that T_γ is a contraction.

Theorem 3.1 extends these results, by means of some new concepts, and states that G is ϕ-monotone operator if and only if there is $\gamma > 0$ such that T_γ is ϕ-contraction.

1. Comparison functions. Various concepts of comparison functions was defined and

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intensively studied in connection with the generalized contraction mapping principle, see, for example RUS,A.I. [7], [8], BERINDE,V. [1], [2]. In the present paper we need comparison functions defined without the monotone increasing condition.

DEFINITION 1.1. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a *comparison function* if

(i) φ is continuous;

(ii) $0 < \varphi(t) < t$, for $t > 0$.

Let's denote by ϕ the set of all comparison functions. Obviously, ϕ is nonempty and contains both linear and nonlinear functions as shown by

Example 1.1. If $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) = at$, $0 \le a < 1$, $t \in \mathbf{R}_+$, then $\varphi \in \phi$.

Exampme 1.2. If $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) = t(1-t)$, for $0 \le t < 1$ and $\varphi(t) = t-1$, for $t \ge 1$, then $\varphi \in \phi$, but φ is nonlinear.

LEMMA 1.1. *Let $\varphi \in \phi$ be a comparison function. Then*

a) $\varphi(0) = 0$;

b) $0 < 2t\varphi(t) - \varphi^2(t) < t^2$, *for* $t > 0$.

Proof.

a) From (ii) we obtain $\lim_{t \to 0} \varphi(t) = 0$, hence, by (i), $\varphi(0) = 0$.

b) Since $\varphi \in \phi$, we have $\varphi(t) \ge 0$ and $t - \varphi(t) \ge 0$, $t \in \mathbf{R}_+$. Then, for every $t \in \mathbf{R}_+$,

$$2t\varphi(t) - \varphi^2(t) = \varphi(t)[t + (t - \varphi(t))] \ge 0$$

Finally, for $t > 0$, $t - \varphi(t) > 0$, then $2t\varphi(t) - \varphi^2(t) > 0$ and $(t - \varphi(t))^2 > 0$, that is, $2t\varphi(t) - \varphi^2(t) < t^2$, which completes the proof.

We are now able to give the following

DEFINITION 1.2. Let $\varphi \in \phi$ be a comparison function. A function $r_\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ given by $r_\varphi(t) = \sqrt{2t\varphi(t) - \varphi^2(t)}$ is called the *transformate* of φ.

LEMMA 1.2. *Let $\varphi \in \phi$ be a comparison function and r_φ its transformate. Then*

a) $r_\varphi \in \phi$;

b) *The mapping $r:\phi \to \phi$, $r(\psi) = r_\psi$ is bijective;*

c) $\varphi(t) < r_\varphi(t)$, *for each* $t > 0$.

Proof.

a) follows from Lemma 1.1.

b) It suffices to show that for any $\psi \in \phi$ there exists a unique $\varphi \in \phi$ there a unique $\varphi \in \phi$ such that

$$2t\varphi(t) - \varphi^2(t) = \psi^2(t), \quad t \in \mathbb{R}_+. \tag{1}$$

First, we observe that for $t = 0$, it follows $\varphi(0) = 0$. Then, let $t \neq 0$ be arbitrary but fixed. Denote $a = \frac{\psi(t)}{t}$, $x = \frac{\varphi(t)}{t}$. Since $\psi \in \phi$, $a \in (0,1)$. From (1) we obtain the equation

$$x^2 - 2x + a^2 = 0$$

which has a unique solution $x \in (0,1)$, $x = 1 - \sqrt{1-a^2}$.

Hence

$$\varphi(t) = t - \sqrt{t^2 - \psi^2(t)}, \quad t \in \mathbb{R}_+,$$

is the unique solution of (1), that is, r is bijective.

c) It is obvious.

DEFINITION 1.3. Let (X,d) be a metric space. A mapping $f:X \to X$ is called *φ-contraction* if there exists a comparison function $\varphi \in \phi$ such that

$$d(f(x),f(y)) \le d(x,y) - \varphi(d(x,y)), \quad \forall\, x,y \in X. \tag{2}$$

We need the following generalization of the contraction mapping principle

THEOREM 1.1. (KRASNOSELSKI and STECENKO [6], RUS,A.I. [7]).

Let (X,d) be a complete metric space and $f: X \to X$ a φ-contraction. Then f has a *unique fixed point that can be found by using functional iteration starting at an arbitrary point x_0 in X.*

2. ϕ-monotone operators. Let H be a real Hilbert space whose norm and inner product are denoted as usually by $\|\cdot\|$ and $<\cdot,\cdot>$, respectively. An operator $G : H \to H$ is called *monotone operator* if

$$<Gu - Gv, u - v> \geq 0, \quad \forall u, v \in H. \tag{3}$$

G is called *strictly monotone operator* if equality in (3) implies $u = v$.

G is said to be *strongly monotone operator* if there exists $m > 0$ such that

$$<Gu - Gv, u - v> \geq m \cdot \|u - v\|^2, \quad \forall u, v \in H. \tag{4}$$

The operator G is called *Lipschitz operator* if there exists $M > 0$ such that

$$\|Gu - Gv\| \leq M \cdot \|u - v\|, \quad \forall u, v \in H. \tag{5}$$

LEMMA 2.1. *If $G: H \to H$ is an operator which satisfies* (4) *and* (5) *then $m \leq M$.*

Proof. Since G is strongly monotone, hence monotone, the Cauchy-Schwarz inequality yields.

$$<Gu - Gv, u - v> \leq \|Gu - Gv\| \cdot \|u - v\|, \quad \forall u, v \in H$$

which together with (4) and (5) gives $m\|u - v\|^2 \leq M \cdot \|u - v\|^2, \ \forall u, v \in H$, that is $m \leq M$.

Remark. If $G: H \to H$ is strongly monotone, then G is injective. For an injective operator G and a given comparison function ψ, let denote by $t_1 = t_1(G,\psi)$, $t_2 = t_2(G,\psi)$ $t_1 < t_2$, the (assumed) real roots of the quadratic equation

$$\|Gu - Gv\|^2 t^2 - 2 \cdot <Gu - Gv, u - v> \cdot t + \psi^2(\|u - v\|) = 0, \ u, v \in H, u \neq v. \tag{6}$$

DEFINITION 2.1. We say that an injective operator $G: H \to H$ is a *ϕ-monotone* *operator* if there exists $\psi \in \phi$ such that

$$(m_1) \quad \langle Gu - Gv, u - v\rangle \geq \|Gu - Gv\| \cdot \psi(\|u - v\|), \ \forall\, u, v \in H;$$

$$(m_2) \quad \bigcap_{\substack{u,v \in H \\ u \neq v}} [t_1(G,\psi), t_2(G,\psi)] \neq \phi .$$

LEMMA 2.2. *Any strongly monotone and Lipschitz operator is a ϕ-monotone operator.*

Proof. Assume G satisfies (4) and (5). To prove (m_1) it suffices to show that there exists $\psi \in \phi$ such that

$$m \cdot \|u - v\|^2 \geq M \cdot \|u - v\| \cdot \psi(\|u - v\|), \ \forall\, u, v \in H$$

or equivalently,

$$m\|u - v\| \geq M \cdot \psi(\|u - v\|), \ \forall\, u, v \in H. \qquad (*)$$

If $m = M$, (*) holds for any $\psi(t) = at$, $0 < a < 1$, $t \in \mathbf{R}$, and if $m < M$, (*) holds for $\psi(t) = \frac{m}{M} t$. For the second part of the proof, let us observe that $[t_1, t_2]$ is the solution of the inequation obtained from (6) replacing "=" by "≤", hence (m_2) is equivalent to the following condition: there exists $\gamma > 0$ such that

$$\|Gu - Gv\|^2 \gamma^2 - 2 \cdot \langle Gu - Gv, u - v\rangle \cdot \gamma + \varphi^2(\|u - v\|) \leq 0, \ \forall\, u, v \in H.$$

Using (4) and (5) we have

$$\begin{aligned} \|Gu - Gv\|^2 \gamma^2 - 2 \cdot \langle Gu - Gv, u - v\rangle \cdot \gamma + \varphi^2(\|u - v\|) \leq \\ \leq (M^2\gamma^2 - 2\gamma m) \cdot \|u - v\|^2 + \varphi^2(\|u - v\|), \end{aligned} \qquad (7)$$

hence, to prove (7) it suffices to show that for certain $\gamma > 0$ and $\varphi = r_\psi$ the inequality

$$(M^2\gamma^2 - 2\gamma m) \cdot \|u - v\|^2 + \varphi^2(\|u - v\|) \leq 0, \ \forall\, u, v \in H \qquad (8)$$

holds.

If $m = M$, then $\psi(t) = at$, $0 < a < 1$ and (8) holds for $\gamma = \frac{1}{M}$, since

$$(m^2\gamma^2 - 2\gamma M)\|u - v\|^2 + a^2\|u - v\|^2 < (M^2\gamma^2 - 2\gamma M + 1)\|u - v\|^2 = 0 .$$

If $m < M$, when $\varphi(t) = \frac{m}{M}t$, (8) holds for $\gamma = \frac{m}{M^2}$.

Indeed, in this case we have

$$(M^2\cdot\gamma^2 - 2\gamma m)\,\|u - v\|^2 + \varphi^2(\,\|u - v\|\,) = \left(M^2\cdot\frac{m^2}{M^4} - 2\frac{m^2}{M^2} + \frac{m^2}{M^2}\right)\cdot\|u - v\|^2 = 0\,.$$

Remark. The class of ϕ-monotone operators is larger than the class of strongly monotone and Lipschitz operators as shown by theorem 3.1 together with theorem 1.1 and theorem 3.2.

3. Fixed points. Let H be, as in the previous section, a real Hilbert space and let G: $H \to H$ be a given operator. For every $\gamma > 0$, let us define the operator T_γ: $H \to H$, given by

$$T_\gamma = I - \gamma G\,, \tag{9}$$

where I is the identity operator.

Such a procedure plays an important role in many practical problems, when the problem of solving the operatorial equation

$$G(x) = 0$$

is reduced (if possibly) to the fixed point problem:

$$T_\gamma(x) = x \tag{10}$$

Thus we are interested to convert the monotonic hypothesis on G in adequate conditions of contractive type on T_γ, in order to obtain an iterative method to solve (10).

The main result of this paper is given by the following

THEOREM 3.1. *Let H be a real Hilbert space, G: $H \to H$ a given operator and let T_γ be the operator defined by* (9).

Then, G is a ϕ-monotone operator if and only if there exists $\gamma > 0$ such that T_γ is a ϕ-contraction.

Proof. Assume T_γ is a φ-contraction. We shall prove that there exists $\psi \in \phi$ such that (m_1) and (m_2) holds. We have from (9)

$$\|T_\gamma u - T_\gamma v\|^2 = \|u - v - \gamma(Gu - Gv)\|^2 = \\ = \|u - v\|^2 - 2\gamma \cdot < Gu - Gv, u - v > + \gamma^2 \cdot \|u - v\|^2, \forall u, v \in H \tag{11}$$

But T_γ is φ-contraction if and only if there exists $\varphi \in \phi$ such that

$$\| T_\gamma u - T_\gamma v \| \le \| u - v \| - \varphi(\| u - v \|), \ \forall u, v \in H \tag{12}$$

Thus from (11) and (12) we deduce that there exists $\varphi \in \phi$ and $\gamma > 0$ such that

$$\|Gu - v\|^2 \cdot \gamma^2 - 2 \cdot < Gu - Gv, u - v > \gamma + r_\varphi^2(\| u - v \|) \le 0, \ \forall u, v \in H, \tag{13}$$

that is, the inequation

$$\|Gu - Gv\|^2 \cdot t^2 - 2 \cdot < Gu - Gv, u - v > \cdot t + r_\varphi^2(\| u - v \|) \le 0, \ u, v \in H \tag{14}$$

has a positive solutions $t = \gamma$ and γ does not depend on $u, v \in H$.

This implies, by a hand, that

$$< Gu - Gv, u - v > \ge 0, \ \forall u, v \in H$$

(otherwise (13) is impossible for $u \ne v$), and on the other hand that

$$< Gu - Gv, u - v >^2 - \| Gu - Gv \|^2 \cdot r_\varphi^2(\| u - v \|) \ge 0, \ \forall u, v \in H \tag{15}$$

From (14) and (15) we obtain (m_1) with $\psi = r_\varphi$.

Let $t_1(G, \psi)$, $t_2(G, \psi)$, $t_1(G, \psi) < t_2(G, \psi)$ the real roots of the equation associate to (14). Then $\gamma \in [t_1(G, \psi), t_2(G, \psi)]$, for each $u, v \in H$, $u \ne v$, that is (m_2) holds. Hence G is φ-monotone operator. The converse is obvious. The proof is now complete.

Remark. From the proof of Lemma 2.2 it results that if G is strongly monotone and Lipschitz operator then G is ϕ_1-monotone, where $\phi_1 = \{\varphi \in \phi \,/\, \varphi(t) = at, \ 0 < a < 1\}$ is the class of linear comparison functions. Thus we obtain from theorem 3.1 the results of DINCĂ ([5], theorem 12.32, p.520; see also DINCĂ, BLEBEA [4]).

THEOREM 3.2. *Let H be a real Hilbert space, $G: H \to H$ a given operator and T_γ:*

$H \to H$ defined by (9). *Then, G is strongly monotone and Lipschitz operator if and only if there exists* $\gamma > 0$ *such that* T_γ *is a contraction.*

Remark. There exist nonlinear comparison functions, see example 1.2, hence the class of ϕ-monotone operators is larger than the class of strongly monotone and Lipschitz operators. From theorem 3.1 we obtain following.

CORROLARY 3.1. *Let H be a real Hilbert space and* $G: H \to H$ *a* ϕ*-monotone operator. Then the equation*

$$G(x) = 0$$

has a unique solution $x^* \in H$ *and there exists* $\gamma > 0$ *such that the sequence of successive approximations* $(x_n)_{n \in N}$ *defined by*

$$x_{n+1} = x_n - \gamma \cdot G(x_n),\ n \geq 0$$

converges to x^*, *for each* $x_0 \in H$.

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