STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 2, June 1999

GENERALIZED CONTRACTIONS FOR SOLVING RIGHT FOCAL POINT BOUNDARY VALUE PROBLEMS

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Abstract. The main goal of the present paper is to use the generalized contraction mapping principle [4] instead of the classical contraction mapping principle, in order to obtain a more general existence and uniqueness theorem for the nth order ordinary differential equation with deviating arguments (1.1) - (1.3).

1. Introduction

Second order as well as higher order boundary value problems with deviating arguments arise naturally in several engineering applications. In spite of their practical importance, only a few papers are devoted to boundary value problems(see [2] and references therein), even if initial value problems for higher order differential equations with deviating arguments have been studied intensively. Consequently, let us consider, as in [2] (all concepts and notations related to ODE are taken from this paper), the nth order ordinary differential equation with deviating arguments

$$x^{(n)}(t) = f(t, x \circ w(t)), t \in [a, b], \tag{1.1}$$

where $x \circ w(t)$ stands for $(x(w_{0,1}(t)), ..., x(w_{0,p(0)}(t)), ..., x^{(q)}(w_{q,p(q)}(t)))$, $0 \leq q \leq n-1$ (but fixed), and $p(i), 0 \leq i \leq q$, are positive integers.

The function $f(t, < x >)$ is assumed to be continuous on $[a, b] \times \mathbf{R}^N$,where $< x >$ represents $(x_{0,1}, ..., x_{0,p(0)}, ..., x_{q,p(q)})$ and $N = \sum_{i=0}^{q} p(i)$. The functions

$$w_{i,j}, 1 \leq j \leq p(i), 0 \leq i \leq q,$$

1991 *Mathematics Subject Classification.* 34K10.

Key words and phrases. generalized contraction, deviating argument, boundary value.

are continuous on $[a,b]$ and $w_{i,j}(t) \le b$ for all $t \in [a,b]$;

Also, they assume the value a at most a finite number of times as t ranges over $[a,b]$.

Let

$$\alpha = \min\{a, \inf_{a \le t \le b} w_{i,j}(t), \quad 1 \le j \le p(i), \quad 0 \le i \le q\}.$$

If $\alpha < a$, we assume that a function $\varphi \in C^{(q)}[\alpha, a]$ is given.

Let k be a fixed integer such that $1 \le k \le n-1$ and let $r = \min\{q, k-1\}$.

We seek a function

$$x \in B = C^{(r)}[\alpha, b] \cap C^{(q)}[\alpha, a] \cap C^{(q)}[a, b],$$

having at least a piecewise continuous nth derivative on $[a,b]$, and such that:

if

$$\alpha < a \quad and \quad q \ge k-1, then \quad x^{(i)}(t) = \varphi^{(i)}(t), 0 \le i \le q, \quad t \in [\alpha, a]; \tag{1.2}$$

if $\alpha < a$ and $q < k-1$, then

$$x^{(i)}(t) = \varphi^{(i)}(t), \quad 0 \le i \le q, \quad t \in [\alpha, a];$$
$$x^{(i)}(a) = A_i, \quad q+1 \le i \le k-1;$$

if $\alpha = a$, then

$$x^{(i)}(a) = A_i, \quad 0 \le i \le k-1$$

and

$$x^{(i)}(b) = B_i, \quad k \le i \le n-1; \tag{1.3}$$

Also, x is a solution of (1.1) on $[a,b]$.

2. Equivalent integral equation

To obtain an existence and uniqueness theorem for the boundary value problem (1.1)-(1.3) we shall convert it into its equivalent integral equation representation. To this end we need the Green's function expression, $g(t,s)$, for the boundary value problem

$$x^{(n)} = 0, \quad x^{(i)}(a) = 0, \quad 0 \le i \le k-1, \quad x^{(i)}(b) = 0, \quad k \le i \le n-1. \tag{2.1}$$

From Lemma 2.1 [2], we have that $g(t,s)$ is given by

$$g(t,s)=\begin{cases} \frac{1}{(n-1)!}\sum\limits_{i=0}^{k-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & if \quad s\le t,\\ -\frac{1}{(n-1)!}\sum\limits_{i=k}^{n-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & if \quad s\ge t. \end{cases}$$

It is known [2] that

$$(-1)^{n-k}g^{(i)}(t,s)\ge 0,\quad 0\le i\le k,\qquad (t,s)\in[a,b]\times[a,b];$$

$$(-1)^{n-i}g^{(i)}(t,s)\ge 0,\quad k+1\le i\le n-1,\qquad (t,s)\in[a,b]\times[a,b];$$

$$\sup_{a\le t\le b}\int_a^b |\, g^{(i)}(t,s)\,|\, ds\le C_{n,i}(b-a)^{n-i},\qquad 0\le i\le n-1,$$

where $g^{(i)}(t,s)=\partial^i g(t,s)/\partial t^i$ and

$$C_{n,i}=\begin{cases} \frac{1}{(n-1)!}\,|\sum\limits_{j=0}^{k-i-1}\binom{n-1}{j}(-1)^{n-j-1}\,|, & 0\le i\le k-1,\\ \frac{1}{(n-1)!}, & k\le i\le n-1. \end{cases}$$

The boundary value problem (1.1)-(1.3) is equivalent to the integral equation

$$x(t)=\psi(t)+\theta(t)\int_a^b g(t,s)f(s,x\circ w(s))ds, \tag{2.2}$$

where

$$\theta(t)=\begin{cases} 0, & t\in[\alpha,a]\\ 1, & otherwise, \end{cases}$$

and the function ψ is defined as follows.

If $\alpha<a$ and $q\ge k-1$, then

$$\psi(t)=\begin{cases} \varphi(t), & t\in[\alpha,a],\\ P_{n-1}(t), & t\in[a,b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a),\ 0 \leq i \leq k-1,\quad \beta_i = B_i,\ k \leq i \leq n-1$,and $p_{n-1}(t)$ is the unique polynomial (see Lemma 2.2,[2]) of degree $n-1$ satisfying

$$P_{n-1}^{(i)}(a) = \alpha_i, \quad 0 \leq i \leq k-1 \text{ and } P_{n-1}^{(i)}(b) = \beta_i, k \leq i \leq n-1.$$

If $\alpha < a$ and $q < k-1$,then

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a), 0 \leq i \leq q,\ \alpha_i = A_i,\ q+1 \leq i \leq k-1$, and $\beta_i = B_i,\ k \leq i \leq n-1$.

If $\alpha = a$, then $\psi(t) = P_{n-1}(t), t \in [\alpha, a]$, where

$$\alpha_i = A_i, 0 \leq i \leq k-1 \text{ and } \beta_i = B_i, k \leq i \leq n-1.$$

It is easy to see that $\psi \in B$, and for all $t \in [a, b]$,with

$$w_{i,j}(t) = a,\ \psi^{(i)}(w_{i,j}(t)) = P_{n-1}^{(i)}(a+0).$$

3. Generalized contraction mapping principle and main result

We shall use a local variant of the generalized contraction mapping principle [4, Theorem 1.5.1.] to state our main result.

Lemma 3.1. (Generalized contraction mapping principle [4]). *Let (X, d) be a complete metric space and let $\mu > 0,\ \mu \in \mathbf{R},\ \overline{S}(u_0, \mu) = \{u \in X : d(u, u_0) \leq \mu\}$. Further, let T be an operator which maps $\overline{S}(u_0, \mu)$ into X, and*

(i) for all $u, v \in \overline{S}(u_0, \mu), d(Tu, Tv) \leq \phi(d(u, v))$, where ϕ is a (c)-comparison function;

(ii) $\mu_0 = d(Tu_0, u_0) \leq \mu - \phi(\mu)$.

Then

(1) T has a fixed point u^* in $\overline{S}(u_0, \mu_0)$;

(2) u^* is the unique fixed point of T in $\overline{S}(u_0, \mu_0)$;

(3) the sequence $\{u_m\}$, where $u_{m+1} = Tu_m, m = 0, 1, \dots$, converges to u^* with

$$d(u^*, u_m) \leq s(\phi^m(d(u_0, u_1)))$$

and

$$d(u^*, u_m) \leq s(d(u_m, u_{m+1}));$$

where $s(t)$ is the sum of the series $\sum_{k=0}^{\infty} \phi^k(t)$.

(4) for any $u \in \overline{S}(u_0, \mu_0), u^* = \lim_{m\to\infty} T^m u$.

Remark. For the notion of (c)-comparison function we refer to [4]. A typical comparison function is

$$\phi(t) = \lambda t, \qquad 0 \leq \lambda < 1, \qquad t \in [0, \infty). \tag{3.1}$$

For ϕ given by (3.1), from Lemma 3.1 we obtain Lemma 2.3 [2].

Let $\overline{A}_i,\ 0 \leq i \leq k-1$ and $\overline{B}_i,\ k \leq i \leq n-1$, be given fixed numbers and $\psi_2 \in B$ the function defined in [2], Section 4. Following [2], a function $\overline{x} \in B$ is called an *approximate solution* of (2.2) if there exist nonnegative constants ϵ and δ such that wherever $\psi^{(i)}(t)$, $\psi_2^{(i)}(t)$ and $\overline{x}^{(i)}(t)$ are defined,

$$\sup_{\alpha\leq t\leq b} | \psi_2^{(i)}(t) - \psi^{(i)}(t) | \leq \epsilon C_{n,i}(b-a)^{n-i}, \quad 0 \leq i \leq q, \tag{3.2}$$

$$\sup_{\alpha\leq t\leq b} | \overline{x}^{(i)}(t) - \psi_2^{(i)}(t) - \theta(t) \int_a^b g^{(i)}(s,t) f(s, \overline{x} \circ w(s)) ds | \leq \delta C_{n,i}(b-a)^{n-i}, 0 \leq i \leq q. \tag{3.3}$$

If we consider the following norm on the space B:

$$\| x \| = \max_{0\leq i\leq q} \left\{ \left(\tfrac{C_{n,0}(b-a)^i}{C_{n,i}}\right) \sup_{\alpha\leq t\leq b} | x^{(i)}(t) | \textit{ wherever } \quad x^{(i)}(t) \quad \textit{exists} \right\}$$

and apply Lemma 3.1 we can prove in a standard way.

Theorem 3.1.. *Suppose that (2.2) has an approximate solution* $\overline{x} \in B$ *and*

(i) f satisfies the Lipschitz condition

$$| f(t, < x >) - f(t, < y >) | \leq \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} | x_{i,j} - y_{i,j} |,$$

for all $(t, < x >), (t, < y >) \in [a, b] \times D_1$, *where*

$$D_1=\left\{<x>: \mid x_{i,j}-x^{(i)}(w_{i,j}(t)) \mid \leq \mu \cdot \frac{C_{n,i}}{C_{n,0}(b-a)^i},\quad 1\leq j\leq p(i),\quad 0\leq i\leq q\right\};$$

(ii) ϕ *is a (c)-comparison function and*

$$(\epsilon+\delta)C_{n,0}(b-a)^n \leq \mu-\phi(\mu). \tag{3.4}$$

Then

(1) There exists a solution $x^*(t)$ *of (1.1)-(1.3) in* $\overline{S}(\overline{x},\mu_0)$;

(2) $x^*(t)$ *is the unique solution of (1.1)-(1.3) in* $\overline{S}(\overline{x},\mu_0)$;

(3) The sequence $\{x_m(t)\}$ *of successive approximations, defined by*

$$x_{m+1}(t)=\psi(t)+\theta(t)\int_a^b g(t,s)f(s,x_m\circ w(s))ds,\quad m=0,1,...$$

and $x_0(t)=\overline{x}(t)$, *converges to* $x^*(t)$ *with*

$$\| x^* - x_m \| \leq s(\phi^m(\| u_0-u_1 \|)),$$

$$\| x^* - x_m \| \leq s(\| u_m-u_{m+1} \|);$$

(4) for any $x_0(t)=x(t)$, *where* $x\in\overline{S}(\overline{x},\mu_0)$, *the iterative process converges to* $x^*(t)$.

Remarks

1) For $\phi(t)$ as given by (3.1), from Theorem 3.1 we obtain Theorem 4.1 in [2];

2) If , for instance , we take the comparison function $\phi:\mathbf{R}_+\to\mathbf{R}_+$, given by :

$$\phi(t)=\begin{cases}\frac{1}{2}t, & 0\leq t\leq 1\\ t-\frac{1}{3}, & t>1,\end{cases}$$

then an operator T, which satisfies all assumptions in Theorem 3.1, will be generally not a contractive operator (with respect to the norm , see [4]), that is, an operator satisfying for all $u,v\in\overline{S}(\overline{x},\mu_0)$, the classical contraction condition

$$\| Tu-Tv \| \leq \lambda \| u-v \|,\quad 0<\lambda<1,$$

but T is a generalized contractive operator. Consequently, Theorem 4.1 from [2] does not apply, while Theorem 3.1 apply to this class of higher order differential equation with deviating argument.

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