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SEQUENCES OF OPERATORS AND FIXED POINTS IN QUASIMETRIC SPACES

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Rezumat. Siruri de operatori si puncte fixe in spatii quasimetrice.
Pentru un sir de operatori (f_n) pe un spatiu quasimetric, care converge catre un operator φ -contractiv f, este data o teorema privind convergenta sirului punctelor fixe ale lui $f_n, (x_n)$, catre unicul punct fix al lui f .
In particular, se obtine un rezultat al lui S.B.Nadler.

Abstract. For a sequence of certain selfoperators (f_n) of a quasimetric space X, uniformly convergent to a φ-contraction f, we establish a convergence theorem for the sequence of fixed points of $f_n, (x_n^*)$, to the unique fixed point of f.

This paper is in continuation with our investigations concerning the generalized contraction mapping principle in quasimetric spaces [2]. Its main goal is to extend a result of S.B.NADLER [3] for contractions in usual metric spaces to generalized contractions in quasimetric spaces. In order to state

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Theorem 3 (the main result of this paper) we need some definitions , examples and results from [1]-[4] which we summarize here.

A quasimetric space is a nonempty set X endowed with a **quasimetric**, that is a function $d : X \times X \to \mathbf{R}_+$,satisfying the following conditions:

d1) $d(x,y) = 0$ if and only if $x = y$,

d2) $d(x,y) = d(y,x), \forall x,y \in X$;

d3) $d(x,z) \leq a[d(x,y) + d(y,z)], \forall x,y,z \in X$,

where $a \geq 1$ is a given real number, see [1].

Obviously, when $a = 1$ we obtain the usual notion of **metric** (**space**).

Example 1. [1] The space $l_p (0 < p < 1)$,

$$l_p = \left\{ (x_n) \subset \mathbf{R} \; / \; \sum_{\mathbf{n=1}}^{\infty} |\mathbf{x_n}|^{\mathbf{p}} < \infty \right\},$$

together with the function $d : l_p \times l_p \to \mathbf{R}$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = (x_n), y = (y_n) \in l_p$, is a quasimetric space. Indeed, by an elementary calculation we obtain

$$d(x,z) \leq 2^{\frac{1}{p}} [d(x,y) + d(y,z)],$$

hence $a = 2^{\frac{1}{p}} > 1$ in this case.

Example 2.[1]. The space $L_p (0 < p < 1)$ of all real functions $x(t)$, $t \in [0,1]$,such that

$$\int_0^1 |x(t)|^p \, dt < \infty,$$

becomes a quasimetric space if we take

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt \right)^{1/p}, \text{for each } x,y \in L_p.$$

The constant a is the same as in the previous example,

$$a = 2^{\frac{1}{p}}.$$

DEFINITION 1. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is called **comparison function** if

(i) φ is monotone increasing;

(ii) $\varphi^n(t) \to 0$, as $n \to \infty$,for each $t \in \mathbf{R}_+$.

Example 3. The mapping $\varphi(t) = \alpha t, t \in \mathbf{R}_+$,where $0 \leq \alpha < 1$, is a comparison function.

DEFINITION 2. Let (X, d) be a quasimetric space. A mapping $f : X \to X$ is called **φ-contraction** if there exists a comparison function φ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \forall x, y \in X. \tag{1}$$

Remark. A $\varphi-$contraction with φ as in Example 3 is an usual contraction.

DEFINITION 3. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ which satisfies:

(i) φ is monotone increasing (isotone);

(ii) There exist a convergent series of positive terms $\sum\limits_{n=0}^{\infty} v_n$ and a real number $\alpha, 0 \leq \alpha < 1$, such that

$$\varphi^{k+1}(t) \leq \alpha\varphi^k(t) + v_k, \text{ for each t} \in \mathbf{R}_+ \text{ and n} \geq \text{N (fixed)} \tag{2}$$

is called **(c)-comparison function** (φ^k stands for the k the iterate of φ).

Remark.1) Using a generalization of the ratio test [5],[7], it results that if φ is a (c)-comparison function then the series

$$\sum_{k=0}^{\infty} \varphi^k(t) \tag{3}$$

converges for each $t \in \mathbf{R}_+$,therefore

$$\varphi^k(t) \to 0, \text{ as } k \to \infty,$$

that is any (c)-comparison function is a comparison function too.

2) If we denote by $s(t)$ the sum of the series (4) then, s is monotone increasing and continuous at zero.

Example 4. The function given in example 3 is a (c)-comparison function but, generally, a comparison function is not a (c)-comparison function, see [2]-[3].

THEOREM 1.([2]) *Let (X,d) be a complete quasimetric space and $f : X \to X$ a $\varphi-$contraction. Then f has a unique fixed point if and only if there exists $x_0 \in X$, such that the sequence $(x_n)_{n\in\mathbf{N}}$ of the succesive approximations,*

$$x_n = f(x_{n-1}), n \in \mathbf{N},$$

is bounded.

In order to establish a generalized fixed point principle which furnishes an approximation method to the fixed point,we have to consider a stronger concept than that of comparison function.

DEFINITION 4. Let $a \geq 1$ be a given real number. A mapping $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$is called **a-comparison function** if f satisfies (i) and

(iii) there are a convergent series of positive terms $\sum\limits_{n=0}^{\infty} v_n$ and a real number $\alpha, 0 \leq \alpha < 1$ such that

$$a^{k+1}\varphi^{k+1}(t) \leq \alpha \cdot a^k\varphi^k(t) + v_k, \text{for t} \in \text{R}_+ \text{ and for k} \geq \text{N (fixed)} \quad (4)$$

Remark. In view of the generalized ratio test [2] , the series

$$\sum_{k=0}^{\infty} a^k\varphi^k(t) \quad (5)$$

converges for each $t \in \mathbf{R}_+$ and its sum, denoted by $s_a(t)$, is monotone increasing and continuous at zero.

Obviously , for $a = 1$ (that is, d is a metric on X) such an a-comparison function is a actually (c)-comparison function.

Based on this concept, in [2] we established the following generalized contraction principle in quasimetric spaces

THEOREM 2. *Let (X,d) be a complete quasimetric space , $f : X \to X$ a φ - contraction with φ an a-comparison function. If $x_0 \in X$ is such that the sequence (x_n),*

$$x_n = f(x_{n-1}), n \in N^*,$$

is bounded and $F_f = \{x^*\}$, *then we have*

$$d(x_n, x^*) \leq a \cdot s_a(d(x_n, x_{n+1})), n \geq 0, \tag{6}$$

where $s_a(t)$ *is the sum of the series (5).*

Remark.1) If $a = 1$,Theorem 2 is just the generalized contraction principle, given in [2]-[4];

2) When $\varphi(t)$ is as in example 3, the condition (5) is satisfied if $\alpha \in [0; 1[$ is such that

$$\alpha a < 1.$$

In concrete problems we need to compute the fixed point x^*. An usual way is to seek for a sequence of operators (f_n) which approximates (uniformly) the given operator f such that $F_{fn} \neq \emptyset$, for each $n \in \mathbf{N}^*$ and the set F_{fn} is easier to be determined.

The following question then arise: in what conditions over f and f_n , from $x_n^* \in F_{fn}$,it results that

$$x_n^* \to x^*, \text{ as } n \to \infty \ ?$$

If the answer is in the affirmative, then x^* may be approximated by x_n^*, for n sufficiently large.

The main result of this paper is given by.

THEOREM 3. *Let* (X, d) *be a complete quasimetric space,* f, $f_n : X \to X (n \in \mathbf{N}^*)$ *such that*

1) f *satisfies the assumptions of Theorem 2 and, in addition,* φ *is subadditive;*

2) (f_n) *converges uniformly to* f *on* X;

3) $x_n^* \in F_{fn}$, *for each* $n \in \mathbf{N}$.

Then (x_n^*) *converges to* x^*, *the unique fixed point of* f.

Proof. From (d_3) we have

$$d(x_n^*, x^*) = d(f_n(x_n^*), f(x^*)) \leq a[d(f(x_n^*), f(x^*) + d(f_n(x_n^*), f(x_n^*))],$$

and, in view of the contraction condition, that is,

$$d(f(x_n^*), f(x^*) \leq \varphi(d(x_n^*, x^*)),$$

we obtain

$$d(x_n^*, x^*) \leq a\varphi(d(x_n^*, x^*)) + ad(f_n(x_n^*), f(x_n^*)).$$

Based on the subadditivity of φ we then obtain by induction

$$d(x_n^*, x^*) \leq a^{k+1}\varphi^{k+1}(d(x_n^*, x^*)) + \sum_{i=0}^{k} a^i\varphi^i(d(f_n(x_n^*), f(x_n^*))), k \geq 0 \qquad (7)$$

But φ is a-comparison function, hence the series (3) is convergent. This implies

$$a^{k+1}\varphi^{k+1}(d(x_n^*, x^*)) \to 0, \text{ as } k \to \infty$$

and then from (7) we obtain

$$d(x_n^*, x^*) \leq s_a(d(f_n(x_n^*), f(x_n^*))).$$

But s_a is continuous at zero and then,from 2) we deduce that

$$d(f_n(x_n^*), f(x_n^*)) \to 0, \text{ as } n \to \infty,$$

therefore

$$d(x_n^*, x^*) \to 0, \text{ as } n \to \infty$$

that is $x_n^* \to x^*, n \to \infty$, as required.

Remark. 1) For φ as in Example 3, from Theorem 3 we obtain a theorem of Nadler type in quasimetric spaces;

2) For $a = 1$ and φ as in Example 3, from Theorem 3 we obtain a result of Nadler [3].

REFERENCES

1.BAHTIN, I.A., The contraction mapping principle in quasimetric spaces (Russian), *Func.An.*, No. 30, 26-37,Unianowsk,Gos.Ped.Ins.,1989

2.BERINDE,V., Generalized contractions in quasimetric spaces, *Seminar on Fixed Point Theory,* Preprint nr.3,1993, pp 3-9, "Babes-Bolyai" University

3.NADLER,S.B., Sequences of contractions and fixed points, *Pacific J. Math.,* vol 27 (1968), No3, 579-585

4.RUS,A.I.,Generalized contractions, *Seminar on Fixed Point Theory,* Preprint No.3, 1983,1-130,"Babes Bolyai" University of Cluj - Napoca