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Approaching the split common solution problem for nonlinear demicontractive mappings by means of averaged iterative algorithms

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Abstract. We consider new iterative algorithms for solving split common solution problems in the class of demicontractive mappings. These algorithms are obtained by inserting an averaged term into the algorithms previously used in [He, Z. and Du, W-S., Nonlinear algorithms approach to split common solution problems, *Fixed Point Theory Appl.* **2012**, 2012:130, 14 pp] for the case of quasi-nonexpansive mappings. In this way, we are able to solve the split common solution problem in the larger class of demicontractive mappings, which strictly includes the class of quasi-nonexpansive mappings. Our investigation is based on the embedding of demicontractive operators in the class of quasi-nonexpansive operators by means of averaged mappings. For the considered algorithms we prove weak and strong convergence theorems in the setting of a real Hilbert space.

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1. Introduction

Let $\mathcal{C}$ and $\mathcal{D}$ be nonempty subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear bounded operator. Let also $f : C \times C \to \mathbb{R}$ and $F : D \times D \to \mathbb{R}$ be two bi-functions.

The *split equilibrium problem* (SEP), see [10], is asking to find a point $\overline{c} \in C$ such that

$$f(\overline{c}, c) \geq 0, \quad \text{for all } c \in C \tag{1.1}$$

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and

$$\overline{d} = A\overline{c} \in D \text{ is such that } F(\overline{d}, d) \geq 0, \text{ for all } d \in D. \tag{1.2}$$

Problem (1.1) alone is the classical equilibrium problem (EP) and its solution set is usually denoted by $EP(f)$.

Several important problems in nonlinear analysis, e.g., the optimization problems, variational inequalities problems, saddle point problems, the Nash equilibrium problems, fixed point problems, complementary problems, bilevel problems, and semi-infinite problems, are special cases of the classical equilibrium problem and have relevant applications in mathematical programming with equilibrium constraint, see [11] and references therein.

In turn, the split equilibrium problem (SEP) (1.1)+(1.2) defines a way to split the solution between two different subsets such that the solution of the equilibrium problem (1.1) and its image by the linear bounded operator A leads to the solution of the second equilibrium problem (1.2).

Let $G : \mathcal{C} \to \mathcal{C}$ be a mapping with $Fix(G) := \{v \in \mathcal{D} \ : \ Gv = v\} \neq \emptyset$ and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ a bi-function.

In this paper, our interest is to study the following *split common solution problem* (SCSP) for equilibrium problems and fixed point problems:

$$\text{find } u \in \mathcal{C} \text{ such that } u \in Fix\,(G)$$

and

$$Au \in \mathcal{D} \text{ with } f(Au, v) \geq 0, \quad \text{for all } \ v \in \mathcal{D}.$$

Denote the set of solutions of this problem by

$$\Omega := \{u \in Fix(G) : Au \in EP(f)\}.$$

Example 1.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, $\mathcal{C} = [0, 1]$ and $\mathcal{D} = [-100, -7/8]$. Let $Au = -u$ for all $u \in \mathbb{R}$ and

$$Gu = \begin{cases} 7/8, & \text{if } 0 \leq u < 1 \\ 1/4, & \text{if } u = 1. \end{cases} \tag{1.3}$$

The mapping G defined on $\mathcal{C}$ is $\frac{2}{3}$-demicontractive but it is neither quasi-nonexpansive nor nonexpansive [3]. Define $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by $f(u, v) = u - v$ for all $u, v \in \mathcal{D}$. It is clear that A is a linear bounded operator, $Fix(G) = \left\{\frac{7}{8}\right\}$ and $A\left(\frac{7}{8}\right) = -\frac{7}{8} \in EP(f)$. Thus, $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$.

Example 1.2. Let $\mathcal{H}_1 = \mathbb{R}^2$, $\mathcal{H}_2 = \mathbb{R}$ with the standard norms. Let $\mathcal{C} = \{u \in \mathbb{R}^2 : \|u\| \leq 1\}$ and $\mathcal{D} = [-100, -5/6]$. Let $Au = -u_2$ for all $u = (u_1, u_2) \in \mathbb{R}^2$ and

$$Gu = \begin{cases} (0, 5/6), & \text{if } u \neq (0, 1) \\ (0, 1/3), & \text{if } u = (0, 1). \end{cases} \tag{1.4}$$

It is easy to see that G defined on $\mathcal{C}$ is a $\frac{1}{2}$-demicontractive mapping.
Define $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by $f(v, w) = v - w$ for all $v, w \in \mathcal{D}$. It is clear that A is a linear bounded operator, $Fix(G) = \left\{(0, \frac{5}{6})\right\}$ and $A\left(0, \frac{5}{6}\right) = -\frac{5}{6} \in EP(f)$. Thus, $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$.

He and Du [11] presented some new iterative algorithms for solving the split common solution problems for equilibrium problems and fixed point problems of nonlinear quasi-nonexpansive mappings.

Our aim in this paper is to construct new averaged iterative algorithms for solving the split common solutions problem in the setting of Hilbert spaces for the the larger class of demicontractive mappings, thus extending the main results in He and Du [11].

Our results are obtained by considering new averaged iterative algorithms for which we prove weak and strong convergence theorems.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{D}\subset\mathcal{H}$ be a closed convex set, and consider the operator $G:\mathcal{D}\to\mathcal{D}$.

Recall that the mapping G is said to be

(a) *nonexpansive* if
$$\|Gu-Gv\|\leq\|u-v\|,\quad \text{for all } u,v\in\mathcal{D}; \tag{2.1}$$

(b) *quasi-nonexpansive* if $Fix(G)\neq\emptyset$ and
$$\|Gv-v^*\|\leq\|v-v^*\|,\quad \text{for all } v\in\mathcal{D} \text{ and } v^*\in Fix(G); \tag{2.2}$$

(c) *α-demicontractive* if $Fix(G)\neq\emptyset$ and there exists a positive number $\alpha<1$ such that
$$\|Gv-v^*\|^2\leq\|v-v^*\|^2+\alpha\|v-Gv\|^2, \tag{2.3}$$
for all $v\in\mathcal{D}$ and $v^*\in Fix(G)$;

(d) *firmly nonexpansive* if
$$\|Gu-Gv\|^2\leq\|u-v\|^2-\|u-v-(Gu-Gv)\|^2, \tag{2.4}$$
for all $u,v\in\mathcal{D}$.

By the above definitions, it is clear that any firmly nonexpansive mapping is nonexpansive, any nonexpansive mapping G with $Fix(G)\neq\emptyset$ is demicontractive and that any quasi-nonexpansive mapping is demicontractive, too, but the reverses are no longer true, as illustrated by the previous Examples 1.1 and 1.2.

It is well known, see [15], that any Hilbert space H satisfies the Opial's condition, that is, if $\{u_p\}$ is a sequence in $\mathcal{H}$ which converges weakly to a point $u\in\mathcal{H}$, then we have
$$\liminf_{p\to\infty}\|u_p-u\|<\liminf_{p\to\infty}\|u_p-v\|,\qquad \text{for all } v\in\mathcal{H}, v\neq u.$$
The following lemmas and proposition are very important in the proof our main results.

Lemma 2.1. [2] *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{D}\subset\mathcal{H}$ a closed and convex set. If $G:\mathcal{D}\to\mathcal{D}$ is α-demicontractive, then for any $\varphi\in(0,1-\alpha)$, the map*
$$G_\varphi=(1-\varphi)I+\varphi G$$
is quasi-nonexpansive.

Lemma 2.2. [11] *Let $\mathcal{H}$ be a real Hilbert space, $D \subset \mathcal{H}$ a closed and convex set and $G : D \to D$ a mapping. Then, for any $\varphi \in (0,1)$, we have $Fix(G_\varphi) = Fix(G)$.*

Definition 2.3. [13] Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ and G a mapping from $\mathcal{D}$ into $\mathcal{D}$. The mapping G is said to be zero-demiclosed if, for any sequence $\{u_p\}$ which weakly converges to u, and if the sequence $\{Gu_p\}$ strongly converges to zero, then $Gu = 0$.

Proposition 2.4. [11] *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a real Hilbert space with zero vector $\mathbf{0}$ and G a mapping from $\mathcal{D}$ into $\mathcal{D}$. Then the following assertions hold.*

- (i) *G is zero-demiclosed if and only if $I - G$ is demiclosed at $\mathbf{0}$;*
- (ii) *If G is a nonexpansive mappings and there is a bounded sequence $\{u_p\} \subset \mathcal{H}$ such that $\|u_p - Gu_p\| \to 0$ as $p \to 0$, then G is zero-demiclosed.*

Lemma 2.5. [7] *Let D be a nonempty, closed, and convex subset of $\mathcal{H}$ and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function that satisfies the following conditions.*

- (C1) *$f(v,v) = 0$ for all $v \in \mathcal{D}$;*
- (C2) *f is monotone, that is, $f(u,v) + f(v,u) \geq 0$;*
- (C3) *for every $u, v, w \in \mathcal{D}$, $\limsup_{t\to 0} f(tw + (1-t)u, v) \leq f(u,v)$;*
- (C4) *for every $u \in \mathcal{D}$, $F(v) \equiv f(u,v)$ is convex and lower semi-continuous.*

Let $\mu > 0$ and $u \in \mathcal{H}$. Then there exists $w \in \mathcal{D}$ such that

$$f(w,v) + \frac{1}{\mu}\langle v - w, w - u\rangle \geq 0$$

for all $v \in \mathcal{D}$.

Lemma 2.6. [9] *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{H}$ and let f be a bi-function from $\mathcal{D} \times \mathcal{D}$ into $\mathbb{R}$ that satisfies* (C1)-(C3)*. For $\mu > 0$ and $u \in \mathcal{H}$, define a mapping*

$$T_\mu^f(u) = \left\{ w \in \mathcal{D} : f(w,v) + \frac{1}{\mu}\langle v - w, w - u\rangle \geq 0, \text{ for all } v \in \mathcal{D} \right\}. \tag{2.5}$$

Then the following assertions hold:

- (a) *T_μ^f is single-valued and $f(T_\mu^f) = EP(f)$ for any $\mu > 0$ and $EP(f)$ is closed and convex;*
- (b) *T_μ^f is firmly nonexpansive.*

Lemma 2.7. [8] *The following assertions hold for all $u, v \in \mathcal{H}$.*

- (a) *$\|u+v\|^2 \leq \|v\|^2 + 2\langle u, u+v\rangle$ and $\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle$.*
- (b) *$\|au + (1-a)v\|^2 = a\|u\|^2 + (1-a)\|v\|^2 - a(1-a)\|u-v\|^2$ for $a \in [0,1]$.*

Lemma 2.8. [10] *Let T_μ^f be as in* (2.5)*. Then for $\mu, \tau > 0$ and $u, v \in \mathcal{H}$,*

$$\|T_\mu^f u - T_\tau^f v\| \leq \|u - v\| + \frac{|\tau - \mu|}{\tau}\left\|T_\tau^f v - v\right\|.$$

In particular, T_μ^f is nonexpansive for any $\mu > 0$.

The next lemma is due to Li and He [12] and will be useful in proving our main results.

Lemma 2.9. [12] *Let $F_1, \cdots, F_n : \mathcal{H}_1 \to \mathcal{H}_1$ be quasi-nonexpansive mappings and set $T = \sum_{i=1}^n b_i F_{a_i}$, where $b_i \in (0,1)$ with $\sum_{i=1}^n b_i = 1$, and $F_{a_i} = (1-a_i)I + a_i F_i$ with $a_i \in (0,1), i = 1, 2, \cdots, n$. Then T is quasi-nonexpansive and*

$$Fix(T) = \bigcap_{i=1}^n Fix(F_i) = \bigcap_{i=1}^n Fix(F_{a_i}).$$

3. Split common solutions in the class of demicotractive mappings

In this section we prove convergence theorems for averaged algorithms used for finding split common solutions for demicontractive mappings. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces.

In the following theorem, we prove the weak convergence of an averaged algorithm used for solving the split common solution for equilibrium problems and fixed point problems of nonlinear demicontractive mappings.

Theorem 3.1. *Let $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{D} \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $G : \mathcal{C} \to \mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function with*

$$\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1 \in \mathcal{C}, & \\ u_p = T^f_{\mu_p} A v_p, & \{\mu_p\} \subset (0,\infty), \\ v_{p+1} = (1-a_p)w_p + a_p\left[(1-\varphi)w_p + \varphi G w_p\right], & \varphi \in (0, 1-\alpha), \\ w_p = P_{\mathcal{C}}(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), & \beta \in \left(0, \frac{1}{\|A^*\|}\right), \quad \text{for all } p \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\liminf_{p\to\infty} \mu_p > 0$, $P_{\mathcal{C}}$ is the projection operator from $\mathcal{H}_1$ onto $\mathcal{C}$ and $\{a_p\}$ is a sequence in $[\varepsilon, 1-\varepsilon]$ with $\varepsilon \in (0, \frac{1}{2})$. Then $\{v_p\}$ converges weakly to $v^* \in \Omega$, and $\{u_p\}$ converges weakly to $Av^* \in EP(f)$.

Proof. Since G is an α-demicontractive mapping, in view of Lemma 2.1 the averaged mapping

$$G_\varphi := (1-\varphi)I + \varphi G \tag{3.2}$$

is quasi-nonexpansive for $\varphi \in (0, 1-\alpha)$. Here I is the identity mapping. Hence, in Algorithm (3.1) we can write

$$v_{p+1} = (1-a_p)w_p + a_p G_\varphi w_p.$$

Let $\Omega_\varphi := \{u \in Fix(G_\varphi) : Au \in EP(f)\} \neq \emptyset$ and $u \in \Omega_\varphi$. Using Lemma 2.6 and Lemma 2.7, it is easy to see that for any $p \in \mathbb{N}$,

$$\|T^f_{\mu_p} Av_p - Au\|^2 \leq \|Av_p - Au\|^2 - \|T^f_{\mu_p} Av_p - Av_p\|^2. \tag{3.3}$$

We also obtain

$$\begin{aligned}
&2\beta\langle v_p - u, A^*\left(T^f_{\mu_p} - I\right) Av_p\rangle \\
&\qquad = 2\beta\langle A(v_p - u) + (T^f_{\mu_p} - I)Av_p - (T^f_{\mu_p} - I)Av_p, (T^f_{\mu_p} - I)Av_p\rangle \\
&\qquad \le 2\beta\left(\frac{1}{2}\|(T^f_{\mu_p} - I)Av_p\|^2 - \|(T^f_{\mu_p} - I)Av_p\|^2\right) \\
&\qquad = -\beta\|(T^f_{\mu_p} - I)Av_p\|^2.
\end{aligned}$$

Since for any $p \in \mathbb{N}$,

$$\|A^*(T^f_{\mu_p} - I)Av_p\|^2 \le \|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2, \tag{3.4}$$

and G_φ is quasi-nonexpansive, we have

$$\begin{aligned}
&\|v_{p+1} - u\|^2 \\
&\ = (1 - a_p)\|w_p - u\|^2 + a_p\|G_\varphi w_p - u\|^2 - (1 - a_p)a_p\|w_p - G_\varphi w_p\|^2 \\
&\ \le \|w_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \qquad (\text{Since } \varepsilon \in [a_p, 1 - a_p]) \\
&\ = \|P_{\mathcal{C}}(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p) - P_{\mathcal{C}}u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\ \le \|v_p + \beta A^*(T^f_{\mu_p} - I)Av_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\ = \|v_p - u\|^2 + \|\beta A^*(T^f_{\mu_p} - I)Av_p\|^2 + 2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle - \varepsilon^2\|w_p - G_\varphi w_p\| \\
&\ \le \|v_p - u\|^2 + \beta^2\|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2 - \beta\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\| \\
&\ = \|v_p - u\|^2 - \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|. \qquad (3.5)
\end{aligned}$$

Since $\beta \in (0, \frac{1}{\|A^*\|^2})$ and $\beta(1 - \beta\|A^*\|^2) > 0$, we have

$$\|v_{p+1} - u\| \le \|w_p - u\| \le \|v_p - u\| \tag{3.6}$$

and by (3.5),

$$\varepsilon^2\|w_p - G_\varphi w_p\|^2 + \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 \le \|v_p - u\|^2 - \|v_{p+1} - u\|^2, \tag{3.7}$$

for any $p \in \mathbb{N}$. Note that since $u \in Fix(G_\varphi)$, it follows that the sequence $\{\|v_p - u\|\}$ is convergent. Inequalities (3.6) and (3.7) imply that

$$\lim_{p\to\infty} \|v_p - u\| = \lim_{p\to\infty} \|w_p - u\|, \tag{3.8}$$

$$\lim_{p\to\infty} \|w_p - G_\varphi w_p\| = 0$$

and

$$\lim_{p\to\infty} \|(T^f_{\mu_p} - I)Av_p\| = 0. \tag{3.9}$$

We obtain

$$\begin{aligned}
\|w_p - v_p\| &= \|P_{\mathcal{C}}\left(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p\right) - P_{\mathcal{C}}v_p\| \\
&\le \beta\|A^*(T^f_{\mu_p} - I)Av_p\| \to 0 \text{ as } p \to \infty.
\end{aligned}$$

Since $\lim_{p\to\infty}\|v_p-u\|$ exists, $\{v_p\}$ is bounded and thus, $\{v_p\}$ has a weakly convergence subsequence $\{v_{p_k}\}$. Let $v^*\in\mathcal{C}$ be the weak limit of $\{v_{p_k}\}$. Hence,

$$Av_{p_k}\rightharpoonup Av^*\in\mathcal{D},\quad y_{p_k}\rightharpoonup v^*$$

and

$$T^f_{\mu_{p_k}}Av_{p_k}\rightharpoonup Av^*.$$

Since G_φ is a zero-demiclosed mapping, and $y_{p_k}\rightharpoonup v^*$, we obtain $v^*\in Fix(G_\varphi)$. Applying Lemma 2.6, $EP(f)=Fix(T^f_\mu)$ for any $\mu>0$. We claim that $T^f_\mu Av^*=Av^*$. Suppose $T^f_\mu Av^*\neq Av^*$. Since $Av_p-T^f_{\mu_p}Av_p=(I-T^f_{\mu_p})Av_p\to 0$ as $p\to\infty$, applying the Opial's property and Lemma 2.8 yields

$$\begin{aligned}\liminf_{j\to\infty}\|Av_{p_k}-Av^*\| &< \liminf_{j\to\infty}\|Av_{p_k}-T^f_\mu Av^*\|\\
&\le \liminf_{j\to\infty}\Big(\|Av_{p_k}-T^f_{\mu_{p_k}}Av_{p_k}\|+\|T^f_{\mu_{p_k}}Av_{p_k}-T^f_\mu Av^*\|\Big)\\
&= \liminf_{j\to\infty}\|T^f_\mu Av^*-T^f_{\mu_{p_k}}Av_{p_k}\|\\
&\le \liminf_{j\to\infty}\left(\|Av_{p_k}-Av^*\|+\frac{|\mu_{p_k}-\mu|}{\mu_{p_k}}\|T^f_{\mu_{p_k}}Av_{p_k}-Av_{p_k}\|\right)\\
&= \liminf_{j\to\infty}\|Av_{p_k}-Av^*\|,\end{aligned}$$

which lead to a contradiction. So $Av^*\in Fix(T^f_\mu)=EP(f)$, and hence

$$v^*\in\Omega_\varphi=\{u\in Fix(G_\varphi):Au\in EP(f)\}.$$

Now we prove that $\{v_p\}$ converges weakly to $v^*\in\Omega_\varphi$. Otherwise, there exists a subsequence $\{v_{p_l}\}$ of $\{v_p\}$ such that $v_{p_l}\rightharpoonup u^*\in\Omega_\varphi$ with $u^*\neq v^*$. By Opial's condition,

$$\liminf_{l\to\infty}\|v_{p_l}-u^*\|<\liminf_{l\to\infty}\|v_{p_l}-v^*\|<\liminf_{l\to\infty}\|v_{p_l}-u^*\|.$$

This is a contradiction. Hence, $\{v_p\}$ converges weakly to an element $v^*\in\Omega_\varphi$.
Finally, we prove that $\{u_p\}$ converges weakly to $Av^*\in EP(f)$. Since $v_p\rightharpoonup v^*$, we have $Av_p\rightharpoonup Av^*$ as $p\to\infty$. Therefore, $u_p:=T^f_{\mu_p}Av_p\rightharpoonup Av^*\in EP(f)$. □

Corollary 3.2. *Let $\mathcal{C}\subset\mathcal{H}_1$ and $\mathcal{D}\subset\mathcal{H}_2$ be two nonempty closed convex sets. Let $G:\mathcal{C}\to\mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ be a bi-function with*

$$\Omega=\{u\in Fix(G):Au\in EP(f)\}\neq\emptyset,$$

where $A:\mathcal{H}_1\to\mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1\in\mathcal{C},\\ u_p=T^f_{\mu_p}Av_p, & \{\mu_p\}\subset(0,\infty),\\ v_{p+1}=(1-a_p)w_p+a_pGw_p,\\ w_p=P_{\mathcal{C}}(v_p+\beta A^*(T^f_{\mu_p}-I)Av_p), & \beta\in\left(0,\frac{1}{\|A^*\|}\right),\ p\in\mathbb{N},\end{cases}\tag{3.10}$$

where $\liminf_{p\to\infty} \mu_p > 0$, $P_{\mathcal{C}}$ *is the projection operator from* $\mathcal{H}_1$ *onto* $\mathcal{C}$ *and* $\{a_p\}$ *is a sequence in* $[\varepsilon, 1-\varepsilon]$ *with* $\varepsilon \in (0,1)$. *Then* $\{v_p\}$ *converges weakly to* $v^* \in \Omega$, *and* $\{u_p\}$ *converges weakly to* $Av^* \in EP(f)$.

Proof. Consider G_φ given in (3.2). By Lemma 2.2, for any $\varphi \in (0,1)$, we have $Fix(G_\varphi) = Fix(G)$. We have

$$\begin{aligned}(1-a_p)w_p + a_p G_\varphi w_p &= (1-a_p)w_p + a_p((1-\varphi)w_p + \varphi G w_p)\\ &= (1-a_p\varphi)w_p + a_p\varphi G w_p.\end{aligned}$$

To obtain exactly the iterative scheme (3.10), we simply denote $a_p := \varphi a_p \in (0,1)$ for all $p \in \mathbb{N}$. □

Next we prove a strong convergence theorem of an iterative method to split common solution for a demicontractive mapping.

Theorem 3.3. *Let* $\mathcal{C} \subset \mathcal{H}_1$ *and* $\mathcal{D} \subset \mathcal{H}_2$ *be two nonempty closed convex sets. Let* $G : \mathcal{C} \to \mathcal{C}$ *be a zero-demiclosed* α*-demicontractive mapping and* $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ *be a bi-function with* $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$, *where* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *is a bounded linear operator with its adjoint* A^*. *Consider the sequence* $\{u_p\}$ *and* $\{v_p\}$ *generated as follows:*

$$\begin{cases} v_1 \in \mathcal{C}, \\ u_p = T^f_{\mu_p} A v_p, \quad \{r_n\} \subset (0,\infty), \\ y_p = (1-a_p)w_p + a_p[(1-\varphi)w_p + \varphi G w_p], \quad \varphi \in (0, 1-\alpha), \\ w_p = P_C(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), \quad \beta \in \left(0, \frac{1}{\|A^*\|^2}\right) \\ C_{p+1} = \{v \in C_p : \|y_p - v\| \le \|w_p - v\| \le \|v_p - v\|\}, \quad \textit{with } C_1 = C, \\ v_{p+1} = P_{C_{p+1}}(v_1), \quad p \in \mathbb{N}. \end{cases} \tag{3.11}$$

where $\liminf_{p\to\infty} r_n > 0$, P_C *is the projection operator from* $\mathcal{H}_1$ *onto* C, *and* $\{a_p\}$ *is a sequence in* $[\varepsilon, 1-\varepsilon], \varepsilon \in (0,1)$. *Then* $v_p \to v^* \in \Omega$ *and* $u_p \to Av^* \in EP(f)$.

Proof. Consider the mapping G_φ given in (3.2). Similar to the proof of Theorem 3.1, the sequence $\{y_n\}$ in Algorithm (3.11) can be written as

$$y_p = (1-a_p)w_p + a_p G_\varphi w_p.$$

Let $\Omega_\varphi = \{u \in Fix(G_\varphi) : Au \in EP(f)\} \neq \emptyset$. We claim that $\Omega_\varphi \subset C_p$ for $p \in \mathbb{N}$. In fact, let $u \in \Omega_\varphi$. Following the same argument as in the proof of Theorem 3.1, we have

$$2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle \le -\beta\|(T^f_{\mu_p} - I)Av_p\|^2, \tag{3.12}$$

and for any $p \in \mathbb{N}$,

$$\|A^*(T^f_{\mu_p} - I)Av_p\|^2 \le \|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2. \tag{3.13}$$

For any $p \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|y_p - u\| &\le \|w_p - u\|^2 - (1-a_p)a_p\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p + \beta A^*(T^f_{\mu_p} - I)Av_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&= \|v_p - u\|^2 + \|\beta A^*(T^f_{\mu_p} - I)Av_p\|^2 + 2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p - u\|^2 + \beta^2\|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2 - \beta\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p - u\|^2 - \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2.
\end{aligned}$$

Since $\beta \in \left(0, \frac{1}{\|A^*\|^2}\right)$, $\beta\left(1 - \beta\|A^*\|^2\right) > 0$, it follows that

$$\|y_p - u\| \le \|w_p - u\| \le \|v_p - u\|, \tag{3.14}$$

and thus $p \in C_p$ for all $p \in \mathbb{N}$. Hence, $\Omega \subset C_p$ and $C_p \neq \emptyset$ for all $p \in \mathbb{N}$.

Now we prove that C_p is a closed convex set for each $p \in \mathbb{N}$. It is not hard to verify that C_p is closed for each p, so it suffices to verify that C_p is convex for each $p \in \mathbb{N}$. Indeed, let $x_1, x_2 \in C_{p+1}$. For any $\gamma \in (0,1)$, since

$$\begin{aligned}
&\|y_p - (\gamma x_1 + (1-\gamma)x_2\|^2 \\
&\quad = \|\gamma(y_p - x_1) + (1-\gamma)(y_p - x_2)\|^2 \\
&\quad = \gamma\|y_p - x_1\|^2 + (1-\gamma)\|y_p - x_2\|^2 - \gamma(1-\gamma)\|x_1 - x_2\|^2 \\
&\quad \le \gamma\|w_p - x_1\|^2 + (1-\gamma)\|w_p - x_2\|^2 - \gamma(1-\gamma)\|x_1 - x_2\|^2 \\
&\quad = \|w_p - (\gamma x_1 + (1-\gamma)x_2\|^2,
\end{aligned}$$

the following inequality holds

$$\|y_p - (\gamma x_1 + (1-\gamma)x_2)\| \le \|w_p - (\gamma x_1 + (1-\gamma)x_2)\|.$$

Similarly, we also have

$$\|w_p - (\gamma x_1 + (1-\gamma)x_2)\| \le \|v_p - (\gamma x_1 + (1-\gamma)x_2)\|,$$

which implies that $\gamma x_1 + (1-\gamma)x_2 \in C_{p+1}$. Hence, C_{p+1} is convex.

Notice that $C_{p+1} \subset C_p$ and $v_{p+1} = P_{C_{p+1}}(v_1) \subset C_p$. Then $\|v_{p+1} - v_1\| \le \|v_p - v_1\|$ for $n > 2$. It follows that $\lim_{p\to\infty}\|v_p - v_1\|$ exists. Hence $\{v_p\}$ is bounded, which yields $\{w_p\}$ and $\{y_p\}$ are bounded. For any $k, p \in \mathbb{N}$ with $k > p$, from $v_k = P_{C_k}(v_1) \subset C_p$ and the character (iii) of the projection operator P, we have

$$\|v_p - v_k\|^2 + \|v_1 - v_k\|^2 = \|v_p - P_{C_k}(v_1)\|^2 + \|v_1 - P_{C_k}(v_1)\|^2 \le \|v_p - v_1\|^2. \tag{3.15}$$

Since $\lim_{p\to\infty}\|v_p - v_1\|$ exists, it follows that $\lim_{p\to\infty}\|v_p - v_k\| = 0$, which implies that $\{v_p\}$ is a Cauchy sequence.

Let $v_p \to v^*$. One can claim that $v^* \in \Omega$. Firstly, by the fact that

$$v_{p+1} = P_{C_{p+1}}(v_1) \in C_{p+1} \subset C_p,$$

we have

$$\|y_p - v_p\| \le \|y_p - v_{p+1}\| + \|v_{p+1} - v_p\| \le 2\|v_{p+1} - v_p\| \to 0, \quad \text{as } p \to \infty \tag{3.16}$$

and

$$\|w_p - v_p\| \le \|w_p - v_{p+1}\| + \|v_{p+1} - v_p\| \le 2\|v_{p+1} - v_p\| \to 0, \quad \text{as } p \to \infty. \tag{3.17}$$

Setting $\rho = \beta(1-\beta\|A^*\|^2)$, we obtain

$$\begin{aligned}\rho\|(T^f_{\mu_p} - I)Av_p\|^2 + \varepsilon^2\|w_p - Tw_p\|^2 &\leq \|v_p - v^*\|^2 - \|y_p - v^*\|^2 \\ &\leq \|v_p - y_p\| \left(\|v_p - v^*\| + \|y_p - v^*\|\right).\end{aligned}$$

So

$$\lim_{p\to\infty} \|G_\varphi w_p - w_p\| = 0$$

and

$$\lim_{p\to\infty} \|(T^f_{\mu_p} - I)Av_p\| = 0.$$

Let $r > 0$. Since $v_p \to v^*$ as $p \to \infty$, Lemma 2.8 implies that

$$\begin{aligned}\|T^f_{\mu_p}Av^* - Av^*\| &\leq \|T^f_{\mu_p}Av^* - T^f_{\mu_p}Av_p\| + \|T^f_{\mu_p}Av_p - Av_p\| + \|Av_p - Av^*\| \\ &\leq 2\|Av_p - Av^*\| + \left(1 + \frac{|r_n - r|}{r_n}\right)\|T^f_{\mu_p}Av_p - Av_p\| \to 0, \text{ as } p \to \infty.\end{aligned}$$

So $T^f_\mu Av^* = Av^*$, which says that $Av^* \in Fix(T^f_{\mu_p}) = EP(f)$. On the other hand, since $v_p - w_p \to 0$ and $v_p \to v^*$, we conclude that $w_p \to v^*$. Notice that G_φ is zero-demiclosed quasi-nonexpansive, $G_\varphi v^* = v^*$. We also deduce that $\{u_p\} := \{T^f_{\mu_p}Av_p\}$ converges strongly to $Av^* \in EP(f)$. □

Corollary 3.4. *Let $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{D} \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $G : \mathcal{C} \to \mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function with $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$, where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:*

$$\begin{cases} v_1 \in \mathcal{C}, \\ u_p = T^f_{\mu_p}Av_p, \quad \{r_n\} \subset (0,\infty), \\ y_p = (1-a_p)w_p + a_pGw_p, \\ w_p = P_C(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), \quad \beta \in \left(0, \frac{1}{\|A^*\|^2}\right) \\ C_{p+1} = \{v \in C_p : \|y_p - v\| \leq \|w_p - v\| \leq \|v_p - v\|\}, \quad \textit{with } C_1 = C, \\ v_{p+1} = P_{C_{p+1}}(v_1), \quad p \in \mathbb{N}. \end{cases} \tag{3.18}$$

where $\liminf_{p\to\infty} r_n > 0$, P_C is the projection operator from $\mathcal{H}_1$ onto C, and $\{a_p\}$ is a sequence in $[\varepsilon, 1-\varepsilon], \varepsilon \in (0,1)$. Then $v_p \to v^* \in \Omega$ and $u_p \to Av^* \in EP(f)$.

Proof. Consider the mapping G_φ given in (3.2). By Lemma 2.2, for any $\varphi \in (0,1)$, we have $Fix(G_\varphi) = Fix(G)$. We have

$$(1-a_p)w_p + a_pG_\varphi w_p = (1-a_p)w_p + a_p((1-\varphi)w_p + \varphi Gw_p) = (1-a_p\varphi)w_p + a_p\varphi Gw_p.$$

To obtain exactly the iterative scheme (3.18), we simply denote $a_p := \varphi a_p \in (0,1)$ for all $p \in \mathbb{N}$. □

We close this section by stating the strong convergence of an iterative scheme for a split common solutions problem with a finite number of demicontractive mappings.

Theorem 3.5. *Let $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{D} \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let*

$$G_1, \cdots, G_n : \mathcal{C} \to \mathcal{C}$$

be a finite number of zero-demiclosed α-demicontractive mappings with

$$\bigcap_{i=1}^{n} Fix(G_i) \neq \emptyset$$

and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function with

$$\Omega = \left\{ u \in \bigcap_{i=1}^{n} Fix(G_i) : Au \in EP(f) \right\} \neq \emptyset,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1 \in \mathcal{C}, \\ u_p = T^f_{\mu_p} A v_p, \quad \{r_n\} \subset (0, \infty), \\ y_p = (1 - a_p) w_p + a_p \sum_{i=1}^{n} c_i [(1 - \varphi_i) w_p + \varphi_i G w_p], \quad c_i, \varphi_i \in (0,1), \quad \sum_{i=1}^{n} c_i = 1, \\ w_p = P_C(v_p + \beta A^*(T^f_{\mu_p} - I) A v_p), \quad \beta \in \left(0, \frac{1}{\|A^*\|^2}\right) \\ C_{p+1} = \{ v \in C_p : \|y_p - v\| \leq \|w_p - v\| \leq \|v_p - v\| \}, \quad \text{with } C_1 = C, \\ v_{p+1} = P_{C_{p+1}}(v_1), \quad p \in \mathbb{N}. \end{cases} \tag{3.19}$$

where $\liminf_{p \to \infty} r_n > 0$, P_C is the projection operator from $\mathcal{H}_1$ onto C, and $\{a_p\}$ is a sequence in $[\varepsilon, 1 - \varepsilon], \varepsilon \in (0,1)$. Then $v_p \to v^* \in \Omega$ and $u_p \to Av^* \in EP(f)$.

Proof. Let $F = \sum_{i=1}^{n} c_i G_{\varphi_i}$, where $G_{\varphi_i} = (1 - \varphi_i) I + \varphi_i G$. Lemma 2.9 implies that F is a quasi-nonexpansive mapping. Furthermore,

$$Fix(F) = \bigcap_{i=1}^{n} Fix(G_{\varphi_i}) = \bigcap_{i=1}^{n} Fix(G_i) \neq \emptyset.$$

It is straightforward to see that F is zero-demiclosed. The rest of the proof is similar to that of Theorem 3.3. □

4. Conclusion

(1) We have proven a weak convergence theorem for an iteration scheme used to approximate split common solutions for demicontractive mappings in Hilbert spaces, which is derived from an associated weak convergence theorem in the class of a quasi-nonexpansive operators.

(2) We also have established a strong convergence theorem for an iteration scheme used to approximate split common solutions of demicontractive mappings in Hilbert spaces, which is derived from a corresponding strong convergence theorem in the class of a quasi-nonexpansive operators.

(3) Our investigation is based on an embedding technique by means of an averaged mapping: if G is α-demicontractive, then for any $\varphi \in (0, 1-\alpha)$,

$$G_\varphi = (1-\varphi)I + \varphi G$$

is a quasi-nonexpansive mapping.

(4) For some very recent developments on related topics we refer the reader to Alakoya et al. [1], Berinde and Saleh [5] [6], Berinde and Păcurar [4], Onah et al. [14], Rathee and Swami [16], Wang and Pan [17],Yao et al. [18], Zhu et al. [19], etc., to which a similar approach seems to be applicable.

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