ON THE RAABE-DUHAMEL AND GENERALIZED RATIO TESTS

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Abstract
In this paper we give a partial answer to the following question: is the generalized ratio test equivalent to the Raabe-Duhamel test?

Key words: *series of positive terms, ratio test, generalized ratio test, Raabe-Duhamel test.*

1. INTRODUCTION

The well known ratio test was extended in our previous papers [2] - [4] to the so - called *generalized ratio test*, given by

Theorem 1. ([2]) *Let* $\sum_{n=1}^{\infty} u_n$ *be a series with positive terms.*

1) If there exist a convergent series of non - negative terms $\sum_{n=1}^{\infty} v_n$ *and a constant number k such that*

$$\frac{u_{n+1} + v_n - v_{n+1}}{u_n} \le k < 1, \text{ for } n \ge N \text{ (fixed), then the series } \sum_{n=1}^{\infty} u_n \text{ is convergent.}$$

2) If there exists a decreasing sequence of positive numbers such that for $n \ge N$ *(fixed), we have*
(i) $u_n > v_n$ *and*
(ii)

$$\frac{u_{n+1} + v_n - v_{n+1}}{u_n} \ge 1,$$

then the series $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Remarks. 1) If we take as „comparison series" in Theorem 1 the null series, that is $v_n = 0$, $n \ge 1$, then we obtain the well-known ratio (or D´Alembert´s) test which is very useful in studying the convergence of positive series.

However, if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then the ratio test fails. As shown by some examples in [1] and [2], the generalized ratio test is better then the classical ratio test.

Indeed, if we consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the ratio test does not apply, while the generalized ratio test applies with the „comparison series“ $v_n = \frac{1}{n(n+1)}$, $n \ge 1$:

$$\frac{u_{n+1}}{u_n + v_n} = \frac{n^2}{2n^2 + 3n + 1} < \frac{1}{2}, \quad \forall n \ge 1.$$

2) A clasical tool in removing difficulties when the ratio test fails is the Raabe test. C. Avramescu [1] asked the following question : are the generalized ratio and the Raabe test equivalent?

For the „convergence part“ of these two tests, a partial answer is given in the next section.

2. THE RAABE TEST DOES IMPLY THE GENERALIZED RATIO TEST

Proposition 1. *If the convergence of a positive series is obtained by means of the Raabe test, then the convergence may be also obtained by means of the generalized ratio test.*

Proof. Let $\sum_{n=1}^{\infty} u_n$ a positive series whose convergence may be obtained by means of the Raabe test, that is (see [5], for example)

$$n\left(\frac{u_{n+1}}{u_n} - 1\right) \le -\alpha < -1, \qquad \text{for } n \ge N_0 \text{ (fixed).}$$

It follows that $\alpha > 1$ and

$$\frac{u_{n+1}}{u_n} \le 1 - \frac{\alpha}{n}, \quad n \ge N_0,$$

and if we denote $k = 1 - \frac{\alpha}{N_0}$ we have

$$\frac{u_{n+1}}{u_n} \le k < 1, \quad n \ge N_0,$$

and then

$$\frac{u_{n+1}}{u_n + v_n} \le \frac{u_{n+1}}{u_n} \le k < 1, \quad n \ge N_0,$$

hence the convergence of $\sum_{n=1}^{\infty} u_n$ may be also obtained by means of the generalized ratio test.

Remark. Having in view the fact that from Theorem 1 we obtain a necessary and sufficient convergence test for the series of decreasing positive terms [3], [4], we expect that the reverse of proposition 1 is true.

References

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