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COMPARING KRASNOSELSKIJ AND MANN ITERATIVE METHODS FOR LIPSCHITZIAN GENERALIZED PSEUDO-CONTRACTIONS

Vasile Berinde

Abstract

A comparison of the convergence rate of the so called Krasnoselskij and Mann iterative methods, both known to converge to a fixed point of Lipschitzian generalized pseudo-contractive operators, is obtained.

1 Introduction

In order to approximate fixed points of certain classes of operators which satisfy weak contractive type conditions that do not guarantee the convergence of Picard iterative process (or method of successive approximations, as it is also known), many authors have used certain mean value fixed point iterations, like Krasnoselskij, Mann or Ishikawa iterative methods, see for example Berinde [3], for a recent survey.

In [7], Verma approximated fixed points of Lipschitzian and generalized pseudo-contractive operators in Hilbert spaces by both Krasnoselskij and Mann type iterative methods. When, for a certain class of mappings, two or more fixed point iteration procedures can be used to approximate their fixed points, it is of theoretical and practical importance to compare the rate of convergence of these methods, and to find out, if possible, which of them converges faster.

Inspired by the recent works [1, 4] of the author, where certain fixed point iterative procedures are compared, this paper compares Krasnoselskij and Mann iterations for the class of Lipschitzian and generalized pseudo-contractive operators in Hilbert spaces. The main result will show that, for any Mann iteration, there is a Krasnoselskij iteration which converges faster to the unique fixed point of the operator in question.

2 Preliminaries

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $<\cdot,\cdot>$, and K be a nonempty subset of H.

An operator $T : K \longrightarrow K$ is said to be a *generalized pseudo-contraction* if, for all x, y in K, there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \leq r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2 . \qquad (2.1)$$

Condition (2.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq r \cdot \|x - y\|^2 , \qquad (2.2)$$

or to

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq (1 - r)\|x - y\|^2 ,$$

where I is the identity map.

Clearly, if T is generalized pseudo-contractive with $r < 1$, then $I-T$ is strongly monotone.

For $r = 1$ in (2.1), T is called *pseudo-contraction*, a concept introduced and studied by Browder and Petryshyn [5] and thereafter by many authors, in connection with the problem of approximating fixed points, see for example Berinde [3].

The operator T is said to be *Lipschitzian* (or Lipschitz continuous) if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \cdot \|x - y\| , \quad \text{for all } x, y \text{ in } K. \qquad (2.3)$$

As shown in Berinde [1], Lemma 2.1, any Lipschitzian operator in a Hilbert space is also generalized pseudo-contractive with the same constant. Consequently, for a Lipschitzian operator T with constant $L > 0$, the only reason to consider also a generalized pseudo-contractive condition of the form (2.1) is that r could be smaller than L.

Example 1. ([1]) Let H be the real line with usual norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \longrightarrow K$ given by $Tx = \frac{1}{x}$, for all x in K. Then T is Lipschitzian with constant $L = 4$ and generalized pseudo-contractive with constant $r > 0$, arbitrary. The Picard iteration, $x_{n+1} = Tx_n$, $n \geq 0$ does not converge, for any initial guess $x_0 \neq 1$ (which is the unique fixed point of T).

In order to approximate fixed points of the operators considered in this paper we shall make use of other two well known iterative methods.

1. The Krasnoselskij iteration method. For $x_0 \in K$ and $\lambda \in [0, 1]$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n \,, \quad n = 0, 1, 2, \ldots \tag{2.4}$$

is called *Krasnoselskij iterative method* or *Krasnoselskij iteration* and is denoted by $K_n(x_0, \lambda, T)$.

2. The Mann iterative method. For $x_0 \in K$ and $\{\alpha_n\}_{n=0}^{\infty}$ a sequence in $[0, 1]$, the sequence $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n \,, \quad n = 0, 1, 2, \ldots \tag{2.5}$$

is called *Mann iterative method* or *Mann iteration* and will be denoted by $M_n(y_0, \alpha_n, T)$.

Remark 1. It is obvious that, for $\lambda = 1$, the Krasnoselskij iteration reduces to the method of successive approximations, while for $a_n = \lambda(\text{const})$, the Mann iteration reduces to the Krasnoselskij method.

In [7], Verma stated two convergence theorems, one for the Krasnoselskij iterative method (Theorem 2.1) and the other for a Mann type iterative method (Theorem 2.2). As they will be used in the proof of our main result, we give their statements here, slightly reformulated.

Theorem 1. [1] *Let K be a non-empty closed convex subset of a real Hilbert space H, and let $T : K \longrightarrow K$ be Lipschitzian and generalized* *pseudo-contractive, with the corresponding constants $L > 0$ and $r > 0$ satisfying*

$$0 < r < 1 \quad and \quad r \leq L\,. \tag{2.6}$$

Then:

(i) T has a unique fixed point p in K;

(ii) The Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \lambda, T)$ converges strongly to p, for any $x_0 \in K$ and all $\lambda \in (0, a) \cap (0, 1)$, where

$$a = 2(1-r)/(1-2r+L^2)\,. \tag{2.7}$$

Theorem 2. [7] *Let H be a real Hilbert space and K a non-empty closed convex subset of H. Let $T : K \longrightarrow K$ be a Lipschitzian and generalized pseudo-contractive operator with the corresponding constants $L \geq 1$ and $r > 0$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be an increasing sequence in $[0, 1]$, such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty\,. \tag{2.8}$$

Then:

(i) T has a unique fixed point p in K;

(ii) The Mann iteration $\{y_n\}_{n=0}^{\infty} = M_n(y_0, t\alpha_n, T)$ converges strongly to p, for any $y_0 \in K$ and all t in $(0, a)$ that satisfy

$$0 \leq (1-t)^2 - 2t(1-t)r + t^2L^2 < 1\,,$$

where a is given by (2.7).

In order to compare two fixed point iteration procedures, we shall make use of the following concept of rate of convergence introduced and studied in the papers Berinde [1, 2, 3, 4]. This concept is slightly different from the one considered by Rhoades [6], but more suitable for our purposes.

Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}\,.$$

a) If $l = 0$, then it is said that $\{a_n\}_{n=0}^{\infty}$ converges *faster* to a than $\{b_n\}_{n=0}^{\infty}$ to b;

b) If $0 < l < \infty$, than we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Remark 2. 1) In the case a), the notation $a_n - a = o(b_n - b)$ is commonly used;

2) If $l = \infty$, that is $b_n - b = o(a_n - a)$, then $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$.

Suppose that for two fixed point procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to a certain fixed point p, the error estimates of the form

$$\|u_n - p\| \leq a_n\,, \quad n = 0, 1, 2, \ldots \tag{2.9}$$

and

$$\|v_n - p\| \leq b_n\,, \quad n = 0, 1, 2, \ldots \tag{2.10}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of positive numbers (both converging to zero).

Definition 1. *[4]. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p such that (2.9) and (2.10) are satisfied.*

If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we say that $\{u_n\}_{n=0}^{\infty}$ converges faster *than $\{v_n\}_{n=0}^{\infty}$ to p.*

Example 2. Consider $p = 0$, $u_n = \dfrac{1}{n+1}$ and $v_n = \dfrac{1}{n}$, $n \geq 1$.
Then $\{u_n\}$ is better than $\{v_n\}$, in the sense of Rhoades' definition [6], i.e.,

$$\|u_n - p\| \leq \|v_n - p\|, \quad \text{for all } n,$$

although $\{u_n\}$ and $\{v_n\}$ have actually the same rate of convergence, in the sense of Definition 1, since $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

The previous example also shows that the concept used by Rhoades [6] is independent of that given by Definition 1.

3 The main result

Theorem 3 in this section unifies and completes the results in Verma [7], by showing that the Krasnoselskij iteration is more suitable than the Mann iteration for approximating fixed points of Lipschitzian and generalized pseudo-contractive operators.

Moreover, adapting a result from Berinde [3], we are able to prove that amongst all Krasnoselskij iterations (2.4), with $\lambda \in (0, a)$, where a is given by (2.7), there exists one which is the fastest with respect to the concept of rate of convergence given by Definition 1.

Theorem 3. *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \longrightarrow K$ be a Lipschitzian and generalized pseudo-contractive operator with corresponding constants $L \geq 1$ and $0 < r < 1$.*

Then:

1) *T has a unique fixed point p in K;*

2) *For any $x_0 \in K$ and $\lambda \in (0, a)$, with a given by (2.7), the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \lambda, T)$ converges strongly to p;*

3) *For any $y_0 \in K$ and $\{\alpha_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying (2.8), the Mann iteration $\{y_n\}_{n=0}^{\infty} = M_n(y_0, \alpha_n, T)$ converges strongly to p;*

4) *For any Mann iteration converging to p, with $0 \leq \alpha_n \leq b < 1$, there exists a Krasnoselskij iteration that converges faster to p.*

Proof. 1)-2) For all $\lambda \in [0, 1]$, consider the operator T_λ on K given by

$$T_\lambda x = (1 - \lambda)x + \lambda T x\,, \quad x \in K\,. \tag{3.1}$$

Since K is convex, we have $T_\lambda(K) \subset K$, for all $\lambda \in [0, 1]$.

From the generalized pseudo-contractive and Lipschitzian conditions on T and

$$\|T_\lambda x - T_\lambda y\|^2 = \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2 = (1-\lambda)^2\|x-y\|^2 + \\ + 2\lambda(1-\lambda)\cdot\langle Tx - Ty, x - y\rangle + \lambda^2 \cdot \|Tx - Ty\|^2$$

we find that

$$\|T_\lambda x - T_\lambda y\|^2 \leq \left[(1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 L^2\right] \cdot \|x - y\|^2\,,$$

so

$$\|T_\lambda x - T_\lambda y\| \leq \theta \cdot \|x-y\|\,, \quad \text{for all } x,y \text{ in } K\,, \tag{3.2}$$

where $0<\theta = \left[(1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2L^2\right]^{1/2} < 1$, as $\lambda < a$.

Since K is a closed subset of a Hilbert space, K is a complete metric space. Then by Banach's contraction mapping principle, T_λ has a unique fixed point q in K and the Picard iteration associated to T_λ,

$$x_{n+1} = T_\lambda x_n\,, \quad n \geq 0\,, \tag{3.3}$$

converges strongly to q, for any $x_0 \in K$.

Now using the fact that $\{x_n\}_{n=0}^\infty$ given by (3.3) is exactly the Krasnoselskij iteration $K_n(x_0,\lambda,T)$ associated to T, on the one hand, and the fact that $F(T)=F(T_\lambda)$, for all $\lambda\in(0,1)$, that is, $p=q$ is the unique fixed point of T, on the other hand, we obtain 1) and 2).

3) Let $\{y_n\}_{n=0}^\infty$ be the Mann iteration given by (2.5), with $\{\alpha_n\}_{n=0}^\infty \subset [0,1]$ satisfying (2.8). Consider t, $0<t<1$, and denote $a_n = \dfrac{1}{t}\alpha_n$, $n=0,1,2,\ldots$.
Then the Mann iteration will be given by

$$y_{n+1} = (1-ta_n)y_n + ta_n\,Ty_n\,, \quad n=0,1,2,\ldots\,.$$

We have

$$\begin{aligned}\|y_{n+1}-p\| &= \|(1-a_n)y_n + a_n\left[(1-t)y_n + t\,Ty_n\right] - p\| \leq \\ &\leq (1-a_n)\|y_n-p\| + a_n\|(1-t)(y_n-p)+t(Ty_n-Tp)\|. \end{aligned}\tag{3.4}$$

Using the properties of T we find that

$$\begin{aligned}\|t(Ty_n-Tp)+(1-t)(y_n-p)\|^2 &= (1-t)^2\|y_n-p\|^2 + \\ &+ 2t(1-t)\left\langle Ty_n-p, y_n-p\right\rangle + t^2\|Ty_n-p\|^2 \leq \\ &\leq (1-t)^2\|y_n-p\|^2 + 2t(1-t)r\|y_n-p\|^2 + t^2L^2\|y_n-p\|^2 = \\ &= \left[(1-t)^2+2t(1-t)r+t^2L^2\right]\|y_n-p\|^2\,. \end{aligned}\tag{3.5}$$

By (3.4) and (3.5) we get

$$\begin{aligned}\|y_{n+1}-p\| &\le \left\{1-a_n+a_n\left[(1-t)^2+2t(1-t)r+t^2L^2\right]^{\frac{1}{2}}\right\}\cdot\|y_n-p\| \\ &= \big(1-(1-\theta)a_n\big)\|y_n-p\| \\ &\le \prod_{k=1}^{n}\big(1-(1-\theta)a_k\big)\|y_1-p\|, \qquad (3.6)\end{aligned}$$

where

$$0\le\theta=\left[(1-t)^2+2t(1-t)r+t^2L^2\right]^{1/2}<1,$$

for all t such that $0<t<2(1-r)/(1-2r+L^2)$.

Since by (2.8) $\sum\limits_{n=0}^{\infty}\alpha_n$ diverges, it follows that $\sum\limits_{n=0}^{\infty}a_n$ diverges, too, and in view of $\theta<1$ we get that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1-(1-\theta)a_k\right]=0,$$

which by (3.6) shows that $\{y_n\}$ converges strongly to p.

4) Take $x:=x_n$, $y:=x_{n-1}$ in (3.2) to obtain

$$\|x_{n+1}-x_n\|\le\theta\cdot\|x_n-x_{n-1}\|,$$

which inductively yields

$$\|x_{n+1}-x_n\|\le\theta^n\|x_1-x_0\|$$

and hence by triangle rule we obtain

$$\|x_{n+p}-x_n\|\le\theta^n\left(1+\theta+\cdots+\theta^{p-1}\right)\|x_1-x_0\|, \qquad (3.7)$$

valid for all $n,p\in\mathbb{N}^*$.

Now letting $p\to\infty$ in (3.7) and using part 2), we get

$$\|x_n-x^*\|\le\frac{\theta^n}{1-\theta}\|x_1-x_0\|. \qquad (3.8)$$

Therefore, in view of (3.6) and (3.8), in order to compare the Krasnoselskij and Mann iterations, we have to compare

$$\theta^n \quad\text{and}\quad \prod_{k=1}^{n}\left[1-(1-\theta)a_k\right].$$

Let $\{y_n\}_{n=0}^{\infty}$ be a certain Mann iteration converging to p, with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying $0 \le \alpha_n \le b < 1$. Then, $a_k \le b/t$ (denote b/t by b) and for any m, $0 < m < 1$, we may find $\theta \in (0,1)$ such that

$$b < \frac{1 - \dfrac{\theta}{m}}{1-\theta}.$$

Indeed, to this end it is enough to take $\theta < \dfrac{m(1-b)}{1-m}$.
Using the fact that $a_k \le b$, it results

$$\frac{\theta}{1-(1-\theta)a_k} \le m < 1,$$

which shows that

$$\lim \frac{\theta}{\prod\limits_{k=1}^{n} \left[1-(1-\theta)a_k\right]} \le \lim_{n\to\infty} m^n = 0\,,$$

so the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \theta, T)$ converges faster than the considered Mann iteration, $\{y_n\}_{n=0}^{\infty} = M_n(y_0, \alpha_n, T)$.

To end the proof we still need to show that the intervals $(0,a)$, with a given by (2.7), and $\left(0, \dfrac{m(1-b)}{1-m}\right)$ have nonempty intersection. But this is immediate, since $\dfrac{m(1-b)}{1-m} > 0$ and $0 < a = \dfrac{2(1-r)}{1-2r+L^2} \le 1$, under the hypotheses of Theorem 3. □

Remark 3. 1) Part 4) in Theorem 3 shows that, in order to approximate the fixed points of a Lipschitzian and pseudo-contractive operator T, it is always more convenient to use a certain Krasnoselskij iteration in the family (2.4), with $\lambda \in (0,a)$ and a given by (2.7).

2) Moreover, as shown by Theorem 3.2 in Berinde [1], amongst the Krasnoselskij iterations in that family there exists one which is the fastest in the sense of Definition 1.

Theorem 4. [1] *Let all assumptions in Theorem 1 be satisfied. Then the fastest iteration* $\{x_n\}_{n=0}^{\infty}$ *in the family* (2.4) *with* $\lambda \in (0, a)$ *is that obtained for*

$$\lambda = (1 - r)/(1 - 2r + L^2).$$

Remark 4. 1) Theorem 4 shows that the fastest iteration is commonly obtained for λ situated in the middle of the interval to which the parameter belongs.

2) Note that, in view of the condition $\lambda < 1$, the convergence of the Picard iteration cannot be obtained from Theorems 1-4. Actually, as shown by Example 1, the Picard iteration does not generally converge and this is the reason we need to consider other fixed point iteration procedures, like Krasnoselskij or Mann, in order to approximate fixed points of Lipschitzian and generalized pseudo-contractions.

We end this paper with some numerical examples that illustrate the effectiveness of our result.

4 Numerical examples

For the decreasing function T in Example 1, the execution of the program FIXPOINT (see [3]) for some input data leads to the following observations.

1) The Krasnoselskij iteration converges to $p = 1$ for any $\lambda \in (0, 1)$ and any initial guess x_0 (recall that the Picard iteration does not converge for any initial value $x_0 \in [1/2, 2]$ different from the fixed point). The convergence is slow for λ close enough to 0 (that is, for Krasnoselskij iterations close enough to the Picard iteration) or close enough to 1. The closer to $1/2$, the middle point of the interval $(0, 1)$, λ is, the faster the Krasnoselskij iteration converges.

For $\lambda = 0.5$ the Krasnoselskij iteration converges very fast to $p = 1$, the unique fixed point of T. For example, starting with $x_0 = 1.5$, only 4 iterations are necessary in order to obtain p with 6 exact digits: $x_1 = 1.08335$, $x_2 = 1.00325$, $x_3 = 1.000053$, $x_4 = 1$.

For the same value of λ and $x_0 = 2$, 4 iterations are again needed to obtain p with the same precision, even though the initial guess is far away from the fixed point: $x_1 = 1.25$, $x_2 = 1.025$, $x_3 = 1.0003$, $x_4 = 1$;

2) The speed of Mann iteration also depends on the position of $\{\alpha_n\}$ in the interval $(0,1)$.

If we take $x_0 = 1.5$, $\alpha_n = 1/(n+1)$ then the Mann iteration converges (slowly) to $p = 1$: after $n = 35$ iterations we get $x_{35} = 1.000155$.

For $\alpha_n = 1/\sqrt[3]{n+1}$, we obtain the fixed point with 6 exact digits performing 8 iterations. Notice that in this case the Mann iteration converges not monotonically to $p = 1$.

Conditions like $\alpha_n \to 0$ (as $n \to \infty$) are usually involved in many convergence theorems regarding the Mann iteration. The next example shows that these conditions are not necessary for the convergence of Mann iteration to the fixed point.

Indeed, taking

$$x_0 = 2,\ \alpha_n = \frac{n}{2n+1} \nearrow \frac{1}{2},$$

we obtain the following results for the Mann iteration: $x_1 = 2$, $x_2 = 1.5$, $x_3 = 1.166$, $x_4 = 1.034$, $x_5 = 1.0042$, $x_6 = 1.00397$, $x_7 = 1.000031$, $x_8 = 1.000002$ and $x_9 = 1$.

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Vasile Berinde

Department of Mathematics and Computer Science,
North University of Baia Mare
Victoriel 76, 430072 Baia Mare ROMANIA
e-mail address: vasile_berinde@yahoo.com