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Recent developments in the fixed point theory of enriched contractive mappings. A survey

VASILE BERINDE[1,2] and MĂDĂLINA PĂCURAR[3]

ABSTRACT. The aim of this note is threefold: first, to present a few relevant facts about the way in which the technique of enriching contractive mappings was introduced; secondly, to expose the main contributions in the area of enriched mappings established by the authors and their collaborators by using this technique; and third, to survey some related developments in the very recent literature which were authored by other researchers.

1. INTRODUCTION

The concept of *enriched* nonexpansive mapping has been introduced in Berinde [28] in the case of a real Hilbert space and then was extended to the more general case of a Banach space in Berinde [29], see also Berinde [27] where the technique of *enriching contractive type mappings* has been applied to strictly pseudocontractive operators.

Soon after that, the authors and their collaborators applied successfully the same technique for some other classes of contractive type mappings in Hilbert spaces, Banach spaces or convex metric spaces, see Berinde [30], [31], [33], Berinde and Păcurar [36], [38], [39], [37], [40], [41], [42], Berinde et al. [35], Abbas et al. [1], [2], Salisu et al. [84],...

Many other authors were attracted to work in the same area and therefore some interesting developments on enriched mappings were obtained, most of them included in the list of References.

The impressive interest for the use of the technique of enriching contractive type mappings suggested us to undertake the task of offering a comprehensive exposure to date on the subject. So, our main aim in this paper is threefold:

(1) to present a few relevant facts about the way in which the technique of enriching contractive mappings was (re-)discovered;
(2) to expose the main contributions in the area of enriched mappings established by the authors and their collaborators by using this technique;
(3) to survey some related developments in this context which were authored by other researchers.

The paper is organised as follows: in Section 2 we give a brief account on how the technique of enriching contractive type mappings has been (re)-discovered and present some the facts about its origins.

Section 3 is devoted to the exposition of some important classes of enriched contractive type mappings, Section 4 exposes the classes of enriched nonexpansive mappings in Hilbert and Banach spaces, Section 5 gives an account on the concepts of unsaturated and saturated classes of contractive mappings, Section 6 surveys the brand new results

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Corresponding author: Vasile Berinde; vberinde@cunbm.utcluj.ro

on enriched contractions in quasi-Banach spaces, while Section 7 deals with other developments in the area of enriched contractive type mappings due to other authors.

2. THE TECHNIQUE OF ENRICHING CONTRACTIVE TYPE MAPPINGS

We start by presenting how the technique of enriching contractive type mappings was discovered (in fact, re-discovered, see explanations following later in this section).

For a rather long period of time, in connection with the study of various fixed point iterative schemes, partially surveyed in the monograph Berinde [25], we were thinking about finding a way to compare the very many classes of nonexpansive type mappings existing in literature, as such a complete comparison did not exist.

Basically, at the beginning, we were trying to compare, by appropriate examples, the following four important classes of nonexpansive type mappings:

- nonexpansive mappings
- quasi nonexpansive mappings
- strictly pseudocontractive mappings
- demicontractive mappings,

as at that time we were interested to deepen our knowledge on the great generality and value of demicontractive mappings. This task has been completed after a while and was very recently published in the paper Berinde [34].

On the way of performing such a comparison, we discovered by chance a method of deriving new constructive fixed point theorems, which we have called, based on the arguments presented below, as the technique of *enriching contractive type mappings*.

The starting point came from some known facts in the metrical fixed point theory. To present them, let $(X, \|\cdot\|)$ be a real normed space, $C \subset X$ a closed and convex set and $T : C \to C$ a self mapping. Denote by

$$Fix\,(T) = \{x \in C : Tx = x\},$$

the set of fixed points of T. For $\lambda \in (0, 1)$, let us also denote

$$T_\lambda := (1 - \lambda)I + \lambda T.$$

T_λ is usually named as the *averaged* (a term coined in Baillon et al. [20]) perturbation of T. It is easy to see that

$$Fix\,(T) = Fix(T_\lambda) \tag{2.1}$$

for all $\lambda \in (0, 1)$.

A mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C. \tag{2.2}$$

We also recall that a mapping $T : C \to C$ is called *asymptotically regular* (on C) if, for any $x \in C$,

$$\|T^{n+1}x - T^n x\| \to 0 \text{ as } n \to \infty.$$

It is known that if T is nonexpansive, then in general T is not asymptotically regular but its averaged perturbation, T_λ, is asymptotically regular, a result that apparently was first established by Krasnoselskii [65], in uniformly convex Banach spaces, and then used and developed by Browder and Petryshyn [45], [46] in Hilbert spaces.

This property is extremely important as it enables us to compute the fixed points of a nonexpansive mapping T by means of its averaged perturbation T_λ, which, from the point of view of the convergence of its iterations, has **richer** properties than T.

After contemplating for a long time the above enriching property of nonexpansive mappings with respect to asymptotical regularity, by using instead of T its averaged perturbation T_λ, we were naturally conducted to formulate the following

Open problem: If one uses $T_\lambda = (1-\lambda)I + \lambda T$ instead of T in a certain contraction condition from metrical fixed point theory, do we obtain a richer class of mappings ?

Fortunately, the answer was in the affirmative. Indeed, by using $T_\lambda = (1-\lambda)I + \lambda T$ instead of T in some contraction conditions from metrical fixed point theory, we obtained larger classes of mappings, for which we are able to establish constructive fixed point theorems, see the results surveyed in Sections 3-6 (for the term constructive fixed point theorem we refer to Berinde [26]).

We first searched an answer for the above question in the case of nonexpansive mappings, by considering inequality (2.2) with T_λ instead of T, that is, by introducing

$$\|T_\lambda x - T_\lambda y\| \leq \|x - y\|, \text{ for all } x, y \in C, \tag{2.3}$$

and we thus identified the class of *enriched nonexpansive mappings* in Hilbert spaces (in Berinde [28]) and called such a mapping T as being *enriched nonexpansive*. Its definition is given in Section 4 in such a way that its origins from (2.3) are hidden under the following equivalent form

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in X. \tag{2.4}$$

with $b = \dfrac{1}{\lambda} - 1$, like in Definition 4.12.

Of course, at the very first steps, we were convinced that we were the first ones to discover this nice technique but, a few years later, after a careful documentation in the related literature and a close and careful analysis, we realized that the same technique has been applied independently and tacitly by other mathematicians long time ago, e.g., by Browder and Petryshyn [46] to introduce the class of strictly pseudocontractive mappings as enriched nonexpansive mappings and also by Hicks and Kubicek [58] to introduce the class of demicontractive mappings by enriching quasi-nonexpansive operators, see Berinde and Păcurar [41] for more details.

So, without explicitly revealing their method of derivation, Browder and Petryshyn [46] introduced and studied the class of *strictly pseudocontractive mappings*, while Hicks and Kubicek [58] introduced and studied the class of *demicontractive mappings*. As shown by the rich literature developed in the last five decades, these two classes of nonexpansive type mappings represent two important concepts in the iterative approximation of fixed points, see for example Berinde [25]. Recall that a mapping T is said to be

1) *quasi-nonexpansive* if $Fix\,(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C \text{ and } y \in Fix\,(T). \tag{2.5}$$

2) *k-strictly pseudocontractive* of the Browder-Petryshyn type ([45]) if there exists $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2, \forall x, y \in C. \tag{2.6}$$

3) *k-demicontractive* ([58]) or *quasi k-strictly pseudocontractive* (see Berinde et al. [44]) if $Fix\,(T) \neq \emptyset$ and there exists a positive number $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2, \tag{2.7}$$

for all $x \in C$ and $y \in Fix\,(T)$.

If we denote by $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$ the classes of nonexpansive, quasi-nonexpansive, strictly pseudocontractive and demicontractive mappings, respectively, then the relationships between these classes are completely represented in the following diagram

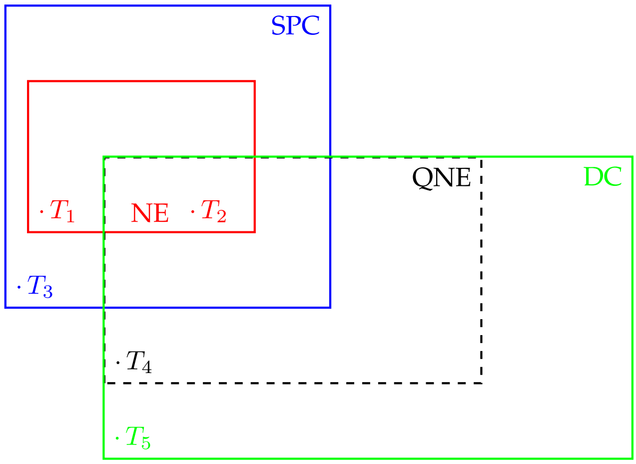

Figure 1. Diagram of the relationships between the classes $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$

The diagram in Figure 1 is taken from Berinde [34], where the mappings T_1-T_5 that differentiate the four classes of mappings are fully treated in Examples 2.1-2.5 [34].

3. Some classes of enriched mappings

Although chronologically, the first class of enriched mappings introduced in literature was the one corresponding to nonexpansive mappings, we start our presentation with enriched Banach contractions, which are related to the mappings appearing in the famous Banach contraction mapping principle - the foundation stone of metrical fixed point theory.

3.1. Enriched contractions in Banach spaces.

The concept of *enriched contraction* was introduced and studied in Berinde and Păcurar [36].

Definition 3.1 ([36]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x-y\|, \forall x, y \in X. \tag{3.8}$$

To indicate the constants involved in (3.8) *we shall also call T a (b, θ)-enriched contraction.*

Example 3.1 ([36]).

(1) A Banach contraction T satisfies (3.8) *with $b = 0$ and $\theta = c \in [0, 1)$.*

(2) Let $X = [0, 1]$ be endowed with the usual norm and let $T : X \to X$ be defined by $Tx = 1-x$, for all $x \in [0, 1]$. Then T is not a Banach contraction but T is a $(b, 1-b)$-enriched contraction for any $b \in (0, 1)$.

An important fixed point theorem and convergence result for enriched contractions is stated in the next theorem.

Theorem 3.1 ([36]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (b, θ)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.9}$$

converges to p, for any $x_0 \in X$;

(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \leq \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\|, \quad n = 0,1,2,\dots;\ i = 1,2,\dots, \tag{3.10}$$

where $c = \dfrac{\theta}{b+1}$.

Remark 3.1. *In the particular case $b = 0$, by Theorem 3.1 we get the classical Banach contraction mapping principle in the original setting of a Banach space.*

It is possible to establish a Maia type fixed point theorem for enriched contractions defined on a linear vector space, by endowing it with a metric d which is subordinated to a norm $\|\cdot\|$. The next result has been established in Berinde [30].

Theorem 3.2 ([30])**.** *Let X be a linear vector space endowed with a metric d and a norm $\|\cdot\|$ satisfying the condition*

$$d(x,y) \leq \|x-y\|, \text{ for all } x,y \in X. \tag{3.11}$$

Suppose

(i) (X,d) is a complete metric space;

(ii) $T : X \to X$ is continuous with respect to d;

(iii) T is an enriched contraction with respect to $\|\cdot\|$, that is, there exist $b \in [0,+\infty)$ and $\theta \in [0, b+1)$ such that

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\|, \forall x,y \in X. \tag{3.12}$$

Then

(i) $Fix\,(T) = \{p\}$, for some $p \in X$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.13}$$

converges in (X,d) to p, for any $x_0 \in X$;

(iii) The estimate

$$d(x_n, p) \leq \frac{c^n}{1-c} \cdot \|x_1 - x_0\|,\ n \geq 1 \tag{3.14}$$

and

$$d(x_n, p) \leq \frac{c}{1-c} \cdot \|x_n - x_{n-1}\|,\ n \geq 1, \tag{3.15}$$

hold with $c = \dfrac{\theta}{b+1}$.

3.2. **Enriched Kannan mappings in Banach spaces.**

The concept of *enriched Kannan mapping* was introduced and studied in Berinde and Păcurar [37], where some applications for solving split feasibility and variational inequality problems were also presented.

Definition 3.2 ([37])**.** *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Kannan mapping if there exist $a \in [0, 1/2)$ and $k \in [0, \infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq a\left[\|x - Tx\| + \|y - Ty\|\right], \text{ for all } x,y \in X. \tag{3.16}$$

To indicate the constants involved in (3.16) *we shall also call T a (k,a)-enriched Kannan mapping.*

Example 3.2.

(1) Any Kannan mapping is a $(0, a)$-enriched Kannan mapping, i.e., it satisfies 3.16 with $k = 0$.

(2) Let $X = [0, 1]$ be endowed with the usual norm and $T : X \to X$ be defined by $Tx = 1 - x$, for all $x \in [0, 1]$. Then T is not a Kannan mapping but T is an enriched Kannan mapping (T is also nonexpansive).

The next result provides a convergence theorem for the Krasnoselskij iterative method used to approximate the fixed points of *enriched* Kannan mappings.

Theorem 3.3 ([37]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, a)-enriched Kannan mapping. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) *There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.17}$$

converges to p, for any $x_0 \in X$;

(iii) *The following estimate holds*

$$\|x_{n+i-1} - p\| \leq \frac{\delta^i}{1-\delta} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.18}$$

where $\delta = \frac{a}{1-a}$.

Remark 3.2. *The notion of enriched Kannan mapping has been extended to* enriched Bianchini mapping*, a class for which corresponding existence and approximation results that generalize Theorem 3.3 were also established in Berinde and Păcurar* [37].

3.3. Enriched Ćirić-Reich-Rus contractions in Banach spaces.

It is possible to unify and extend Theorems 3.1 and 3.3 from the previous sections and thus obtain a fixed point theorem for the so called *enriched Ćirić-Reich-Rus contractions*. This concept has been first introduced in Berinde and Păcurar [39], in a particular case, and then was improved to the currentt version in Berinde and Păcurar [42].

Definition 3.3 ([42]). *Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$ a self mapping. T is a (k, a, b)-enriched Ćirić-Reich-Rus contraction if, for some $k \in [0, \infty)$ and $a, b \geq 0$, satisfying $\dfrac{a}{k+1} + 2b < 1$, the following condition holds.*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b\left(\|x - Tx\| + \|y - Ty\|\right), \text{ for all } x, y \in X. \tag{3.19}$$

Remark 3.3.

1) A Ćirić-Reich-Rus contraction satisfies (3.19) *with $k = 0$.*

2) If $b = 0$, then from (3.19) *we obtain the contraction condition* (3.8) *that defines enriched contractions, with $k \in [0, +\infty)$ and $a \in [0, k + 1)$.*

3) If $a = 0$, then from (3.19) *we obtain the contraction condition* (3.16) *satisfied by an enriched Kannan mapping.*

Theorem 3.4 ([42]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, a, b)-enriched Ćirić-Reich-Rus contraction in the sense of Definition 3.3. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) *There exists $\lambda \in (0, 1]$ such that the iterative method $\{y_n\}_{n=0}^{\infty}$, given by*

$$y_{n+1} = (1 - \lambda)y_n + \lambda T y_n,\ n \geq 0, \tag{3.20}$$

converges to p, for any $y_0 \in X$;

(iii) The following estimates hold

$$\|y_n - p\| \le \begin{cases} \alpha^n \cdot \|y_0 - p\|, n \ge 0 \\ \dfrac{\alpha}{1-\alpha} \cdot \|y_n - y_{n-1}\|, n \ge 1 \end{cases} \tag{3.21}$$

where $\alpha = \dfrac{a+(k+1)b}{(k+1)(1-b)}$.

Remark 3.4.

As mentioned before, a preliminary version of the concept of enriched Ćirić-Reich-Rus contraction in Definition 3.3 has been introduced and studied in Berinde and Păcurar [39].

3.4. **Enriched Chatterjea mappings in Banach spaces.**

The notion of *enriched Chatterjea mapping* was introduced and studied in Berinde and Păcurar [38].

Definition 3.4 ([38]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Chatterjea mapping if there exist $b \in [0, 1/2)$ and $k \in [0, +\infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \le b\,[\|(k+1)(x-y) + y - Ty\| +$$

$$+\|(k+1)(y-x) + x - Tx\|]\,\|, \forall x, y \in X. \tag{3.22}$$

To indicate the constants involved in (3.22) *we shall call T a (k, b)-enriched Chatterjea mapping.*

Example 3.3.

1) A Chatterjea mapping satisfies (3.22) *with $k = 0$.*

2) All Banach contractions with constant $c < \dfrac{1}{3}$, all Kannan mappings with Kannan constant $a < \dfrac{1}{4}$ and all Chatterjea mappings are enriched Chatterjea mappings, i.e., they satisfy (3.22) *with $k = 0$.*

3) T in Example 3.2 (2) is also an enriched Chatterjea mapping.

Theorem 3.5 ([38]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, b)-enriched Chatterjea mapping. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \ge 0, \tag{3.23}$$

converges to p, for any $x_0 \in X$;

(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \le \frac{\delta^i}{1-\delta} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.24}$$

where $\delta = \dfrac{b}{1-b}$.

3.5. **Enriched almost contractions in Banach spaces.**

The notion of *enriched almost contraction* has been introduced and studied in Berinde and Păcurar [43]. It is very general and unifies and extend all the previous concepts of enriched contractive type mappings.

Definition 3.5 ([43]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched almost contraction if there exist $b \in [0, \infty)$, $\theta \in (0, b+1)$ and $L \ge 0$ such that*

$$\|b(x-y) + Tx - Ty\| \le \theta\|x-y\| + L\|b(x-y) + Tx - y\|, \tag{3.25}$$

for all $x, y \in X$. To indicate the constants involved in (3.25) *we shall also call T as an enriched (b, θ, L)-almost contraction.*

Example 3.4.

1) Any (δ, L)-almost contraction is an enriched $(0, \delta, L)$-almost contraction, i.e., it satisfies (3.25) *with $b = 0$ and $\theta = \delta$;*

2) Any (b, θ)-enriched contraction is an enriched $(b, \theta, 0)$-almost contraction;

3) Any (k, a)-enriched Kannan mapping is an enriched $\left(k, \dfrac{a}{1-a}, \dfrac{2a}{1-a}\right)$-almost contraction;

4) Any (k, b)-enriched Chatterjea mapping is an enriched $\left(k, \dfrac{b}{1-b}, \dfrac{2b}{1-b}\right)$-almost contraction.

The next result unifies all main results in the previous subsections, i.e., Theorem 3.1, Theorem 3.3, Theorem 3.4 and Theorem 3.5, and provides a Krasnoselskij iterative method for approximating the fixed points of enriched almost contractions.

Theorem 3.6 ([43])**.** *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be a (b, θ, L)-almost contraction.*

Then

1) *$Fix\,(T) \neq \emptyset$;*

2) *For any $x_0 \in X$, there exists $\lambda \in (0, 1)$ such that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$, defined by,*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.26}$$

converges to some $x^* \in Fix\,(T)$, for any $x_0 \in X$;

3) *The estimate* (3.24) *holds with $\delta = \dfrac{\theta}{b+1}$.*

The next example is remarkable and illustrates the great generality of enriched almost contractions and therefore of Theorem 3.6 itself.

Example 3.5 ([43])**.** *Let $X = \left[0, \frac{4}{3}\right]$ with the usual norm and $T : X \to X$ be given by*

$$Tx = \begin{cases} 1 - x, & \text{if } x \in \left[0, \frac{2}{3}\right) \\ 2 - x, & \text{if } x \in \left[\frac{2}{3}, \frac{4}{3}\right]. \end{cases} \tag{3.27}$$

Then $Fix\,(T) = \left\{\dfrac{1}{2}, 1\right\}$ and:

1) T is a $(1, \theta, 3)$-enriched almost contraction, for any $\theta \in (0, 2)$;

2) T is not an almost contraction;

3) T does not belong to the classes of enriched contractions, enriched Kannan mappings or enriched Chatterjea mappings;

4) T is neither nonexpansive nor quasi-nonexpansive;

5) T is not an enriched nonexpansive mapping.

Remark 3.5. *It is important to note that, in view of Theorems 3.1-3.5, enriched contractions, enriched Kannan mappings, enriched Ćirić-Reich-Rus contractions and enriched Chatterjea mappings have all a unique fixed point, while enriched almost contractions - which include all these classes of mappings - could have two or more fixed points.*

3.6. **Enriched φ-contractions in Banach spaces.**

The notion of *enriched φ-contraction* has been introduced and studied in Berinde et al. [35].

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *comparison function* (see for example [25]), if the following two conditions hold:

(i_φ) φ is nondecreasing, i.e., $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$.

(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \ge 0$.

It is obvious that any comparison function also possesses the following property:

(iii_φ) $\varphi(t) < t$, for $t > 0$.

Some examples of comparison functions are the following:

$$\varphi(t) = \frac{t}{t+1}, t \in [0, \infty);\ \varphi(t) = \frac{t}{2}, t \in [0, 1] \text{ and } \varphi(t) = t - \frac{1}{3}, t \in (1, \infty);$$

(one can note that a comparison function is not necessarily continuous).

Definition 3.6 ([35]). *Consider a linear normed space $(X, \|\cdot\|)$ and let $T : X \to X$ be a self mapping. T is said to be an enriched φ-contraction if one can find a constant $b \in [0, +\infty)$ and a comparison function φ such that*

$$\|b(x-y) + Tx - Ty\| \le (b+1)\varphi(\|x-y\|), \forall x, y \in X. \tag{3.28}$$

We shall also call T a (b, φ)-enriched contraction.

Example 3.6.

1) Any (b, θ)-enriched contraction is an enriched φ-contraction with $\varphi(t) = \dfrac{\theta}{b+1} \cdot t$.

2) Any φ-contraction is a $(0, \varphi)$-enriched contraction.

3) Consider X to be the unit interval $[0, 1]$ of $\mathbb{R}$ endowed with the usual norm and the function $T : X \to X$ given by $Tx = 1 - x$, for all $x \in [0, 1]$. Then T is neither a contraction nor a φ-contraction but T is an enriched ϕ-contraction (as it is an enriched contraction).

Theorem 3.7 ([35]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ an enriched (b, φ)-contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^\infty$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \ge 0, \tag{3.29}$$

and $x_0 \in X$ arbitrary, converges strongly to p;

Remark 3.6.

An enriched (b, φ)-contraction with $b = 0$ is a usual φ-contraction, a concept that was studied previously in Berinde [22], [23], [24] *and many other papers.*

Now, let us consider the auxiliary functions $\psi : \mathbb{R}_+ \to [0, 1)$ satisfying the following property:

(g) If $\{t_n\} \subset \mathbb{R}_+$ and $\psi(t_n) \to 1$ as $n \to \infty$, then $t_n \to 0$ as $n \to \infty$.

Let $\mathcal{P}$ denote the set of all auxiliary functions ψ satisfying condition (g) above. It is easy to check that $\mathcal{P} \ne \emptyset$, as the function $\psi(t) = \exp(-t)$, for $t \ge 0$, belongs to $\mathcal{P}$.

The next result is a very general fixed point theorem, that includes many other fixed point results as particular cases, see Berinde and Păcurar [40].

Theorem 3.8 ([40]). *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be an enriched ψ-contraction, i.e., a mapping for which there exists a function $\psi \in \mathcal{P}$ such that*

$$\|b(x-y) + Tx - Ty\| \le (b+1)\psi(\|x-y\|)\|x-y\|, \forall x, y \in X. \tag{3.30}$$

Then,

(i) $Fix\,(T) = \{p\}$, for some $p \in X$.

(ii) *The sequence* $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.31}$$

and $x_0 \in X$ *arbitrary, converges strongly to* p.

3.7. **Cyclic enriched φ-contractions in Banach spaces.**

The class of enriched φ-contractions has been extended further to *cyclic enriched φ-contractions* in Berinde et al. [35]. To present it, we need the following prerequisites, see Rus [81].

Let X be a nonempty set, m a positive integer and $T : X \to X$ an operator. By definition, $\bigcup_{i=1}^{m} X_i$ is a *cyclic representation of X with respect to T* if

(i) $X_i \neq \emptyset$, $i = 1, 2, ..., m$;

(ii) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Let (X, d) be a metric space, m a positive integer, $A_1, \ldots, A_m$ nonempty and closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T : X \to X$ is called a *cyclic φ-contraction* if

(a) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(b) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)), \tag{3.32}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$.

Definition 3.7. *Consider a linear normed space* $(X, \|\cdot\|)$, $T : X \to X$ *be a self mapping and let* $\bigcup_{i=1}^{m} A_i$ *be a cyclic representation of* X *with respect to* T. *If one can find a constant* $b \in [0, +\infty)$ *and a comparison function* φ *such that*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\varphi(\|x-y\|), \forall x \in A_i \text{ and } \forall y \in A_{i+1}, \tag{3.33}$$

for $i = 1, 2, \ldots, m$, *where* $A_{m+1} = A_1$, *then* T *is said to be a cyclic enriched φ-contraction.*

Example 3.7.

1) Any cyclic φ-contraction is a cyclic enriched φ-contraction (with $b = 0$);

2) Any enriched contraction is a cyclic enriched φ-contraction (with $m = 1$).

A comparison function φ is said to be a (c)-comparison function (see [22]) if there exist $k_0 \in \mathbb{N}$, $\delta \in (0, 1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \leq \delta \varphi^k(t) + v_k,\ k \geq k_0,\ t \in \mathbb{R}_+. \tag{3.34}$$

It is known (see for example Lemma 1.1 [70]) that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then $s : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$s(t) = \sum_{k=1}^{\infty} \varphi^k(t),\ t \in \mathbb{R}_+, \tag{3.35}$$

is increasing and continuous at 0.

Theorem 3.9 ([35]). *Let* $(X, \|\cdot\|)$ *be a Banach space,* m *a positive integer,* $A_1, \ldots, A_m$ *nonempty and closed subsets of* X, $Y = \bigcup_{i=1}^{m} A_i$ *and* $T : X \to X$ *a cyclic enriched φ-contraction with* φ *a (c)-comparison function. Then*

(i) *T has a unique fixed point* $p \in \bigcap_{i=1}^{m} A_i$;

(ii) *there exists* $\lambda \in (0, 1]$ *such that the iterative method* $\{x_n\}_{n=0}^{\infty}$, *given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.36}$$

and $x_0 \in X$ *arbitrary, converges strongly to* p;

(iii) *the following estimates hold*

$$\|x_n - p\| \leq s\left(\varphi^n(\|x_0 - x_1\|)\right), n \geq 1;$$

$$\|x_n - p\| \leq s\left(\varphi(\|x_n - x_{n+1}\|)\right), n \geq 1;$$

(iv) *for any $x \in Y$:*

$$\|x_n - p\| \leq s(\lambda\|x - Tx\|),$$

where s is defined by (3.35).

Remark 3.7. *If $m = 1$, then by Theorem 3.9 one obtains Theorem 3.8 from the previous section.*

3.8. **Enriched contractions in convex metric spaces.**

The notion of *enriched contraction* in convex metric spaces has been introduced and studied in Berinde and Păcurar [40]. To introduce it we need the following

Definition 3.8. *Let (X, d) be a metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0, 1]$,*

(3.37) $$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for any } u \in X.$$

A metric space (X, d) endowed with a convex structure W is called a Takahashi convex metric space and is usually denoted by (X, d, W).

Remark 3.8. *Any linear normed space and each of its convex subsets are convex metric spaces, with the natural convex structure*

(3.38) $$W(x, y; \lambda) = \lambda x + (1 - \lambda)y, x, y \in X; \lambda \in [0, 1].$$

but the reverse is not valid.

Definition 3.9 ([40]). *Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $c \in [0, 1)$ and $\lambda \in [0, 1)$ such that*

(3.39) $$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq cd(x, y), \text{ for all } x, y \in X.$$

To specify the parameters c and λ involved in (3.39)*, we also call T a (λ, c)-enriched contraction.*

Example 3.8.

Any $(0, c)$-enriched contraction is a usual Banach contraction and therefore any enriched contraction.

The next result is a significant extension of Theorem 3.1 from the case of a Banach space setting to that of an arbitrary complete convex metric space.

Theorem 3.10 ([40]). *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a (λ, c)-enriched contraction. Then,*

(i) *$Fix\,(T) = \{p\}$, for some $p \in X$.*

(ii) *The sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process*

(3.40) $$x_{n+1} = W(x_n, Tx_n; \lambda),\ n \geq 0,$$

converges to p, for any $x_0 \in X$.

(iii) *The following estimate holds*

(3.41) $$d(x_{n+i-1}, p) \leq \frac{c^i}{1 - c} \cdot d(x_n, x_{n-1}) \quad n = 1, 2, \ldots;\ i = 1, 2, \ldots$$

3.9. **Enriched Prešić contractions.**

The notion of *enriched Prešić contraction* has been introduced and studied in Păcurar [71].

Definition 3.10 ([71]). *Let $(X,+,\cdot)$ be a linear vector space, k a positive integer and $T: X^k \to X$ an operator. For $\lambda_0, \lambda_1, \dots, \lambda_k \geq 0$, with $\sum\limits_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$, the operator $T_\lambda : X^k \to X$, defined by*

$$T_\lambda(x_0, x_1, \dots, x_{k-1}) = \lambda_0 x_0 + \lambda_1 x_1 + \dots + \lambda_{k-1} x_{k-1} + \lambda_k T(x_0, x_1, \dots, x_{k-1}) \tag{3.42}$$

will be called the averaged mapping corresponding to T.

Remark 3.9. *One can easily see that, for $k = 1$, the above definition reduces to $T_\lambda(x_0) = \lambda_0 x_0 + \lambda_1 T(x_0)$, for $x_0 \in X$, where $\lambda_0 + \lambda_1 = 1$, that is, the averaged mapping $T_\lambda : X \to X$ extensively used in the previous sections.*

Remark 3.10. *As in the case of the averaged mapping corresponding to an operator defined on X, it is not difficult to show that $x^* \in X$ is a fixed point of $T^k : X \to X$ if and only if it is a fixed point of the corresponding $T_\lambda : X^k \to X$, for some $\lambda_i \geq 0$, $i = 0, 1, \dots, k$, with $\sum\limits_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$.*

Indeed, supposing $x^* \in X$ such that $T_\lambda(x^*, x^*, \dots, x^*) = x^*$, it follows that

$$\lambda_0 x^* + \lambda_1 x^* + \dots + \lambda_{k-1} x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*,$$

so

$$(1 - \lambda_k) x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*.$$

Since $\lambda_k \neq 0$, it follows immediately that $T(x^*, x^*, \dots, x^*) = x^*$. The inverse is obvious.

Definition 3.11 ([71]). *Let $(X, \|\cdot\|)$ be a linear normed space and k a positive integer. A mapping $T : X^k \to X$ is said to be an enriched Prešić operator if there exist $b_i \geq 0, i = 0, 1, \dots, k-1$ and $\theta_i \geq 0, i = 0, 1, \dots, k-1$ with $\sum\limits_{i=0}^{k-1} (\theta_i - b_i) < 1$ such that:*

$$\left\| \sum_{i=0}^{k-1} b_i (x_i - x_{i+1}) + T(x_0, x_1, \dots, x_{k-1}) - T(x_1, x_2, \dots, x_k) \right\| \leq \sum_{i=0}^{k-1} \theta_i \|x_i - x_{i+1}\|,$$

for all $x_0, x_1, \dots, x_k \in X$.

Remark 3.11.

1) For $k = 1$ this reduces to the definition of an enriched Banach contraction;

2) If $b_0 = b_1 = \dots = b_{k-1} = 0$ in the above definition, then we obtain the definition of a Prešić operator, see Păcurar [71].

The next result states that an enriched Prešić operator posseses a unique fixed point, which can be obtained by means of some appropriate iterative methods.

Theorem 3.11 ([71]). *Let $(X, \|\cdot\|)$ be a Banach space, k a positive integer and $T : X^k \to X$ an enriched Prešić operator with constants $b_i, \theta_i, i = 0, 1, \dots, k-1$. Then:*

1) *T has a unique fixed point $x^* \in X$ such that $T(x^*, x^*, \dots, x^*)$;*
2) *There exists $a \in (0, 1]$ such that the iterative method $\{y_n\}_{n \geq 0}$ given by*

$$y_n = (1-a) y_{n-1} + a T(y_{n-1}, y_{n-1}, \dots, y_{n-1}), n \geq 1,$$

converges to the unique fixed point x^*, starting from any initial point $y_0 \in X$.

3) *There exist* $\lambda_0, \lambda_1, \dots, \lambda_k \geq 0$ *with* $\sum_{i=0}^{k} \lambda_i = 1$ *and* $\lambda_k \neq 0$ *such that the iterative method* $\{x_n\}_{n\geq 0}$ *given by*

$$x_n = \lambda_0 x_{n-k} + \lambda_1 x_{n-k+1} + \cdots + \lambda_{k-1} x_{n-1} + \lambda_k T(x_{n-k}, x_{n-k+1}, \dots, x_{n-1})$$

or simply

$$x_n = T_\lambda(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), n \geq 1,$$

converges to x^*, *for any initial points* $x_0, x_1, \dots, x_{k-1} \in X$.

4. ENRICHED NONEXPANSIVE MAPPINGS

Before proceeding with the exposure of the class of enriched nonexpansive mappings, we recall the following related concepts.

4.1. Enriched nonexpansive mappings in Hilbert spaces.

The notion of *enriched nonexpansive mapping* has been introduced and studied in Berinde [28].

Definition 4.12 ([28]). *Let* $(X, \|\cdot\|)$ *be a linear normed space. A mapping* $T : X \to X$ *is said to be an enriched nonexpansive mapping if there exists* $b \in [0, \infty)$ *such that*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in X. \tag{4.43}$$

To indicate the constant involved in (4.43) *we shall also call* T *as a b-enriched nonexpansive mapping.*

It is important to note that inequality (4.43) in Definition 4.12 was derived from (2.3) by denoting $b = \frac{1}{\lambda} - 1$.

Any nonexpansive mapping T is an enriched nonexpansive mapping, i.e., it satisfies (4.43) with $b = 0$, but the reverse is not true, as shown by the next example.

Example 4.9 ([28]).

Let $X = \left[\frac{1}{2}, 2\right]$ *be endowed with the usual norm and* $T : X \to X$ *be defined by* $Tx = \frac{1}{x}$, *for all* $x \in \left[\frac{1}{2}, 2\right]$. *Then*

(i) T *is Lipschitz continuous with Lipschitz constant* $L = 4$ *(and so* T *is not nonexpansive);*

(ii) T *is a* $3/2$*-enriched nonexpansive mapping.*

For the sake of completeness, we recall that a mapping $T : C \to H$, where C is a bounded closed convex subset of a Hilbert space H, is called *demicompact* if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.

The next result states that any enriched nonexpansive mapping which is also demicompact has a nonempty convex fixed point set and that one can approximate its fixed points by means of a Krasnoselskij type iterative scheme.

Theorem 4.12 ([28]). *Let* C *be a bounded closed convex subset of a Hilbert space* H *and* $T : C \to C$ *be a b-enriched nonexpansive and demicompact mapping. Then the set* $Fix\,(T)$ *of fixed points of* T *is a nonempty convex set and there exists* $\lambda \in (0, 1)$ *such that, for any given* $x_0 \in C$, *the Krasnoselskij iteration* $\{x_n\}_{n=0}^{\infty}$ *given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{4.44}$$

converges strongly to a fixed point of T.

If we denote by $\mathcal{ENE}$ the class of enriched nonexpansive mappings, then by the previous example it follows that we have the following strict inclusion relationship

$$\mathcal{NE} \subsetneq \mathcal{ENE}$$

and, by the diagram in Figure 1, we also deduce that

$$\mathcal{NE} \subsetneq \mathcal{SPC}.$$

It was then natural to raise the following

Problem. Find the relationship between the classes $\mathcal{ENE}$ and $\mathcal{SPC}$.

For the case of enriched nonexpansive mappings defined on a real Hilbert space, the answer is given by the next theorem which is a reformulated version of Theorem 8 in Berinde and Păcurar [41].

Theorem 4.13. *In a real Hilbert space, $\mathcal{ENE} = \mathcal{SPC}$.*

Proof. Let $T \in \mathcal{SPC}$. Then T satisfies (2.6) with some $k \in (0, 1)$. We have

$$\|x - y - (Tx - Ty)\|^2 = \langle x - y - (Tx - Ty), x - y - (Tx - Ty)\rangle \tag{4.45}$$

$$= \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|Tx - Ty\|^2$$

and so (2.6) is equivalent to

$$\|Tx - Ty\|^2 \leq \frac{1+k}{1-k} \cdot \|x - y\|^2 - \frac{2k}{1-k} \cdot \langle x - y, Tx - Ty\rangle.$$

By adding to both sides of the previous inequality the quantity

$$\left(\frac{k}{1-k}\right)^2 \cdot \|x - y\|^2 + \frac{2k}{1-k}\langle x - y, Tx - Ty\rangle,$$

we deduce that (2.6) is equivalent to

$$\left\|\frac{k}{1-k}(x - y) + Tx - Ty\right\|^2 \leq \left(\left(\frac{k}{1-k}\right)^2 + \frac{1+k}{1-k}\right)\|x - y\|^2. \tag{4.46}$$

Now, by denoting $b = \dfrac{k}{1-k} > 0$, it follows that inequality (4.46) is equivalent to

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \forall x, y \in C,$$

and this shows that $T \in \mathcal{NE}$.

The converse follows by runing backwardly the previous implications. □

4.2. Enriched nonexpansive mappings in Banach spaces.

Remark 4.12. *The equality in Theorem 4.13 is no more valid if we work in a Banach space. The main reason is that in a Banach space we cannot derive the fundamental identity* (4.45) *in the proof of Theorem 4.13, as it is expressed by means of the inner product in the Hilbert space H.*

Hence in a Banach space the class of enriched nonexpansive mappings and that of of strictly pseudocontractive mappings are independent and therefore the next result is an important generalization of several results in literature established for nonexpansive mappings, e.g., in Browder and Petryshyn [46].

Theorem 4.14 ([29]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a b-enriched nonexpansive mapping. Suppose T satisfies Condition I, i.e., there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with the properties $f(0) = 0$ and $f(r) > r$, for $r > 0$, such that*

$$\|x - Tx\| \geq f(d(x, Fix\,(T)), \forall x \in C, \tag{4.47}$$

where

$$d(x, Fix\,(T)) = \inf\{\|x - z\| : z \in Fix\,(T)\}$$

is the distance between the point x and the set $Fix\,(T)$.

Then $Fix\,(T) \neq \emptyset$ and, for any $\lambda \in \left(0, \frac{1}{b+1}\right)$ and for any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{4.48}$$

converges strongly to a fixed point of T.

5. Unsaturated and saturated classes of contractive mappings

After a close examination of the fixed point results in the Sections 3 and 4, it was noted that the technique of enriching a contractive type mapping T, by means of the averaged operator T_λ, cannot effectively enlarge all classes of contractive mappings.

This observation suggested us a new interesting concept, that of *saturated* class of contractive mappings with respect to the averaged operator T_λ, a notion that has been introduced and studied in Berinde and Păcurar [41].

Definition 5.13 ([41]). *Let $(X, \|\cdot\|)$ be a linear normed space and let $\mathcal{C}$ be a subset of the family of all self mappings of X. A mapping $T : X \to X$ is said to be $\mathcal{C}$*-enriched *or* enriched with respect to $\mathcal{C}$ *if there exists $\lambda \in (0, 1]$ such $T_\lambda \in \mathcal{C}$.*

We denote by $\mathcal{C}^e$ the set of all enriched mappings with respect to $\mathcal{C}$.

Remark 5.13. *From Definition 5.13 it immediately follows that $\mathcal{C} \subseteq \mathcal{C}^e$.*

Definition 5.14 ([41]). *Let X be a linear vector space and let $\mathcal{C}$ be a subset of the family of all self mappings of X. If $\mathcal{C} = \mathcal{C}^e$, we say that $\mathcal{C}$ is a* saturated class of mappings, *otherwise $\mathcal{C}$ is said to be* unsaturated.

If we summarize the results surveyed in the previous two sections of this paper, then we can see that the following classes of mappings:

- Banach contractions
- Kannan mappings
- Ćirić-Reich-Rus contractions
- Chatterjea mappings
- almost contractions
- φ-contractions
- cyclic enriched φ-contractions
- Prešić contractions

are all *unsaturated* in the setting of a Banach space.

Also, from the results surveyed in Section 4 we infer that

- the class of nonexpansive mappings is unsaturated in Hilbert spaces;
- the class of nonexpansive mappings is unsaturated in Banach spaces

Let us denote by $\mathcal{E}$ the enriching operator by the average perturbation of a certain class of self mappings of X, i.e., if $\mathcal{C}$ is a class of mappings, then

$$\mathcal{E}(\mathcal{C}) = \mathcal{C}^e.$$

It is easy to prove that $\mathcal{E}$ is idempotent, that is, $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, which means that any class of enriched mappings is *saturated*.

This implies that

- the class of strictly pseudocontractive mappings is saturated in Hilbert spaces;
- the class of enriched nonexpansive mappings is saturated in Hilbert spaces;
- the class of demicontractive mappings is is saturated in Hilbert spaces.

The last claim follows by Theorem 9 in Berinde and Păcurar [41].

6. Enriched contractions in quasi-Banach spaces

The notion of enriched contraction in the setting of a quasi-Banach space has been introduced and studied in Berinde [34]. We recall the following prerequisites.

Definition 6.15. *A quasi-norm on a real vector space X is a map $\|\cdot\| : X \to [0,\infty)$ satisfying the following conditions:*

(QN_0) $\|x\| = 0$ *if and only if* $x = 0$;
(QN_1) $\|\lambda x\| = |\lambda| \cdot \|x\|$, *for all* $x \in X$ *and* $\lambda \in \mathbb{R}$.
(QN_2) $\|x + y\| \leq C\,[\|x\| + \|y\|]$, *for all* $x, y \in X$, *where* $C \geq 1$ *does not depend on* x, y;

The pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a quasi-norm on a real vector space X, is said to be a quasi-normed space. If $(X, \|\cdot\|)$ is complete (with respect to the quasi norm), then is called a *quasi-Banach space*.

Definition 6.16. *Let $(X, \|\cdot\|)$ be a linear quasi-normed space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x - y\|, \forall x, y \in X. \tag{6.49}$$

To indicate the constants involved in (6.49) *we shall also call T a (b,θ)-enriched contraction.*

Remark 6.14.

1) As any Banach space is a quasi-Banach space (with $C = 1$), the enriched contractions T in a Banach space introduced in Berinde and Păcurar [36] *are enriched contractions in the sense of Definition 6.16.*

2) It is worth mentioning that, like in the case of Banach spaces, any (b,θ)-enriched contraction is continuous.

The following result is an extension of Theorem 3.1 from Banach spaces to quasi-Banach spaces.

Theorem 6.15 (Berinde [34]). *Let $(X, \|\cdot\|)$ be a quasi-Banach space and $T : X \to X$ a (b,θ)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$;
(ii) *There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{6.50}$$

converges to p, for any $x_0 \in X$;

7. OTHER DEVELOPMENTS IN THE AREA OF ENRICHED CONTRACTIVE TYPE MAPPINGS

The technique of enriching contractive type mappings, applied by the current authors to several classes of mappings, attracted the interest of many other researchers. In the following we summarise to date some of these contributions.

(1) Abbas et al. [1] introduced the notion of *enriched quasi-contraction*, as a generalization of the clasical Ćirić quasi-contraction, and also the class of *enriched weak contraction mappings* and established and studied the existence and iterative approximation of their fixed points.
(2) Abbas et al. [2] introduced and studied the class of *enriched multivalued contraction mappings* and also considered the data dependence problem and Ulam-Hyers stability of the fixed point problems for enriched multivalued contraction mappings. They also give applications of the obtained results to the problem of the existence of a solution of differential inclusions and dynamic programming.
(3) Abbas et al. [3] introduced the concept of *generalized enriched cyclic contraction mapping* and obtained existence of fixed points and established convergence results for Krasnoselskij iteration used for approximating fixed points of such mappings. As an application of their results, they also established the existence and uniqueness of an attractor for an iterated function system composed of generalized enriched cyclic contraction mappings.
(4) Abbas et al. [5] introduced the concept of *enriched contractive mappings of Suzuki type*. Such a mapping has a fixed point and characterizes the completeness of the underlying normed space.
(5) Abbas et al. [6] introduced the class of *enriched interpolative Kannan type operators* on Banach spaces. This class contains the classes of enriched Kannan operators, interpolative Kannan type contraction operators and some other classes of nonlinear operators. They prove a convergence theorem for the Krasnoselskii iteration method to approximate fixed point of the enriched interpolative Kannan type operatorsand, as an application of the main result, solved a variational inequality problem. The same authors propose in [7] a new class of multi-valued enriched interpolative Ćirić-Reich-Rus type contraction operators, prove a fixed point result, study the data dependence and Ulam-Hyers stability for these operators and obtain a homotopy result as an application of their results.
(6) Hacioğlu and Gürsoy [59] introduced multivalued Górnicki mappings and various other new types of multivalued enriched contractive mappings, like multivalued enriched Kannan mappings, multivalued enriched Chatterjea mappings, and multivalued enriched Ćirić-Reich-Rus mappings, and established existence results for the fixed points of these multivalued contractive type mappings by using the fixed point property of the average operator of the mappings.
(7) Babu and Mounika [19] defined the classes of *enriched Jaggi contraction maps*, *enriched Dass and Gupta contraction maps* and *almost* (k, a, b, λ)*-enriched CRR contraction maps* in Banach spaces and proved the existence and uniqueness of fixed points of these maps.
(8) By using some semi-implicit relations, Mondal et al. [68] introduced *enriched* $\mathcal{A}$*-contractions* and *enriched* $\mathcal{A}'$*-contractions* and studied the existence of fixed points, the well-posedness and limit shadowing property of the fixed point problem involving these contractions. The same authors obtained laterMaia type results for enriched contractions via implicit relations [21] , thus extending the results of Berinde [30].

(9) Chandok [47] obtained convergence and existence results of best proximity points for cyclic enriched contraction maps in Takahashi convex metric spaces.
(10) Popescu [77] introduced a new class of Picard operators, called *Górnicki mappings*, which includes the class of enriched contractions [36], enriched Kannan mappings [37], and enriched Chatterjea mappings [38], and proved some fixed point theorems for these mappings. However, while the fixed points of enriched contractions can be approximated by means of Krasnoselskij iteration, there is no any approximation result in [77] for the case of Górnicki mappings. This rise the challenging problem of finding iterative schemes to approximate the unique fixed point of a Górnicki mapping.
(11) Debnath [48] introduced the notion of Górnicki-type pair of mappings, establish a criterion for existence and uniqueness of common fixed point for such a pair without assuming continuity of the underlying mappings and also establish a common fixed point result for a pair of enriched contractions.
(12) Deshmukh et al. [50], amongst many other related results for enriched non-expansive maps and enriched generalized non-expansive maps, also give stability results for two iterative procedures in the class of enriched contractions.
(13) Faraji and Radenović [53] established fixed point results for enriched contractions and enriched Kannan contractions in partially ordered Banach spaces.
(14) Based on the so-called degree of nondensifiability, García [55] introduced a generalization of the (b, θ)-enriched contractions [36] and established a fixed point existence result for this new class of mappings. From their main result, and under some suitable conditions, they derived a result on the existence of fixed points for the sum of two mappings, one of them being compact.
(15) Khan et al. [64] initiate the study of enriched mappings in modular function spaces, by introducing the concepts of *enriched ρ-contractions* and *enriched ρ-Kannan mappings* and establishing some results on the existence of fixed points of such mappings in this setting.
(16) By introducing the concept of convex structure in rectangular G_b-metric spaces, Li and Cui [66] studied the existence of fixed points of enriched type contractions in such a space.
(17) As a generalization of the main result in [36], Marchiş [67] obtained some common fixed point theorems under an enriched type contraction condition for two single-valued mappings satisfying a weak commutativity condition in Banach spaces and has shown that the unique common fixed point of these mappings can be approximated using the Krasnoselskij iteration.
(18) By using the idea of the orbital contraction condition given in [75] and considering the second iterate of the mapping in the enriched contraction condition, Nithiarayaphaks and Sintunavarat [69] introduced the class of *weak enriched contraction mappings* and approximated their fixed point by Kirk's iterative scheme.
(19) Panicker and Shukla [72] obtained stability results of fixed point sets for a sequence of enriched contraction mappings in the setting of convex metric spaces, by considering two types of convergence of sequences of mappings, namely, $(\mathcal{G})$-convergence and $(\mathcal{H})$-convergence.
(20) Panja et al. [74] introduced a new non-linear semigroup of enriched Kannan type contractions and proved the existence of a common fixed point on a closed, convex, bounded subset of a real Banach space having uniform normal structure.
(21) Among many other related results, Prithvi and Katiyar [78] studied fractals through generalized cyclic enriched Ćirić-Reich-Rus iterated function systems.

(22) Rawat et al. [80] considered enriched ordered contractions in convex noncommutative Banach spaces, while Rawat, Bartwal and Dimri [79] defined and studied interpolative enriched contractions of Kannan type, Hardy-Rogers type and Matkowski type in the setting of a convex metric space.
(23) Ali and Jubair [12] introduced and studied the so called enriched Berinde nonexpansive mappings, which are related to enriched almost contractions.
(24) Anjali and Batra [14] introduced enriched Ćirić's type and enriched Hardy-Rogers contractions for which they established fixed point theorems in Banach spaces and convex metric spaces. They showed that Ćirić's type and Hardy-Rogers contractions are unsaturated classes of mappings and also considered Reich and Bianchini contractions, which were shown to be unsaturated classes of mappings, too.
(25) Babu and Mounika [19] introduced enriched Jaggi contraction maps, enriched Dass and Gupta contraction maps and almost (k, a, b, λ)-enriched CRR contraction maps in Banach spaces and established results on the existence and uniqueness of fixed points of these maps.
(26) Zhou et al. [95] introduced and studied weak enriched $\mathcal{F}$-contractions, weak enriched $\mathcal{F}'$-contraction, and k-fold averaged mapping based on Kirk's iterative algorithm of order k and proved the existence of a unique fixed point of the k-fold averaged mapping associated with weak enriched contractions considered.
(27) Ullah et al. [94] introduced and studied the class of enriched Suzuki nonexpansive mappings, which properly contains the class of Suzuki nonexpansive as well as the class of enriched nonexpansive mappings.
(28) Turcanu and Postolache [93] introduced and studied enriched Suzuki mappings, which in particular include enriched nonexpansive mappings, from Hilbert spaces to Hadamard spaces.
(29) Phairatchatniyoman et al. Phairat used a modified Ishikawa iteration scheme to solve a fixed point problem and a split variational inclusion problem in real Hilbert spaces, for b-enriched nonexpansive mapping, and applied it for solving a split feasibility problem.
(30) Dechboon and Khammahawongwe [49] established the existence and uniqueness of the best proximity point for several classes of generalized cyclic enriched contractions in convex metric spaces.
(31) ..

8. CONCLUSIONS

1. In the first part of this paper we presented a few relevant facts about the way in which the technique of enriching contractive mappings was (re-)discovered.

2. In the main part of the paper we have exposed the main contributions in the area of enriched mappings established by the authors and their collaborators by using this technique.

3. In the last part, we also surveyed some related developments which are authored by other researchers, by considering a list of references to date.

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[1] DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
NORTH UNIVERSITY CENTRE AT BAIA MARE
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
VICTORIEI 76, 430072 BAIA MARE ROMANIA
Email address: vasile.berinde@mi.utcluj.ro

[2] ACADEMY OF ROMANIAN SCIENTISTS, 3 ILFOV, 050044, BUCHAREST, ROMANIA
Email address: vasile.berinde@gmail.com

[3] DEPARTMENT OF ECONOMICS AND BUSSINESS ADMINISTRATION
IN GERMAN LANGUAGE
FACULTY OF ECONOMICS AND BUSSINESS ADMINISTRATION
BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
T. MIHALI 58-60, 400591 CLUJ-NAPOCA, ROMANIA
Email address: madalina.pacurar@econ.ubbcluj.ro