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A FAMILY OF MONOTONIC ITERATIVE METHODS FOR SOLVING ρ-DEMICONTRACTIVE FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITIES INVOLVING PSEUDOMONOTONE OPERATORS

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Abstract. A class of inertial extragradient methods is proposed to find a numerical common solution to the variational inequality problem, including a pseudomonotone and Lipschitz continuous operator, as well as the fixed point problem in real Hilbert spaces with a ρ-demicontractive mapping. Inertial iterative methods use self-adaptive step size rules and therefore do not require previous knowledge of the Lipschitz constant. We show that the proposed methods strongly converge to a solution of the variational inequality and fixed point problems under standard test conditions. Finally, we present numerical examples to compute the effectiveness and validate the proposed techniques. The conclusions of this study on variational inequality and fixed point problems strengthen and expand on previous research in the literature.

1. Introduction

Assume that $\mathcal{C}$ is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. The main contribution of this research is to investigate the convergence analysis of the iterative methods for solving variational inequality problems and fixed point problems in real Hilbert spaces. The motivation to study a common solution problem is its potential applicability to mathematical models whose constraints can be stated as fixed point problems. This is especially true in real applications like signal processing, network resource allocation, and picture recovery. This is especially relevant in applications such as signal processing, composite minimization, optimum control, and image restoration; see, for example, [12, 17]. Let's take a look at both of the problems highlighted by this research. Let $\mathcal{N} : \mathcal{C} \to \mathcal{H}$ be an operator. First, we look at the classic variational inequality problem [26], which is expressed as follows:

$$\text{(VIP)} \qquad \text{Find } r^* \in \mathcal{C} \text{ such that } \left\langle \mathcal{N}(r^*), y - r^* \right\rangle \geq 0, \ \forall\, y \in \mathcal{C}.$$

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Let $VI(\mathcal{C},\mathcal{N})$ be denoted by the solution set of the problem (VIP). This idea of variational inequalities covers several disciplines such as partial differential equations, optimization, optimal control, mechanics, mathematical programming, and finance (see for details [10, 13, 14, 19]). Many authors have proposed various projection methods to solve the problem (VIP) (see for more details: [1, 3, 4, 6–8, 15, 16, 18, 20–22, 24, 30–34]) and others in [5, 27, 37–41]. Korpelevich [15] and Antipin [2] established the extragradient method described below. Their method takes the form of

$$\begin{cases} u_1 \in \mathcal{C}, \\ y_k = P_{\mathcal{C}}[u_k - \eta\mathcal{N}(u_k)], \\ u_{k+1} = P_{\mathcal{C}}[u_k - \eta\mathcal{N}(y_k)], \end{cases} \tag{1.1}$$

where $0 < \eta < \frac{1}{L}$. According to above method, each iteration must estimate two projections on the feasible set $\mathcal{C}$. Of course, if the feasible set $\mathcal{C}$ has a convoluted structure, this might have an impact on the computing efficacy of the approach adopted. On the other hand, Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a mapping, and the fixed point problem (FPP) for the mapping $\mathcal{T}$ is to find $r^* \in \mathcal{H}$ such that

$$\mathcal{T}(r^*) = r^*. \tag{FP}$$

The solution set of problem (FP) is known as the fixed point set of $\mathcal{T}$ and is represented by $Fix(\mathcal{T})$. The most of algorithms for addressing problem (FP) are derived from the standard Mann iteration, specifically, from $u_1 \in \mathcal{H}$, construct sequence $\{u_{k+1}\}$ for all $k \geq 1$ by

$$u_{k+1} = \alpha_k u_k + (1 - \alpha_k)\mathcal{T}u_k, \tag{1.2}$$

where the variable sequence $\{\alpha_k\}$ must meet certain requirements in order to obtain weak convergence. Another iterative method that is effective in infinite-dimensional Hilbert spaces for obtaining strong convergence is the Halpern iteration. The iterative sequence can be written as follows:

$$u_{k+1} = \alpha_k u + (1 - \alpha_k)\mathcal{T}u_k, \tag{1.3}$$

where $u \in \mathcal{H}$ is a fixed element and the sequence $\alpha_k \subset (0; 1)$ is non-summable and slowly diminishing, i.e., $\alpha_k \to 0$ and $\sum_{k=1}^{+\infty} \alpha_k = +\infty$.

Motivated by the work presented in [28,29], we propose a strongly convergent inertial extragradient method without using Mann and Viscosity-type iterative schemes for solving variational inequalities and fixed point problems with a self adaptive step size rule. In real Hilbert spaces, two strong convergence theorems for solving pseudomonotone variational inequalities and the ρ-demicontractive fixed point problem are presented. In order to obtain the strong convergence of these methods, we use double inertial iterative scheme and have a strong convergence.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty, closed, and convex subset of $\mathcal{H}$, the real Hilbert space. $u_k \rightharpoonup u$ and $u_k \to u$ represent the weak and strong convergence of u_k to u, respectively. For each $u, y \in \mathcal{H}$, we have

(1) $\|u + y\|^2 = \|u\|^2 + 2\langle u, y\rangle + \|y\|^2$;
(2) $\|u + y\|^2 \leq \|u\|^2 + 2\langle y, u + y\rangle$;

(3) $\|bu + (1-b)y\|^2 = b\|u\|^2 + (1-b)\|y\|^2 - b(1-b)\|u-y\|^2$.

The definition of *metric projection* $P_{\mathcal{C}}(u)$ of $u \in \mathcal{H}$ is defined by

$$P_{\mathcal{C}}(u) = \arg\min\{\|u-y\| : y \in \mathcal{C}\}.$$

The metric projection $P_{\mathcal{C}}$ satisfies the following properties:

(1) $\langle u - P_{\mathcal{C}}(u), y - P_{\mathcal{C}}(u)\rangle \leq 0,\ \forall\, y \in \mathcal{C}$;

(2) $\|P_{\mathcal{C}}(u) - P_{\mathcal{C}}(y)\|^2 \leq \langle P_{\mathcal{C}}(u) - P_{\mathcal{C}}(y), u - y\rangle,\ \forall\, y \in \mathcal{C}$.

Definition 2.1. Assume that $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator with $Fix(\mathcal{T}) \neq \emptyset$. Then, $I - \mathcal{T}$ is said to be demiclosed at zero if, for all $\{u_k\}$ in $\mathcal{H}$, the following expressions are true:

$$u_k \rightharpoonup u \quad \text{and} \quad (I-\mathcal{T})u_k \to 0 \Rightarrow u \in Fix(\mathcal{T}).$$

Lemma 2.2. [23] *Let* $\{p_k\} \subset [0, +\infty)$, $\{q_k\} \subset (0,1)$ *and* $\{r_k\} \subset \mathbb{R}$ *be three sequences meet the following condition:*

$$p_{k+1} \leq (1-q_k)p_k + q_k r_k,\ \forall\, k \in \mathbb{N} \quad \textit{and} \quad \sum_{k=1}^{+\infty} q_k = +\infty.$$

If $\limsup_{j\to+\infty} r_{k_j} \leq 0$ *for each subsequence* $\{p_{k_j}\}$ *of* $\{p_j\}$ *meet* $\liminf_{j\to+\infty}(p_{k_j+1} - p_{k_j}) \geq 0$. *Then,* $\lim_{k\to+\infty} p_k = 0$.

3. Main Results

In this section, we examine in detail the convergence of two novel inertial extragradient algorithms for solving variational inequality and fixed point problems. It is assumed that the following conditions are satisfied to prove the strong convergence: ($\mathcal{N}1$) The solution set $Fix(\mathcal{T}) \cap VI(\mathcal{C}, \mathcal{N}) \neq \emptyset$; ($\mathcal{N}2$) The mapping $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is said to be *pseudomonotone* if

$$\text{(PM)} \qquad \langle \mathcal{N}(u_1), u_2 - u_1\rangle \geq 0 \Longrightarrow \langle \mathcal{N}(u_2), u_1 - u_2\rangle \leq 0,\ \forall\, u_1, u_2 \in \mathcal{C};$$

($\mathcal{N}3$) The mapping $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is said to be *Lipschitz continuous* with constant $L > 0$ such that

$$\|\mathcal{N}(u_1) - \mathcal{N}(u_2)\| \leq L\|u_1 - u_2\|,\ \forall\, u_1, u_2 \in \mathcal{C};$$

($\mathcal{N}4$) A mapping $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ is said to be *sequentially weakly continuous* if $\{\mathcal{N}(u_k)\}$ weakly converges to $\mathcal{N}(u)$ for any sequence $\{u_k\}$ weakly converges to u; ($\mathcal{N}5$) The mapping $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is ρ-demicontractive if $0 \leq \rho < 1$ such that

$$\|\mathcal{T}(u_1) - u_2\|^2 \leq \|u_1 - u_2\|^2 + \rho\|(I-\mathcal{T})(u_1)\|^2, \quad \forall\, u_2 \in Fix(\mathcal{T}), u_1 \in \mathcal{H};$$

or equivalently

$$\langle \mathcal{T}(u_1) - u_1, u_1 - u_2\rangle \leq \frac{\rho - 1}{2}\|u_1 - \mathcal{T}(u_1)\|^2, \quad \forall\, u_2 \in Fix(\mathcal{T}), u_1 \in \mathcal{H}.$$

Lemma 3.1. *A sequence* $\{\eta_k\}$ *generated by* (3.3) *is convergent to* η *and bounded by* $\min\{\frac{\mu}{L}, \eta_1\}$.

Algorithm 1

Step 0: Take $u_0, u_1 \in \mathcal{C}$, $\theta \in (0,1)$, $0 < \eta < \frac{1}{L}$ and a sequence $\{\beta_k\} \subset (0,1)$ meets the subsequent criteria: $\lim_{k\to+\infty} \beta_k = 0$ and $\sum_{k=1}^{+\infty} \beta_k = +\infty$.
Step 1: Compute $t_k = u_k + \theta_k(u_k - u_{k-1}) - \beta_k\big[u_k + \theta_k(u_k - u_{k-1})\big]$, while θ_k taken as follows:

$$0 \le \theta_k \le \hat{\theta_k} \quad \text{and} \quad \hat{\theta_k} = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \neq u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases} \tag{3.1}$$

Moreover, a positive sequence $\epsilon_k = \circ(\beta_k)$ satisfies $\lim_{k\to+\infty} \frac{\epsilon_k}{\beta_k} = 0$.
Step 2: Compute $y_k = P_{\mathcal{C}}(t_k - \eta\mathcal{N}(t_k))$. If $t_k = y_k$, then STOP. Otherwise
Step 3: Firstly construct a half-space $\mathcal{H}_k = \{z \in \mathcal{H} : \langle t_k - \eta\mathcal{N}(t_k) - y_k, z - y_k\rangle \le 0\}$ and compute $z_k = P_{\mathcal{H}_k}(t_k - \eta\mathcal{N}(y_k))$.
Step 4: Compute $u_{k+1} = (1-\alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$.

Algorithm 2

Step 0: Take $u_0, u_1 \in \mathcal{C}$, $\theta \in (0,1)$, $\mu \in (0,1)$, $\eta_1 > 0$. Moreover, $\{\beta_k\} \subset (0,1)$ meets the subsequent requirements: $\lim_{k\to+\infty} \beta_k = 0$ and $\sum_{k=1}^{+\infty} \beta_k = +\infty$.
Step 1: Compute $t_k = u_k + \theta_k(u_k - u_{k-1}) - \beta_k\big[u_k + \theta_k(u_k - u_{k-1})\big]$, while θ_k taken as follows:

$$0 \le \theta_k \le \hat{\theta_k} \quad \text{and} \quad \hat{\theta_k} = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \neq u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases} \tag{3.2}$$

Moreover, a positive sequence $\epsilon_k = \circ(\beta_k)$ satisfies $\lim_{k\to+\infty} \frac{\epsilon_k}{\beta_k} = 0$.
Step 2: Compute $y_k = P_{\mathcal{C}}(t_k - \eta_k\mathcal{N}(t_k))$. If $t_k = y_k$, then STOP. Otherwise
Step 3: Firstly construct a half-space $\mathcal{H}_k = \{z \in \mathcal{H} : \langle t_k - \eta_k\mathcal{N}(t_k) - y_k, z - y_k\rangle \le 0\}$ and compute $z_k = P_{\mathcal{H}_k}(t_k - \eta_k\mathcal{N}(y_k))$.
Step 4: Compute $u_{k+1} = (1-\alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$.
Step 5: Compute

$$\eta_{k+1} = \begin{cases} \min\left\{\eta_k, \ \frac{\mu\|t_k - y_k\|^2 + \mu\|z_k - y_k\|^2}{2\left[\left\langle \mathcal{N}(t_k) - \mathcal{N}(y_k), z_k - y_k\right\rangle\right]}\right\} & \text{if} \quad \left\langle \mathcal{N}(t_k) - \mathcal{N}(y_k), z_k - y_k\right\rangle > 0, \\ \eta_k, & \textit{otherwise.} \end{cases} \tag{3.3}$$

Proof. Since the mapping $\mathcal{N}$ is Lipschitz continuous such that there exists a positive constant L. It is given that $\left\langle \mathcal{N}(t_k) - \mathcal{N}(y_k), z_k - y_k\right\rangle > 0$, and

$$\begin{aligned} \frac{\mu(\|t_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left\langle \mathcal{N}(t_k) - \mathcal{N}(y_k), z_k - y_k\right\rangle} &\ge \frac{2\mu\|t_k - y_k\|\|z_k - y_k\|}{2\|\mathcal{N}(t_k) - \mathcal{N}(y_k)\|\|z_k - y_k\|} \\ &\ge \frac{2\mu\|t_k - y_k\|\|z_k - y_k\|}{2L\|t_k - y_k\|\|z_k - y_k\|} \\ &\ge \frac{\mu}{L}. \end{aligned} \tag{3.4}$$

Thus, the sequence $\{\eta_k\}$ is convergent to η. This completes the proof. □

Lemma 3.2. *Let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be a mapping meet the items $(\mathcal{N}1)$–$(\mathcal{N}4)$. Let $\{u_k\}$ be a sequence is generated by Algorithm 1. For each $r^* \in VI(\mathcal{C}, \mathcal{N})$, we have*

$$\|z_k - r^*\|^2 \leq \|t_k - r^*\|^2 - \big(1 - \eta L\big)\|t_k - y_k\|^2 - \big(1 - \eta L\big)\|z_k - y_k\|^2.$$

Lemma 3.3. [36] *Let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be a mapping meet the items $(\mathcal{N}1)$–$(\mathcal{N}4)$. Let $\{u_k\}$ be a sequence is generated by Algorithm 2. For each $r^* \in VI(\mathcal{C}, \mathcal{N})$, we have*

$$\|z_k - r^*\|^2 \leq \|t_k - r^*\|^2 - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|t_k - y_k\|^2 - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|z_k - y_k\|^2.$$

Lemma 3.4. *Let a mapping $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions $(\mathcal{N}1)$–$(\mathcal{N}4)$. If there exists a subsequence $\{t_{k_j}\}$ weakly convergent to $\hat{u}$ and $\lim_{j\to+\infty}\|t_{k_j} - y_{k_j}\| = 0$. Then, $\hat{u} \in VI(\mathcal{C}, \mathcal{N})$.*

Proof. Since $\{t_{k_j}\}$ weakly convergent to $\hat{u}$ and due to $\lim_{j\to+\infty}\|t_{k_j} - y_{k_j}\| = 0$, sequence $\{y_{k_j}\}$ also weakly convergent to $\hat{u}$. Next, we need to prove that $\hat{u} \in VI(\mathcal{C}, \mathcal{N})$. It gives that $y_{k_j} = P_{\mathcal{C}}[t_{k_j} - \eta_{k_j}\mathcal{N}(t_{k_j})]$ that is equivalent to

$$\langle t_{k_j} - \eta_{k_j}\mathcal{N}(t_{k_j}) - y_{k_j}, y - y_{k_j}\rangle \leq 0,\ \forall y \in \mathcal{C}. \tag{3.5}$$

Due to the above inequity, we have

$$\langle t_{k_j} - y_{k_j}, y - y_{k_j}\rangle \leq \eta_{k_j}\langle \mathcal{N}(t_{k_j}), y - y_{k_j}\rangle,\ \forall y \in \mathcal{C}. \tag{3.6}$$

Consequently, we obtain

$$\frac{1}{\eta_{k_j}}\langle t_{k_j} - y_{k_j}, y - y_{k_j}\rangle + \langle \mathcal{N}(t_{k_j}), y_{k_j} - t_{k_j}\rangle \leq \langle \mathcal{N}(t_{k_j}), y - t_{k_j}\rangle,\ \forall y \in \mathcal{C}. \tag{3.7}$$

Since $\min\big\{\frac{\mu}{L}, \eta_1\big\} \leq \eta \leq \eta_1$ and $\{t_{k_j}\}$ is a bounded sequence. By the use of $\lim_{j\to+\infty}\|t_{k_j} - y_{k_j}\| = 0$ and $j \to +\infty$ in expression (3.7), we get

$$\liminf_{j\to+\infty}\langle \mathcal{N}(t_{k_j}), y - t_{k_j}\rangle \geq 0,\ \forall y \in \mathcal{C}. \tag{3.8}$$

It follows that

$$\begin{aligned}\langle \mathcal{N}(y_{k_j}), y - y_{k_j}\rangle &= \langle \mathcal{N}(y_{k_j}) - \mathcal{N}(t_{k_j}), y - t_{k_j}\rangle \\ &\quad + \langle \mathcal{N}(t_{k_j}), y - t_{k_j}\rangle + \langle \mathcal{N}(y_{k_j}), t_{k_j} - y_{k_j}\rangle.\end{aligned} \tag{3.9}$$

Since $\lim_{j\to+\infty}\|t_{k_j} - y_{k_j}\| = 0$ and condition on mapping $\mathcal{N}$, we have

$$\lim_{j\to+\infty}\|\mathcal{N}(t_{k_j}) - \mathcal{N}(y_{k_j})\| = 0, \tag{3.10}$$

which together with expressions (3.9) and (3.10), we obtain

$$\liminf_{j\to+\infty}\langle \mathcal{N}(y_{k_j}), y - y_{k_j}\rangle \geq 0,\ \forall y \in \mathcal{C}. \tag{3.11}$$

To prove further, let us take a positive sequence $\{\epsilon_j\}$ that is convergent to zero and decreasing. For each $\{\epsilon_j\}$ and m_j denoted by the least positive integer such that

$$\langle \mathcal{N}(t_{k_i}), y - t_{k_i}\rangle + \epsilon_j > 0,\ \forall i \geq m_j, \tag{3.12}$$

while m_j can be observed by expression (3.11). If there exists an integer $N_0 \in \mathbb{N}$ such that for all $k_{m_n} \geq N_0$ such that $\mathcal{N}(t_{k_{m_n}}) \neq 0$. Thus, we have

$$\Upsilon_{k_{m_n}} = \frac{\mathcal{N}(t_{k_{m_n}})}{\|\mathcal{N}(t_{k_{m_n}})\|^2}, \ \forall\, k_{m_n} \geq N_0. \tag{3.13}$$

As a result of the above definition, we have

$$\langle \mathcal{N}(t_{k_{m_n}}), \Upsilon_{k_{m_n}} \rangle = 1, \ \forall\, k_{m_n} \geq N_0. \tag{3.14}$$

Additionally, expressions (3.12) and (3.14) for all $k_{m_n} \geq N_0$, we have

$$\langle \mathcal{N}(t_{k_{m_n}}),\ y + \epsilon_k \Upsilon_{k_{m_n}} - t_{k_{m_n}} \rangle > 0. \tag{3.15}$$

Since $\mathcal{N}$ is pseudomonotone mapping. Thus, we have

$$\langle \mathcal{N}(y + \epsilon_k \Upsilon_{k_{m_n}}),\ y + \epsilon_k \Upsilon_{k_{m_n}} - t_{k_{m_n}} \rangle > 0. \tag{3.16}$$

For all $k_{m_n} \geq N_0$, we have

$$\begin{aligned} \langle \mathcal{N}(y), y - t_{k_{m_n}} \rangle \geq\ & \langle \mathcal{N}(y) - \mathcal{N}(y + \epsilon_k \Upsilon_{k_{m_n}}),\ y + \epsilon_k \Upsilon_{k_{m_n}} - t_{k_{m_n}} \rangle \\ & - \epsilon_k \langle \mathcal{N}(y), \Upsilon_{k_{m_n}} \rangle. \end{aligned} \tag{3.17}$$

Due to the sequence $\{t_{k_n}\}$ converges weakly to $\hat{u} \in \mathcal{C}$ with an operator $\mathcal{N}$ is weakly sequentially continuous on the set $\mathcal{C}$, we obtain $\{\mathcal{N}(t_{k_n})\}$ weakly converges to $\mathcal{N}(\hat{u})$. Let $\mathcal{N}(\hat{u}) \neq 0$, that implies that

$$\|\mathcal{N}(\hat{u})\| \leq \liminf_{n \to +\infty} \|\mathcal{N}(t_{k_n})\|. \tag{3.18}$$

Since $\{t_{k_{m_n}}\} \subset \{t_{k_n}\}$ and $\lim_{k \to +\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{n \to +\infty} \|\epsilon_k \Upsilon_{k_{m_n}}\| = \lim_{n \to +\infty} \frac{\epsilon_k}{\|\mathcal{N}(t_{k_{m_n}})\|} \leq \frac{0}{\|\mathcal{N}(\hat{u})\|} = 0. \tag{3.19}$$

Next, consider that $n \to +\infty$ in expression (3.17), we obtain

$$\langle \mathcal{N}(y), y - \hat{u} \rangle \geq 0, \ \forall\, y \in \mathcal{C}. \tag{3.20}$$

Let $u \in \mathcal{C}$ be arbitrary element and for $0 < \varpi \leq 1$. Let consider that

$$\hat{u}_\varpi = \varpi u + (1 - \varpi)\hat{u}. \tag{3.21}$$

Then $\hat{u}_\varpi \in \mathcal{C}$ and from expression (3.20), we have

$$\varpi \big\langle \mathcal{N}(\hat{u}_\varpi), u - \hat{u} \big\rangle \geq 0. \tag{3.22}$$

Hence, we have

$$\big\langle \mathcal{N}(\hat{u}_\varpi), u - \hat{u} \big\rangle \geq 0. \tag{3.23}$$

Let $\varpi \to 0$. Then $\hat{u}_\varpi \to \hat{u}$ along a line segment. By the continuity of an operator, $\mathcal{N}(\hat{u}_\varpi)$ converges to $\mathcal{N}(\hat{u})$ as $\varpi \to 0$. It follows from (3.23) that

$$\big\langle \mathcal{N}(\hat{u}), u - \hat{u} \big\rangle \geq 0. \tag{3.24}$$

Thus, $\hat{u}$ is a solution of problem (VIP). □

Theorem 3.5. *Let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be an operator satisfies the conditions ($\mathcal{N}1$)–($\mathcal{N}4$). Then, sequence $\{u_k\}$ generated by Algorithm 2 strongly converges to $r^* \in VI(\mathcal{C}, \mathcal{N}) \cap Fix(\mathcal{T})$ where $r^* = P_{VI(\mathcal{C},\mathcal{N}) \cap Fix(\mathcal{T})}(0)$.*

Proof. **Claim 1: The sequence $\{u_k\}$ is bounded.** Indeed, we have $u_{k+1} = (1-\alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$. By value of the sequence $\{u_{k+1}\}$ we obtain

$$\begin{aligned}\big\|u_{k+1}-r^*\big\|^2 &= \big\|(1-\alpha_k)z_k+\alpha_k\mathcal{T}(z_k)-r^*\big\|^2\\ &= \big\|z_k-r^*\big\|^2+2\alpha_k\big\langle z_k-r^*,\mathcal{T}(z_k)-z_k\big\rangle+\alpha_k^2\big\|\mathcal{T}(z_k)-z_k\big\|^2\\ &\le \big\|z_k-r^*\big\|^2+\alpha_k(\rho-1)\big\|\mathcal{T}(z_k)-z_k\big\|^2+\alpha_k^2\big\|\mathcal{T}(z_k)-z_k\big\|^2\\ &= \big\|z_k-r^*\big\|^2-\alpha_k(1-\rho-\alpha_k)\big\|\mathcal{T}(z_k)-z_k\big\|^2. \end{aligned}\tag{3.25}$$

By the use of value of $\{t_k\}$ we obtain

$$\begin{aligned}\big\|t_k-r^*\big\| &= \big\|u_k+\theta_k(u_k-u_{k-1})-\beta_k u_k-\theta_k\beta_k(u_k-u_{k-1})-r^*\big\|\\ &= \big\|(1-\beta_k)(u_k-r^*)+(1-\beta_k)\theta_k(u_k-u_{k-1})-\beta_k r^*\big\| \quad (3.26)\\ &\le (1-\beta_k)\big\|u_k-r^*\big\|+(1-\beta_k)\theta_k\big\|u_k-u_{k-1}\big\|+\beta_k\big\|r^*\big\|\\ &\le (1-\beta_k)\|u_k-r^*\|+\beta_k K_1, \quad (3.27)\end{aligned}$$

for some K_1 we have $(1-\beta_k)\frac{\theta_k}{\beta_k}\big\|u_k-u_{k-1}\big\|+\big\|r^*\big\| \le K_1$. It is given that $\eta_k \to \eta$ such that that there exists $\vartheta \in (0, 1-\mu)$ such that

$$\lim_{k\to+\infty}\Big(1-\frac{\mu\eta_k}{\eta_{k+1}}\Big) = 1-\mu > \vartheta > 0.$$

Thus, there exists a finite natural number $N_1 \in \mathbb{N}$ such that

$$\Big(1-\frac{\mu\eta_k}{\eta_{k+1}}\Big) > \vartheta > 0,\ \forall\, k \ge N_1. \tag{3.28}$$

By Lemma 3.3, we may rewrite

$$\|z_k-r^*\|^2 \le \|t_k-r^*\|^2,\ \forall\, k \ge N_1. \tag{3.29}$$

From expressions (3.25), (3.27) and (3.29) infer that

$$\|u_{k+1}-r^*\| \le (1-\beta_k)\|u_k-r^*\|+\beta_k K_1-\alpha_k(1-\rho-\alpha_k)\big\|\mathcal{T}(z_k)-z_k\big\|^2. \tag{3.30}$$

Since $\{\alpha_k\} \subset (a, 1-\rho)$ we obtain

$$\begin{aligned}\|u_{k+1}-r^*\| \le (1-\beta_k)\|u_k-r^*\|+\beta_k K_1 &\le \max\big\{\|u_k-r^*\|, K_1\big\}\\ &\ \ \vdots\\ &\le \max\big\{\|u_{N_1}-r^*\|, K_1\big\}. \end{aligned}\tag{3.31}$$

Thus, we can conclude that the sequence $\{u_k\}$ is bounded.

Claim 2:

$$\begin{aligned}\Big(1-\frac{\mu\eta_k}{\eta_{k+1}}\Big)\|t_k-y_k\|^2+\Big(1-\frac{\mu\eta_k}{\eta_{k+1}}\Big)\|z_k-y_k\|^2&\\ +\alpha_k(1-\rho-\alpha_k)\big\|\mathcal{T}(z_k)-z_k\big\|^2&\\ \le \|u_k-r^*\|^2-\|u_{k+1}-r^*\|^2+\beta_k K_2&. \end{aligned}\tag{3.32}$$

for some $K_2 > 0$. Indeed, it follows from definition of $\{u_{k+1}\}$ that

$$\begin{aligned}\big\|u_{k+1}-r^*\big\|^2 &= \big\|(1-\alpha_k)z_k+\alpha_k\mathcal{T}(z_k)-r^*\big\|^2\\ &= \big\|z_k-r^*\big\|^2+2\alpha_k\big\langle z_k-r^*,\mathcal{T}(z_k)-z_k\big\rangle+\alpha_k^2\big\|\mathcal{T}(z_k)-z_k\big\|^2\end{aligned}$$

$$\leq \big\|z_k - r^*\big\|^2 + \alpha_k(\rho - 1)\big\|\mathcal{T}(z_k) - z_k\big\|^2 + \alpha_k^2\big\|\mathcal{T}(z_k) - z_k\big\|^2$$

$$= \big\|z_k - r^*\big\|^2 - \alpha_k(1 - \rho - \alpha_k)\big\|\mathcal{T}(z_k) - z_k\big\|^2. \tag{3.33}$$

By using Lemma 3.3 we have

$$\|z_k - r^*\|^2 \leq \|t_k - r^*\|^2 - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|t_k - y_k\|^2 - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|z_k - y_k\|^2. \tag{3.34}$$

Indeed, it follow from expression (3.27) that

$$\begin{aligned}\big\|t_k - r^*\big\|^2 &\leq (1 - \beta_k)^2\|u_k - r^*\|^2 + \beta_k^2 K_1^2 + 2K_1\beta_k(1 - \beta_k)\|u_k - r^*\| \\ &\leq \|u_k - r^*\|^2 + \beta_k\big[\beta_k K_1^2 + 2K_1(1 - \beta_k)\|u_k - r^*\|\big] \\ &\leq \|u_k - r^*\|^2 + \beta_k K_2, \end{aligned} \tag{3.35}$$

for some $K_2 > 0$. Combining expressions (3.33), (3.34) and (3.35) we obtain

$$\begin{aligned}\|u_{k+1} - r^*\|^2 \leq \|u_k - r^*\|^2 + \beta_k K_2 - \alpha_k(1 - \rho - \alpha_k)\big\|\mathcal{T}(z_k) - z_k\big\|^2 \\ - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|t_k - y_k\|^2 - \Big(1 - \frac{\mu\eta_k}{\eta_{k+1}}\Big)\|z_k - y_k\|^2. \end{aligned} \tag{3.36}$$

Claim 3: From definition of $\{t_k\}$ we can write

$$\begin{aligned}\big\|t_k - r^*\big\|^2 &= \big\|u_k + \theta_k(u_k - u_{k-1}) - \beta_k u_k - \theta_k\beta_k(u_k - u_{k-1}) - r^*\big\|^2 \\ &= \big\|(1 - \beta_k)(u_k - r^*) + (1 - \beta_k)\theta_k(u_k - u_{k-1}) - \beta_k r^*\big\|^2 \\ &\leq \big\|(1 - \beta_k)(u_k - r^*) + (1 - \beta_k)\theta_k(u_k - u_{k-1})\big\|^2 \\ &\quad + 2\beta_k\langle -r^*, t_k - r^*\rangle \\ &= (1 - \beta_k)^2\big\|u_k - r^*\big\|^2 + (1 - \beta_k)^2\theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\ &\quad + 2\theta_k(1 - \beta_k)^2\big\|u_k - r^*\big\|\big\|u_k - u_{k-1}\big\| \\ &\quad + 2\beta_k\langle -r^*, t_k - u_{k+1}\rangle + 2\beta_k\langle -r^*, u_{k+1} - r^*\rangle \\ &\leq (1 - \beta_k)\big\|u_k - r^*\big\|^2 + \theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\ &\quad + 2\theta_k(1 - \beta_k)\big\|u_k - r^*\big\|\big\|u_k - u_{k-1}\big\| \\ &\quad + 2\beta_k\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\beta_k\langle -r^*, u_{k+1} - r^*\rangle \\ &= (1 - \beta_k)\big\|u_k - r^*\big\|^2 \\ &\quad + \beta_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2(1 - \beta_k)\big\|u_k - r^*\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\langle r^*, r^* - u_{k+1}\rangle\Big]. \end{aligned} \tag{3.37}$$

Combining expressions (3.29) and (3.37) we obtain

$$\begin{aligned}\big\|u_{k+1} - r^*\big\|^2 &\leq (1 - \beta_k)\big\|u_k - r^*\big\|^2 \\ &\quad + \beta_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \end{aligned}$$

$$+ 2(1-\beta_k)\big\|u_k - r^*\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\|$$
$$+ 2\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\langle r^*, r^* - u_{k+1}\rangle\Big]. \tag{3.38}$$

Claim 4: The sequence $\big\|u_k - r^*\big\|^2$ converges to zero. Set $p_k := \|u_k - r^*\|^2$ and

$$r_k := \theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| + 2(1-\beta_k)\big\|u_k - r^*\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| + 2\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\langle r^*, r^* - u_{k+1}\rangle.$$

Then, **Claim 4** can be rewritten as follows: $p_{k+1} \le (1-\beta_k)p_k + \beta_k r_k$. Indeed, from Lemma 2.2, it suffices to show that $\limsup_{j\to+\infty} r_{k_j} \le 0$ for every subsequence $\{p_{k_j}\}$ of $\{p_k\}$ satisfying $\liminf_{j\to+\infty}(p_{k_j+1} - p_{k_j}) \ge 0$. This is equivalently to need to show that $\limsup_{j\to+\infty}\langle r^*, r^* - u_{k_j+1}\rangle \le 0$ and $\limsup_{j\to+\infty}\big\|t_{k_j} - u_{k_j+1}\big\| \le 0$, for every subsequence $\{\|u_{k_j} - r^*\|\}$ of $\{\|u_k - r^*\|\}$ satisfying $\liminf_{j\to+\infty}\big(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\big) \ge 0$. Suppose that $\{\|u_{k_j} - r^*\|\}$ is a subsequence of $\{\|u_k - r^*\|\}$ satisfying $\liminf_{j\to+\infty}\big(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\big) \ge 0$. Then

$$\begin{aligned}&\liminf_{j\to+\infty}\big(\|u_{k_j+1} - r^*\|^2 - \|u_{k_j} - r^*\|^2\big)\\ &= \liminf_{j\to+\infty}\big(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\big)\big(\|u_{k_j+1} - r^*\| + \|u_{k_j} - r^*\|\big) \ge 0.\end{aligned} \tag{3.39}$$

It follows from Claim 2 that

$$\begin{aligned}&\limsup_{j\to+\infty}\Big[\Big(1 - \frac{\mu\eta_{k_j}}{\eta_{k_j+1}}\Big)\|t_{k_j} - y_{k_j}\|^2 + \Big(1 - \frac{\mu\eta_{k_j}}{\eta_{k_j+1}}\Big)\|z_{k_j} - y_{k_j}\|^2\\ &\quad + \alpha_{k_j}(1-\rho-\alpha_{k_j})\big\|\mathcal{T}(z_{k_j}) - z_{k_j}\big\|^2\Big]\\ &\le \limsup_{j\to+\infty}\Big[\|u_{k_j} - r^*\|^2 - \|u_{k_j+1} - r^*\|^2\Big] + \limsup_{j\to+\infty}\beta_{k_j}K_2\\ &= -\liminf_{j\to+\infty}\Big[\|u_{k_j+1} - r^*\|^2 - \|u_{k_j} - r^*\|^2\Big] \le 0.\end{aligned} \tag{3.40}$$

The above relation implies that

$$\lim_{j\to+\infty}\|t_{k_j} - y_{k_j}\| = 0, \quad \lim_{j\to+\infty}\|z_{k_j} - y_{k_j}\| = 0, \quad \lim_{j\to+\infty}\big\|\mathcal{T}(z_{k_j}) - z_{k_j}\big\| = 0. \tag{3.41}$$

Thus, we obtain

$$\lim_{j\to+\infty}\|z_{k_j} - t_{k_j}\| = 0. \tag{3.42}$$

Next, we need to compute

$$\begin{aligned}\|t_{k_j} - u_{k_j}\| &= \|u_{k_j} + \theta_{k_j}(u_{k_j} - u_{k_j-1}) - \beta_{k_j}\big[u_{k_j} + \theta_{k_j}(u_{k_j} - u_{k_j-1})\big] - u_{k_j}\|\\ &\le \theta_{k_j}\|u_{k_j} - u_{k_j-1}\| + \beta_{k_j}\|u_{k_j}\| + \theta_{k_j}\beta_{k_j}\|u_{k_j} - u_{k_j-1}\|\\ &= \beta_{k_j}\frac{\theta_{k_j}}{\beta_{k_j}}\|u_{k_j} - u_{k_j-1}\| + \beta_{k_j}\|u_{k_j}\| + \beta_{k_j}^2\frac{\theta_{k_j}}{\beta_{k_j}}\|u_{k_j} - u_{k_j-1}\| \longrightarrow 0.\end{aligned} \tag{3.43}$$

This together with $\lim_{j\to+\infty}\|z_{k_j}-t_{k_j}\|=0$, provides that

$$\lim_{j\to+\infty}\|z_{k_j}-u_{k_j}\|=0. \tag{3.44}$$

From $u_{k_j+1}=(1-\alpha_{k_j})z_{k_j}+\alpha_{k_j}\mathcal{T}(z_{k_j})$, such that

$$\lim_{j\to+\infty}\|u_{k_j+1}-z_{k_j}\|=\alpha_{k_j}\|\mathcal{T}(z_{k_j})-z_{k_j}\|\leq(1-\rho)\|\mathcal{T}(z_{k_j})-z_{k_j}\|. \tag{3.45}$$

Thus, we obtain

$$\lim_{j\to+\infty}\|u_{k_j+1}-z_{k_j}\|=0. \tag{3.46}$$

The above expression implies that

$$\lim_{j\to+\infty}\|u_{k_j}-u_{k_j+1}\|\leq\lim_{j\to+\infty}\|u_{k_j}-z_{k_j}\|+\lim_{j\to+\infty}\|z_{k_j}-u_{k_j+1}\|=0, \tag{3.47}$$

and

$$\lim_{j\to+\infty}\|t_{k_j}-u_{k_j+1}\|\leq\lim_{j\to+\infty}\|t_{k_j}-z_{k_j}\|+\lim_{j\to+\infty}\|z_{k_j}-u_{k_j+1}\|=0. \tag{3.48}$$

Since the sequence $\{u_{k_j}\}$ is a bounded, without loss of generality we assume that $\{u_{k_j}\}$ converges weakly to $\hat{u}\in\mathcal{H}$. From given $r^*=P_{VI(\mathcal{C},\mathcal{N})\cap Fix(\mathcal{T})}(0)$, we have

$$\langle 0-r^*,y-r^*\rangle\leq 0,\ \forall y\in VI(\mathcal{C},\mathcal{N})\cap Fix(\mathcal{T}). \tag{3.49}$$

From (3.43), one gets $\{t_{k_j}\}$ converges weakly to $\hat{u}\in\mathcal{H}$. Combining (3.41), $\lim_{k\to}\eta_k=\eta$ and Lemma 3.4, one concludes that $\hat{u}\in VI(\mathcal{C},\mathcal{N})$. It follows from (3.44) that $\{z_{k_j}\}$ converges weakly to $\hat{u}\in\mathcal{H}$. By the demiclosedness of $(I-\mathcal{T})$, we get that $\hat{u}\in Fix(\mathcal{T})$. Hence, $\hat{u}\in VI(\mathcal{C},\mathcal{N})\cap Fix(\mathcal{T})$. Thus, we have

$$\lim_{j\to+\infty}\langle r^*,r^*-u_{k_j}\rangle=\langle r^*,r^*-\hat{u}\rangle\leq 0. \tag{3.50}$$

By using the fact $\lim_{j\to+\infty}\left\|u_{k_j+1}-u_{k_j}\right\|=0$. Thus, we have

(3.51)
$$\limsup_{j\to+\infty}\langle r^*,r^*-u_{k_j+1}\rangle\leq\limsup_{j\to+\infty}\langle r^*,r^*-u_{k_j}\rangle+\limsup_{j\to+\infty}\langle r^*,u_{k_j}-u_{k_j+1}\rangle\leq 0.$$

By using Claim 3 and in the light of Lemma 2.2, we observe that $u_k\to r^*$ as $k\to+\infty$. The proof of Theorem 3.5 is completed.

□

Lemma 3.6. *The step size sequence* $\{\eta_k\}$ *generated in* (3.54) *is monotonically decreasing with bounded by* $\min\left\{\frac{\mu}{L},\eta_1\right\}$.

Proof. It is given that $\mathcal{N}$ is Lipschitz-continuous with constant $L>0$, we have

$$\frac{\mu\|t_k-y_k\|}{\|\mathcal{N}(t_k)-\mathcal{N}(y_k)\|}\geq\frac{\mu\|t_k-y_k\|}{L\|t_k-y_k\|}\geq\frac{\mu}{L}. \tag{3.55}$$

Thus, the sequence $\{\eta_k\}$ is convergent to η. This completes the proof. □

Lemma 3.7. *Let* $\mathcal{N}:\mathcal{H}\to\mathcal{H}$ *satisfies the items* $(\mathcal{N}1)$–$(\mathcal{N}4)$. *Let* $\{u_k\}$ *be a sequence is generated by Algorithm 3. For each* $r^*\in VI(\mathcal{C},\mathcal{N})$, *we have*

$$\|z_k-r^*\|^2\leq\left\|t_k-r^*\right\|^2-\left(1-\eta^2L^2\right)\left\|t_k-y_k\right\|^2.$$

Algorithm 3

Step 0: Take $u_0, u_1 \in \mathcal{C}$, $\theta \in (0,1)$, $0 < \eta < \frac{1}{L}$ and $\{\beta_k\} \subset (0,1)$ meet the conditions: $\lim_{k\to+\infty} \beta_k = 0$ and $\sum_{k=1}^{+\infty} \beta_k = +\infty$.
Step 1: Compute $t_k = u_k + \theta_k(u_k - u_{k-1}) - \beta_k\big[u_k + \theta_k(u_k - u_{k-1})\big]$, while θ_k taken as follows:

$$0 \le \theta_k \le \hat{\theta_k} \quad \text{and} \quad \hat{\theta_k} = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \neq u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases} \tag{3.52}$$

Moreover, a positive sequence $\epsilon_k = \circ(\beta_k)$ satisfies $\lim_{k\to+\infty} \frac{\epsilon_k}{\beta_k} = 0$.
Step 2: Compute $y_k = P_{\mathcal{C}}(t_k - \eta\mathcal{N}(t_k))$. If $t_k = y_k$, then STOP. Otherwise
Step 3: Compute $z_k = y_k + \eta\big[\mathcal{N}(t_k) - \mathcal{N}(y_k)\big]$.
Step 4: Compute $u_{k+1} = (1 - \alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$.

Algorithm 4

Step 0: Take $u_0, u_1 \in \mathcal{C}$, $\theta \in (0,1)$, $\mu \in (0,1)$, $\eta_1 > 0$. Select a sequence $\{\beta_k\} \subset (0,1)$ meet the conditions: $\lim_{k\to+\infty} \beta_k = 0$ and $\sum_{k=1}^{+\infty} \beta_k = +\infty$.
Step 1: Compute $t_k = u_k + \theta_k(u_k - u_{k-1}) - \beta_k\big[u_k + \theta_k(u_k - u_{k-1})\big]$, while θ_k taken as follows:

$$0 \le \theta_k \le \hat{\theta_k} \quad \text{and} \quad \hat{\theta_k} = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \neq u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases} \tag{3.53}$$

Moreover, a positive sequence $\epsilon_k = \circ(\beta_k)$ satisfies $\lim_{k\to+\infty} \frac{\epsilon_k}{\beta_k} = 0$.
Step 2: Compute $y_k = P_{\mathcal{C}}(t_k - \eta_k\mathcal{N}(t_k))$.
Step 3: Compute $z_k = y_k + \eta_k\big[\mathcal{N}(t_k) - \mathcal{N}(y_k)\big]$.
Step 4: Compute $u_{k+1} = (1 - \alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$.
Step 5: Compute

$$\begin{cases} \min\left\{\eta_k, \ \frac{\mu\|t_k - y_k\|}{\|\mathcal{N}(t_k) - A(y_k)\|}\right\} & \text{if} \quad \mathcal{N}(t_k) \neq \mathcal{N}(y_k), \\ \eta_k, & \text{otherwise.} \end{cases} \tag{3.54}$$

Lemma 3.8. [35] *Let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ satisfies the items $(\mathcal{N}1)$–$(\mathcal{N}4)$. Let $\{u_k\}$ be a sequence is generated by Algorithm 4. Then, for each $r^* \in VI(\mathcal{C}, \mathcal{N})$, we have*

$$\|z_k - r^*\|^2 \le \big\|t_k - r^*\big\|^2 - \left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right)\big\|t_k - y_k\big\|^2.$$

Theorem 3.9. *Let $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ be an operator satisfies the items $(\mathcal{N}1)$–$(\mathcal{N}4)$. Then, sequence $\{u_k\}$ generated by Algorithm 4 strongly converges to $r^* \in VI(\mathcal{C}, \mathcal{N}) \cap Fix(\mathcal{T})$ where $r^* = P_{VI(\mathcal{C},\mathcal{N})\cap Fix(\mathcal{T})}(0)$.*

Proof. **Claim 1: The sequence $\{u_k\}$ is bounded.** Indeed, we have $u_{k+1} = (1 - \alpha_k)z_k + \alpha_k\mathcal{T}(z_k)$. By using the value of sequence $\{u_{k+1}\}$ we have

$$\big\|u_{k+1} - r^*\big\|^2 = \big\|(1 - \alpha_k)z_k + \alpha_k\mathcal{T}(z_k) - r^*\big\|^2$$

$$= \|z_k - r^*\|^2 + 2\alpha_k\langle z_k - r^*, \mathcal{T}(z_k) - z_k\rangle + \alpha_k^2\|\mathcal{T}(z_k) - z_k\|^2$$
$$\leq \|z_k - r^*\|^2 + \alpha_k(\rho - 1)\|\mathcal{T}(z_k) - z_k\|^2 + \alpha_k^2\|\mathcal{T}(z_k) - z_k\|^2$$
$$= \|z_k - r^*\|^2 - \alpha_k(1 - \rho - \alpha_k)\|\mathcal{T}(z_k) - z_k\|^2. \tag{3.56}$$

By value of $\{t_k\}$ we obtain

$$\|t_k - r^*\| = \|u_k + \theta_k(u_k - u_{k-1}) - \beta_k u_k - \theta_k\beta_k(u_k - u_{k-1}) - r^*\|$$
$$= \|(1 - \beta_k)(u_k - r^*) + (1 - \beta_k)\theta_k(u_k - u_{k-1}) - \beta_k r^*\| \tag{3.57}$$
$$\leq (1 - \beta_k)\|u_k - r^*\| + (1 - \beta_k)\theta_k\|u_k - u_{k-1}\| + \beta_k\|r^*\|$$
$$\leq (1 - \beta_k)\|u_k - r^*\| + \beta_k M_1, \tag{3.58}$$

for some M_1 we have $(1 - \beta_k)\frac{\theta_k}{\beta_k}\|u_k - u_{k-1}\| + \|r^*\| \leq M_1$. It is given that $\eta_k \to \eta$ such that $\epsilon \in (0, 1 - \mu^2)$ and

$$\lim_{k\to+\infty}\left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right) = 1 - \mu^2 > \epsilon > 0.$$

Thus, there is a finite number $k_0 \in \mathbb{N}$ such that

$$\left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right) > \epsilon > 0, \ \forall\, k \geq k_0. \tag{3.59}$$

By the use of Lemma 3.8, we may rewrite

$$\|z_k - r^*\|^2 \leq \|t_k - r^*\|^2, \ \forall\, k \geq k_0. \tag{3.60}$$

From expressions (3.56), (3.58) and (3.60) infer that

$$\|u_{k+1} - r^*\| \leq (1 - \beta_k)\|u_k - r^*\| + \beta_k M_1 - \alpha_k(1 - \rho - \alpha_k)\|\mathcal{T}(z_k) - z_k\|^2. \tag{3.61}$$

Since $\{\alpha_k\} \subset (a, 1 - \rho)$, we obtain

$$\|u_{k+1} - r^*\| \leq (1 - \beta_k)\|u_k - r^*\| + \beta_k M_1$$
$$\leq \max\{\|u_k - r^*\|, M_1\} \leq \max\{\|u_{k_0} - r^*\|, M_1\}. \tag{3.62}$$

Finally, we can conclude that the sequence $\{u_k\}$ is bounded.

Claim 2:

$$\left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right)\|t_k - y_k\|^2 + \alpha_k(1 - \rho - \alpha_k)\|\mathcal{T}(z_k) - z_k\|^2$$
$$\leq \|u_k - r^*\|^2 - \|u_{k+1} - r^*\|^2 + \beta_k M_2, \tag{3.63}$$

for some $M_2 > 0$. Indeed, it follows from definition of $\{u_{k+1}\}$ that

$$\|u_{k+1} - r^*\|^2 = \|(1 - \alpha_k)z_k + \alpha_k\mathcal{T}(z_k) - r^*\|^2$$
$$= \|z_k - r^*\|^2 + 2\alpha_k\langle z_k - r^*, \mathcal{T}(z_k) - z_k\rangle + \alpha_k^2\|\mathcal{T}(z_k) - z_k\|^2$$
$$\leq \|z_k - r^*\|^2 + \alpha_k(\rho - 1)\|\mathcal{T}(z_k) - z_k\|^2 + \alpha_k^2\|\mathcal{T}(z_k) - z_k\|^2$$
$$= \|z_k - r^*\|^2 - \alpha_k(1 - \rho - \alpha_k)\|\mathcal{T}(z_k) - z_k\|^2. \tag{3.64}$$

By using Lemma 3.8, we have

$$\|z_k - r^*\|^2 \le \big\|t_k - r^*\big\|^2 - \left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right)\big\|t_k - y_k\big\|^2. \tag{3.65}$$

Indeed, it follow from expression (3.58) that

$$\begin{aligned}\big\|t_k - r^*\big\|^2 &\le (1-\beta_k)^2\|u_k - r^*\|^2 + \beta_k^2 M_1^2 + 2M_1\beta_k(1-\beta_k)\|u_k - r^*\| \\ &\le \|u_k - r^*\|^2 + \beta_k\big[\beta_k M_1^2 + 2M_1(1-\beta_k)\|u_k - r^*\|\big] \\ &\le \|u_k - r^*\|^2 + \beta_k M_2, \end{aligned} \tag{3.66}$$

for some $M_2 > 0$. Combining expressions (3.64), (3.65) and (3.66) we obtain

$$\begin{aligned}\|u_{k+1} - r^*\|^2 \le \|u_k - r^*\|^2 + \beta_k M_2 - \alpha_k(1-\rho-\alpha_k)\big\|\mathcal{T}(z_k) - z_k\big\|^2 \\ - \left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right)\big\|t_k - y_k\big\|^2. \end{aligned} \tag{3.67}$$

Claim 3: From definition of $\{t_k\}$ we can write

$$\begin{aligned}\big\|t_k - r^*\big\|^2 &= \big\|u_k + \theta_k(u_k - u_{k-1}) - \beta_k u_k - \theta_k\beta_k(u_k - u_{k-1}) - r^*\big\|^2 \\ &= \big\|(1-\beta_k)(u_k - r^*) + (1-\beta_k)\theta_k(u_k - u_{k-1}) - \beta_k r^*\big\|^2 \\ &\le \big\|(1-\beta_k)(u_k - r^*) + (1-\beta_k)\theta_k(u_k - u_{k-1})\big\|^2 + 2\beta_k\langle -r^*, t_k - r^*\rangle \\ &= (1-\beta_k)^2\big\|u_k - r^*\big\|^2 + (1-\beta_k)^2\theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\ &\quad + 2\theta_k(1-\beta_k)^2\big\|u_k - r^*\big\|\big\|u_k - u_{k-1}\big\| \\ &\quad + 2\beta_k\langle -r^*, t_k - u_{k+1}\rangle + 2\beta_k\langle -r^*, u_{k+1} - r^*\rangle \\ &\le (1-\beta_k)\big\|u_k - r^*\big\|^2 + \theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\ &\quad + 2\theta_k(1-\beta_k)\big\|u_k - r^*\big\|\big\|u_k - u_{k-1}\big\| \\ &\quad + 2\beta_k\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\beta_k\langle -r^*, u_{k+1} - r^*\rangle \\ &= (1-\beta_k)\big\|u_k - r^*\big\|^2 \\ &\quad + \beta_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2(1-\beta_k)\big\|u_k - r^*\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\langle r^*, r^* - u_{k+1}\rangle\Big]. \end{aligned} \tag{3.68}$$

Combining expressions (3.60) and (3.68) we obtain

$$\begin{aligned}\big\|u_{k+1} - r^*\big\|^2 &\le (1-\beta_k)\big\|u_k - r^*\big\|^2 \\ &\quad + \beta_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2(1-\beta_k)\big\|u_k - r^*\big\|\frac{\theta_k}{\beta_k}\big\|u_k - u_{k-1}\big\| \\ &\qquad + 2\big\|r^*\big\|\big\|t_k - u_{k+1}\big\| + 2\langle r^*, r^* - u_{k+1}\rangle\Big]. \end{aligned} \tag{3.69}$$

Claim 4: The sequence $\|u_k - r^*\|^2$ converges to zero. Set $p_k := \|u_k - r^*\|^2$ and

$$r_k := \theta_k \|u_k - u_{k-1}\| \frac{\theta_k}{\beta_k} \|u_k - u_{k-1}\| + 2(1-\beta_k)\|u_k - r^*\| \frac{\theta_k}{\beta_k} \|u_k - u_{k-1}\| + 2\|r^*\| \|t_k - u_{k+1}\| + 2\langle r^*, r^* - u_{k+1}\rangle.$$

Then, **Claim 4** can be rewritten as follows: $p_{k+1} \leq (1-\beta_k)p_k + \beta_k r_k$. Indeed, from Lemma 2.2, it suffices to show that $\limsup_{j\to+\infty} r_{k_j} \leq 0$ for every subsequence $\{p_{k_j}\}$ of $\{p_k\}$ satisfying $\liminf_{j\to+\infty}(p_{k_j+1} - p_{k_j}) \geq 0$. This is equivalently to need to show that $\limsup_{j\to+\infty}\langle r^*, r^* - u_{k_j+1}\rangle \leq 0$ and $\limsup_{j\to+\infty} \|t_{k_j} - u_{k_j+1}\| \leq 0$, for every subsequence $\{\|u_{k_j} - r^*\|\}$ of $\{\|u_k - r^*\|\}$ satisfying

$$\liminf_{j\to+\infty} \left(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\right) \geq 0.$$

Suppose that $\{\|u_{k_j} - r^*\|\}$ is a subsequence of $\{\|u_k - r^*\|\}$ satisfying

$$\liminf_{j\to+\infty} \left(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\right) \geq 0.$$

Then

$$\begin{aligned} &\liminf_{j\to+\infty} \left(\|u_{k_j+1} - r^*\|^2 - \|u_{k_j} - r^*\|^2\right) \\ &= \liminf_{j\to+\infty} \left(\|u_{k_j+1} - r^*\| - \|u_{k_j} - r^*\|\right)\left(\|u_{k_j+1} - r^*\| + \|u_{k_j} - r^*\|\right) \geq 0. \end{aligned} \tag{3.70}$$

It follows from Claim 2 that

$$\begin{aligned} &\limsup_{j\to+\infty} \left[\left(1 - \frac{\mu^2 \eta_{k_j}^2}{\eta_{k_j+1}^2}\right)\|t_{k_j} - y_{k_j}\|^2 + \alpha_{k_j}(1-\rho-\alpha_{k_j})\|\mathcal{T}(z_{k_j}) - z_{k_j}\|^2\right] \\ &\leq \limsup_{j\to+\infty} \left[\|u_{k_j} - r^*\|^2 - \|u_{k_j+1} - r^*\|^2\right] + \limsup_{j\to+\infty} \beta_{k_j} K_2 \\ &= -\liminf_{j\to+\infty} \left[\|u_{k_j+1} - r^*\|^2 - \|u_{k_j} - r^*\|^2\right] \leq 0. \end{aligned} \tag{3.71}$$

The above relation implies that

$$\lim_{j\to+\infty} \|t_{k_j} - y_{k_j}\| = 0, \quad \lim_{j\to+\infty} \|\mathcal{T}(z_{k_j}) - z_{k_j}\| = 0. \tag{3.72}$$

It follows that

$$\|z_{k_j} - y_{k_j}\| = \|y_{k_j} + \eta_{k_j}[\mathcal{N}(t_{k_j}) - \mathcal{N}(y_{k_j})] - y_{k_j}\| \leq \eta_{k_j} L \|t_{k_j} - y_{k_j}\|. \tag{3.73}$$

The above expression implies that

$$\lim_{j\to+\infty} \|z_{k_j} - y_{k_j}\| = 0. \tag{3.74}$$

The proof is similar to the Claim 4 of Theorem 3.5. So we omit it here. □

4. Numerical Illustrations

In contrast to some previous work in the literature, this part describes the algorithmic repercussions of the presented techniques, as well as an analysis of how differences in control parameters affect the numerical efficacy of the proposed algorithms. All calculations are performed in MATLAB R2018b on an HP i5 Core (TM) i5-6200 laptop with 8.00 GB (7.78 GB usable) RAM.

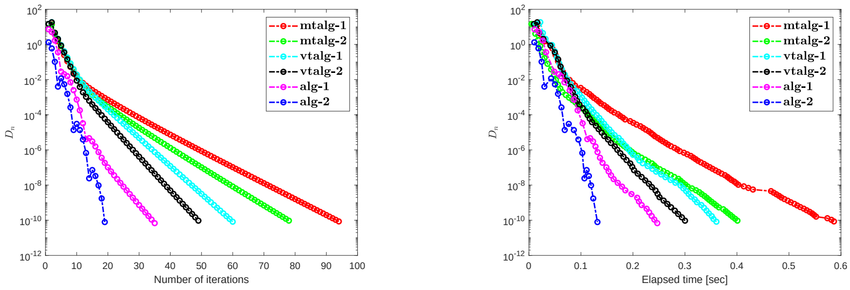
Example 4.1. Consider the HpHard problem and many researchers have considered this example for numerical experiments (see for details, [9, 11, 25]). Let's say a mapping $\mathcal{N} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\mathcal{N}(u) = Mu + q$ and $q = 0$ where $M = NN^T + B + D$. We used $N = rand(m)$ as a random matrix and $B = 0.5K - 0.5K^T$ as a skew-symmetric matrix with $K = rand(m)$ and $D = diag(rand(m, 1))$ during this experiment denotes a diagonal matrix. The practicable set $\mathcal{C}$ is interpreted as follows: $\mathcal{C} = \{u \in \mathbb{R}^m : -10 \leq u_i \leq 10\}$. It is obvious that $\mathcal{N}$ is monotone and that Lipschitz is continuous by $L = \|M\|$. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be provided by $\mathcal{T}u = \frac{1}{2}u$. The starting point for this experiment are $u_0 = u_1 = (2, 2, \ldots, 2)$ and dimension of the space is taken differently with stopping criterion $D_k = \|t_k - y_k\| \leq 10^{-10}$. Numerical observations for Example 4.1 are shown in Figures 1–2 and Table 1–2. Control criteria apply are as follows: (1) Algorithm 2 (shortly, **alg-1**): $\eta_1 = 0.55, \theta = 0.45, \mu = 0.44, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(2k+4)}, \alpha_k = \frac{k}{(2k+1)}$. (2) Algorithm 4 (shortly, **alg-2**): $\eta_1 = 0.55, \theta = 0.45, \mu = 0.44, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(2k+4)}, \alpha_k = \frac{k}{(2k+1)}$. (3) Algorithm 1 in [28] (shortly, **mtalg-1**): $\gamma_1 = 0.55, \delta = 0.45, \phi = 0.44, \theta_k = \frac{1}{(2k+4)}, \eta_k = \frac{1}{2}(1 - \theta_k), \epsilon_k = \frac{100}{(1+k)^2}$. (4) Algorithm 2 in [28] (shortly, **mtalg-2**): $\gamma_1 = 0.55, \delta = 0.45, \phi = 0.44, \theta_k = \frac{1}{(2k+4)}, \eta_k = \frac{1}{2}(1 - \theta_k), \epsilon_k = \frac{100}{(1+k)^2}$. (5) Algorithm 1 in [29] (shortly, **vtalg-1**): $\tau_1 = 0.55, \theta = 0.45, \mu = 0.44, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(2k+4)}, \alpha_k = \frac{k}{(2k+1)}, f(u) = \frac{u}{2}$. (6) Algorithm 2 in [29] (shortly, **vtalg-2**): $\tau_1 = 0.55, \theta = 0.45, \mu = 0.44, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(2k+4)}, \alpha_k = \frac{k}{(2k+1)}, f(u) = \frac{u}{2}$.

Table 1. Example 4.1 obtained numerical values.

			Total number of iterations			
m	**alg-1**	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
5	35	19	94	78	60	49
10	46	24	102	80	62	51
20	42	25	93	85	59	53
50	39	29	86	87	57	55
100	37	33	84	88	56	56
200	38	36	84	94	56	62

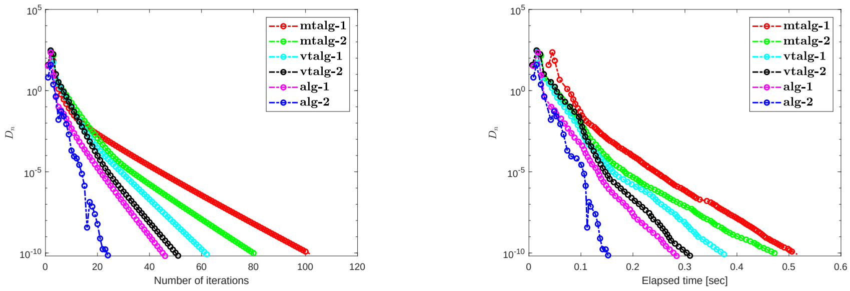
Table 2. Example 4.1 obtained numerical values.

			Required CPU time			
m	**alg-1**	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
5	0.246841	0.1317703	0.5865135	0.4009539	0.360533465	0.3001653
10	0.284076	0.1523123	0.5159276	0.4722816	0.375091336	0.3097725
20	0.2602246	0.1633652	0.4246998	0.4630932	0.393142367	0.3358743
50	0.293302	0.1854808	0.4320612	0.5335381	0.331728156	0.3663686
100	0.2566573	0.2301228	0.4752024	0.5067862	0.358997537	0.3936471
200	0.3544296	0.3695034	0.7371152	0.8441844	0.516623963	0.6142675



(A) (B)

FIGURE 1. Computational illustration of Algorithm 2 and Algorithm 4 with Algorithm 1 in [28], Algorithm 2 in [28] and Algorithm 1 in [29], Algorithm 2 in [29] when $m = 5$.



(A) (B)

FIGURE 2. Computational illustration of Algorithm 2 and Algorithm 4 with Algorithm 1 in [28], Algorithm 2 in [28] and Algorithm 1 in [29], Algorithm 2 in [29] when $m = 10$.

Example 4.2. Consider a nonlinear operator $\mathcal{N} : \mathcal{R}^2 \to \mathcal{R}^2$ is defined by

$$\mathcal{N}(u, y) = (u + y + \sin u; -u + y + \sin y)$$

and the feasible set $\mathcal{C}$ is a set expressed by $\mathcal{C} = [-1, 1] \times [-1, 1]$. It is easy to check that $\mathcal{N}$ is monotone and Lipschitz continuous with the constant $L = 3$. Let E be a 2×2 matrix defined by $E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We consider the mapping $\mathcal{T} : \mathcal{R}^2 \to \mathcal{R}^2$ by $\mathcal{T}z = \|E\|^{-1}Ez$, where $z = (u, y)^T$. It is obvious to see that $\mathcal{T}$ is 0-demicontractive and thus $\rho = 0$. The solution of the problem is $r^* = (0, 0)^T$. The starting point for this experiment are taken differently with stopping criterion $D_k = \|t_k - y_k\| \leq 10^{-10}$. Numerical observations for Example 4.2 are shown in Table 3–4. Control criteria apply are as follows: (1) Algorithm 2 (shortly, **alg-1**): $\eta_1 = 0.45, \theta = 0.35, \mu = 0.33, \epsilon_k = \frac{10}{(1+k)^2}, \beta_k = \frac{1}{(3k+6)}, \alpha_k = \frac{k}{(3k+1)}$. (2) Algorithm 4 (shortly, **alg-2**): $\eta_1 = 0.45, \theta = 0.35, \mu = 0.33, \epsilon_k = \frac{10}{(1+k)^2}, \beta_k = \frac{1}{(3k+4)}, \alpha_k = \frac{k}{(3k+1)}$. (3) Algorithm 1 in [28] (shortly, **mtalg-1**): $\gamma_1 = 0.45, \delta = 0.35, \phi = 0.33, \theta_k =$

$\frac{1}{(3k+6)}, \eta_k = \frac{1}{2.5}(1-\theta_k), \epsilon_k = \frac{10}{(1+k)^2}$. (4) Algorithm 2 in [28] (shortly, **mtalg-2**): $\gamma_1 = 0.45, \delta = 0.35, \phi = 0.33, \theta_k = \frac{1}{(3k+6)}, \eta_k = \frac{1}{2.5}(1-\theta_k), \epsilon_k = \frac{10}{(1+k)^2}$. (5) Algorithm 1 in [29] (shortly, **vtalg-1**): $\tau_1 = 0.45, \theta = 0.35, \mu = 0.33, \epsilon_k = \frac{10}{(1+k)^2}, \beta_k = \frac{1}{(3k+6)}, \alpha_k = \frac{k}{(3k+1)}, f(u) = \frac{u}{2}$. (6) Algorithm 2 in [29] (shortly, **vtalg-2**): $\tau_1 = 0.45, \theta = 0.35, \mu = 0.33, \epsilon_k = \frac{10}{(1+k)^2}, \beta_k = \frac{1}{(3k+6)}, \alpha_k = \frac{k}{(3k+1)}, f(u) = \frac{u}{2}$.

TABLE 3. Example 4.2 obtained numerical values.

	Total number of iterations					
$u_0 = u_1$	**alg-1**	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
$(1,1)^T$	49	35	68	75	82	85
$(2,2)^T$	48	36	61	65	77	78
$(1,-1)^T$	44	33	72	83	86	92
$(-2,3)^T$	51	37	65	70	81	79

TABLE 4. Example 4.2 obtained numerical values.

	Required CPU time					
$u_0 = u_1$	**alg-1**	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
$(1,1)^T$	0.2284193	0.1631707	0.2969821	0.3224385	0.3469049	0.3625844
$(2,2)^T$	0.2297859	0.1757931	0.3720656	0.3078242	0.3847476	0.4105755
$(1,-1)^T$	0.1986126	0.1512495	0.3220028	0.3729462	0.3787876	0.4068135
$(-2,3)^T$	0.2380988	0.1703252	0.2690971	0.3069672	0.3448697	0.3428332

Example 4.3. Suppose that $\mathcal{H} = l_2$ is a real Hilbert space including the elements as a sequences of real numbers meet the following criteria

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2 + \cdots < +\infty. \tag{4.1}$$

Next, consider that an operator $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is defined by $G(u) = (5-\|u\|)u, \ \forall\, u \in \mathcal{H}$ while $\mathcal{C} = \{u \in \mathcal{H} : \|u\| \leq 3\}$. We can easily seen that $\mathcal{N}$ is weakly sequentially continuous on $\mathcal{H}$ and the solution set is nonempty with $VI(\mathcal{C}, \mathcal{N}) = \{0\}$. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be provided by $\mathcal{T}u = \frac{1}{2}u$. For any $u, y \in \mathcal{H}$, we have

$$\begin{aligned}\big\|\mathcal{N}(u) - \mathcal{N}(y)\big\| &= \big\|(5 - \|u\|)u - (5 - \|y\|)y\big\| \\ &= \big\|5(u-y) - \|u\|(u-y) - (\|u\| - \|y\|)y\big\| \\ &\leq 5\|u-y\| + \|u\|\|u-y\| + \big|\|u\| - \|y\|\big|\|y\| \\ &\leq 5\|u-y\| + 3\|u-y\| + 3\|u-y\| \\ &\leq 11\|u-y\|. \end{aligned} \tag{4.2}$$

Hence $\mathcal{N}$ is L-Lipschitz continuous with $L = 11$. For any $u, y \in \mathcal{H}$ and let $\langle \mathcal{N}(u), y-u\rangle \geq 0$, such that $(5 - \|u\|)\langle u, y-u\rangle \geq 0$. Since $\|u\| \leq 3$ and it implies that $\langle u, y-u\rangle \geq 0$. Consider that

$$\begin{aligned}\langle \mathcal{N}(y), y-u\rangle &= (5 - \|y\|)\langle y, y-u\rangle \\ &\geq (5 - \|y\|)\langle y, y-u\rangle - (5 - \|y\|)\langle u, y-u\rangle\end{aligned}$$

$$\geq 2\|u-y\|^2 \geq 0. \tag{4.3}$$

Hence a mapping $\mathcal{N}$ is pseudomonotone on $\mathcal{C}$. Let $u = (\frac{5}{2}, 0, 0, \ldots, 0, \ldots)$ and $y = (3, 0, 0, \ldots, 0, \ldots)$ such that

$$\big\langle \mathcal{N}(u) - \mathcal{N}(y), u - y \big\rangle = \Big(\frac{5}{2} - 3\Big)^3 < 0.$$

Take a look at the following projection formula:

$$P_{\mathcal{C}}(u) = \begin{cases} u & \text{if} \quad \|u\| \leq 3, \\ \frac{3u}{\|u\|}, & \text{otherwise.} \end{cases}$$

The starting points for this experiment are taken differently with stopping criterion $D_k = \|t_k - y_k\| \leq 10^{-3}$. Numerical observations for Example 4.3 are shown in Table 5. Control criteria apply are as follows: (1) Algorithm 2 (shortly, **alg-1**): $\eta_1 = 0.76, \theta = 0.54, \mu = 0.33, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(3k+4)}, \alpha_k = \frac{k}{(2k+1)}$. (2) Algorithm 4 (shortly, **alg-2**): $\eta_1 = 0.76, \theta = 0.54, \mu = 0.33, \epsilon_k = \frac{100}{(1+k)^2}, \beta_k = \frac{1}{(3k+4)}, \alpha_k = \frac{k}{(2k+1)}$.

Table 5. Numerical values for Example 4.3.

	Number of iterations		Execution time in seconds	
u_1	**alg-1**	**alg-2**	**alg-1**	**alg-2**
$(2, 2, \ldots, 2_{10000}, 0, 0, \ldots)$	45	33	2.7557399	1.9794786
$(1, 2, \ldots, 10000, 0, 0, \ldots)$	47	34	3.0437379	2.0083004
$(10, 10, \ldots, 10_{10000}, 0, 0, \ldots)$	53	38	3.5634891	2.8256418

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