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EXISTENCE AND APPROXIMATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ITERATIVE DIFFERENTIAL EQUATIONS*

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Dedicated to the memory of Professor Emeritus Dr. Mihail Megan (1947-2025)

Abstract

We study the existence and approximation of solutions to the second-order iterative boundary-value problem

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \quad 0 \leq t \leq 1,$$

where $x^{[2]}(t); = x(x(t))$, with solutions satisfying one of the two sets of boundary conditions: (i) $x(0) = 0$, $x(1) = 1$; (ii) $x(0) = 1$, $x(1) = 0$.

Keywords: ordinary differential equation, boundary value problem, fixed point, contraction mapping, nonexpansive mapping.

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1 Introduction

Iterative differential equations represent a special class of differential equations with deviating arguments depending on both the state variable x and the time t.

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Iterative differential equations are important in theory and applications, see for example Yang [41] and the papers quoted there, where applications to insect population dynamics models in ecology, models of hematopoiesis in hematology, epidemiological models in epidemiology, two-body equations of motion in classical electrodynamics, the motion of charged particles with retarded interaction are indicated, and son on.

One of the first papers devoted to first order iterative differential equations appears to be due to Pethukov [28] who studied the existence of solutions for the equation

$$x'(t) = a\, x(x(t)), \tag{1}$$

with $a > 0$, by constructing ε-approximate solutions and then letting $\varepsilon \to 0$.

Dunkel [11] studied the more general equation

$$x'(t) = f\left(x(h(x(t)))\right), \tag{2}$$

where f is continuous and increasing on $[0, +\infty)$ and $f(0) = 0$, h is continuous, increasing and such that $0 < t_1 < h(t_1)$, for some t_1, and $h(x) \leq x$.

Dunkel [11] obtained existence and uniqueness theorems for solutions of such equations, as well as bounds on their growth and criteria for their nonexistence, using successive approximations and differential inequalities. We also note that Dunkel [11] and Agarwal [1] used the more suggestive term "nested" to designate iterative differential equations.

Apparently unaware of the previous contributions by Pethukov [28], Dunkel [11] and Agarwal [1], Eder [13] considered the functional differential equation (1) for the particular case $a = 1$

$$x'(t) = x(x(t)),\ t \in A \subset \mathbb{R},$$

and has shown that every solution either vanishes identically or is strictly monotonic.

Subsequently, Fečkan [17] studied a first order iterative differential equation of the more general form similar to (2) but not obtainable from (2)

$$x'(t) = f(x(x(t))),$$

with $f \in C^1(\mathbb{R})$.

For some other developments on this topic, see also Buică [9], Berinde [7], Eder [13], Egri [14], [15], [16], Li and Zhang [26], Li and Cheng [27], Rus [30], Yang [41] and references therein.

The present author Berinde [7] studied the existence of solutions of iterative differential equations of the general form

$$x'(t) = f(t, x(x(t))),$$

and also illustrated how one can approximate the non-unique solutions of such kind of iterative differential equations by means of iterative fixed point techniques associated to nonexpansive type mappings.

Second-order iterative differential equations were also studied by some authors. For example, Khemis et al. [23] established the existence of periodic solutions to a class of second-order iterative differential equations. Their method is based on the equivalence of the original problem with a certain integral equation and then by applying Schauder's fixed point theorem and the Green's functions method.

On the other hand, Kaufmann [21], studied the existence and uniqueness of solutions to the second-order iterative boundary-value problem

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \; a \leq t \leq b, \tag{3}$$

with the solutions satisfying one of the two sets of boundary conditions:

$$\begin{cases} x(a) = a \\ x(b) = b, \end{cases} \tag{4}$$

and

$$\begin{cases} x(a) = b \\ x(b) = a. \end{cases} \tag{5}$$

Starting from this background, our aim in this paper is to complement the results in Kaufmann [21] by providing results on the existence and *approximation* of solutions of the problems (3)+(4) and (3)+(5) under significantly weaker conditions than the ones in Kaufmann [21].

To this end, we use the powerful and more reliable technique of contractive operators, presented in the next Section. More advanced convergence theorems from the theory of iterative approximation of fixed points of non-expansive mappings, illustrated in Berinde [7], see also Berinde [6] and Chidume [10] for more details, are also used.

2 Basics of fixed point theory of contractive mappings

In order to prepare the main theoretical tools we should use in the next section, we present some basic facts about the contraction mapping principle. Although we should need its Banach space version for our applications, we present it in the more general setting of a metric space.

Let (X, d) be a complete metric space equipped with the distance function $d(x, y)$. A mapping $T : X \to X$ for which there exists a constant $k \in [0, 1)$ such that $d(T(x), T(y)) \leq k \cdot d(x, y)$, for all $x, y \in X$, is called a *k-contraction* or simply a *strict contraction*.

An element $\overline{x} \in X$ is called a *fixed point* of T, provided that $\overline{x} = T(\overline{x})$. In many instances, in order to solve a certain nonlinear equation / problem, it is advantageous to transform the nonlinear equation / problem into an equivalent fixed point problem $x = T(x)$ and therefore the task of solving the original nonlinear equation / problem is converted into that of finding a fixed point of the mapping T.

One fundamental tool in doing so is the famous Banach contraction mapping principle, which in its complete form can be stated as follows.

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ a δ-contraction. Then*

1) *$Fix\,(T) = \{\overline{x}\}$;*

2) *For any $x_0 \in X$, Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to $\overline{x}$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, \overline{x}) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{6}$$

Remark 1. *The estimate (6), which incorporates both* a priori *and* a posteriori *error estimates, is very important in applications, as it offers stopping criteria for the Picard iteration, see for example Berinde [6] for more details.*

We note that Theorem 1 could be stated with additional conclusions, related to data dependence on the operator perturbations, Ulam-Hyers stability, De Blasi well-posedness and so on, see for example Rus [31].

An important case of contractive mappings is obtained within the limit case $k = 1$ of the definition of a k-contraction. A mapping $T : X \to X$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y),\ \forall\, x, y \in X. \tag{7}$$

Despite the fact that the contractive condition (7) is obviously related to that of strict contractions, the problem of the existence and approximation of fixed points for nonexpansive mappings is essentially different from the case of contractions.

This means that a nonexpansive mapping defined on a complete metric space (even on a Banach space) need not have a fixed point. Even if a nonexpansive mapping T has a (unique) fixed point, then Picard iteration does not converge in general to that fixed point.

In order to get positive answers to the problem of existence and approximation of fixed points of nonexpansive mappings, we need a more appropriate setting that the one of a metric space, i.e., that of a Banach space (complete linear normed space).

As we have mentioned before, although the non-expansive mappings are natural generalizations of α-contractions, they do not inherit the fixed point properties of contractive mappings. In the following example, T is a nonexpansive mapping which is fixed point free.

Example 1. ([18], Example 3.3, pp. 30) *In the space $c_0(\mathbb{N})$ the isometry T defined by*

$$T(x_1, x_2, \dots) = (1, x_1, x_2, \dots)$$

maps the unit ball into its boundary but T has not fixed points.

As is shown by the next example, even in the cases where the nonexpansive mapping T has a unique fixed point, the Picard iteration associated to T (i.e., the Picard iteration $\{x_n\}$, for a given $x_0 \in K$), may fail to converge to the fixed point.

Example 2. *Let $[0, 1]$ be the unit interval with the usual norm. The function $T : [0, 1] \to [0, 1]$ given by $Tx = 1 - x$, for all $x \in [0, 1]$ has a unique fixed point, $\overline{x} = \frac{1}{2}$ but, except for the trivial case $x_0 = \frac{1}{2}$, the Picard iteration starting from x_0 yields an oscillatory sequence.*

The previous examples indicate that, in order to have ensured the existence and approximation of fixed points for nonexpansive mappings, we need some richer geometrical properties of the ambient space E.

One such setting is essential in the following fixed point theorem for nonexpansive mappings, due to Browder, Göhde and Kirk, see e.g. [6], is stated as follows.

Theorem 2. *If K is a nonempty closed convex and bounded subset of a uniformly convex Banach space E then any non-expansive mapping $T : K \to K$ has a fixed point.*

Remark 2. *As one can see, Theorem 2 does not provide information on the approximation of fixed points of a nonexpansive mapping T. Also, by Example 2, we see that Picard iteration does not resolve the problem of approximating the fixed points of such a mapping. Therefore, we need more reliable fixed point iteration procedures, like Krasnoselskij iteration, Mann or Ishikawa iterations, see Berinde [6] and Chidume [10].*

Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a self-mapping. Given $x_0 \in K$ given and a real number $\lambda \in [0, 1]$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n = 0, 1, 2, \cdots \tag{8}$$

is usually called *Krasnoselskij iteration*, or *Krasnoselskij-Mann iteration*. Clearly, (8) reduces to Picard iteration for $\lambda = 1$.

For $x_0 \in K$ given, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n,\ n = 0, 1, 2, \cdots \tag{9}$$

where $\{\lambda_n\} \subset [0, 1]$ is a sequence of real numbers satisfying some appropriate conditions, is called *Mann iteration*.

It was shown by Krasnoselskij [24] in the case $\lambda = 1/2$, and latter by Schaefer [32] for $\lambda \in (0, 1)$ arbitrary, that if E is a uniformly convex Banach space and K is a convex and compact subset of E containing fixed points of T, then the Krasnoselskij iteration converges to a fixed point of the nonexpansive mapping T.

Moreover, Edelstein [12] proved that strict convexity of E suffices for the same conclusion. The question of whether or not strict convexity can be removed has been answered in the affirmative by Ishikawa [20] by the following result.

Theorem 3. *Let K be a subset of a Banach space E and let $T : K \to K$ be a non-expansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process $\{x_n\}$ given by (9) under the following assumptions*
(a) $x_n \in K$ for all positive integers n;
(b) $0 \le \lambda_n \le b < 1$ for all positive integers n;
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.
If $\{x_n\}$ is bounded, then $x_n - T x_n \to 0$ as $n \to \infty$.

The following corollaries of Theorem 3 will be particularly important for the application part of our paper.

Corollary 1. *Let K be a convex and compact subset of a Banach space E and let $T : K \to K$ be a non-expansive mapping. If the Mann iteration process $\{x_n\}$ given by (9) satisfies assumptions (a)-(c) in Theorem 3, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. See Theorem 6.17 in Chidume [10].

□

Corollary 2. *Let E be a real normed space, K a closed bounded convex subset of E and let $T : K \to K$ be a non-expansive mapping. If $I - T$ maps closed bounded subsets of E into closed subsets of E and $\{x_n\}$ is the Mann iteration defined by (9), with $\{\lambda_n\}$ satisfying assumptions (a)-(c) in Theorem 3, then $\{x_n\}$ converges strongly to a fixed point of T in K.*

Proof. See Corollary 6.19 in Chidume [10]. □

3 Second order boundary value problems

Our aim in this section is to obtain existence and approximation results for the two point boundary value problem associated to the second order iterative differential equation

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \; 0 \le t \le 1, \tag{10}$$

with the solutions satisfying one of the two sets of boundary conditions:

$$\begin{cases} x(0) = 0 \\ x(1) = 1, \end{cases} \tag{11}$$

and

$$\begin{cases} x(0) = 1 \\ x(1) = 0. \end{cases} \tag{12}$$

We shall need the following two auxiliary lemmas.

Lemma 1. *Problem (10)+(11) is equivalent to the integral equation*

$$x(t) = t - \int_0^1 G(t,s) f\left(s, x(s), x^{[2]}(s)\right) ds, \; 0 \le t \le 1, \tag{13}$$

where $G : [0,1] \times [0,1] \to \mathbb{R}$ is the Green's function

$$G(t,s) = \begin{cases} (1-t) \cdot s, & \text{if } 0 \le s \le t \le 1, \\ (1-s) \cdot t, & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{14}$$

Proof. Let $x \in C^2[0,1]$ be a solution of (10)+(11). By integrating twice equation (10),

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \; 0 \le t \le 1,$$

we get

$$x(t) = x'(0)t + \int_0^t (t-s)f\left(s, x(s), x^{[2]}(s)\right) ds, \; 0 \le t \le 1, \tag{15}$$

Taking $t = 1$ in (15) and solving the resulted identity for $x'(0)$ we obtain

$$\begin{aligned} x'(0) = 1 - &\int_0^1 (1-s)f\left(s, x(s), x^{[2]}(s)\right) ds \\ &+ \int_0^t (t-s)f\left(s, x(s), x^{[2]}(s)\right) ds. \end{aligned} \tag{16}$$

Now, by combining (15) and (16) we get

$$\begin{aligned} x(t) = t - t \cdot &\int_0^1 (1-s)f\left(s, x(s), x^{[2]}(s)\right) ds \\ &+ \int_0^t (t-s)f\left(s, x(s), x^{[2]}(s)\right) ds. \end{aligned} \tag{17}$$

By splitting the first integral in (17) on $[0, t]$ and $[t, 1]$ and then by grouping the two integrals on $[0, t]$, we obtain

$$\begin{aligned} x(t) = t + &\int_0^t (t-1)sf\left(s, x(s), x^{[2]}(s)\right) ds \\ &+ \int_t^1 (1-s)tf\left(s, x(s), x^{[2]}(s)\right) ds, \end{aligned} \tag{18}$$

which is exactly the integral equation (13).

This proves that any solution of the problem (10)+(11) is a solution of the integral equation (13) and vice versa.

□

Lemma 2. *The Green's function $G : [0, 1] \times [0, 1] \to \mathbb{R}$, defined by (14), has the following properties:*

(i) $0 \le 4G(t, s) \le 1, \forall t, s \in [0, 1]$;

(ii) $8 \int_0^1 G(t, s)ds \le 1$.

Proof. See for example [4]. □

Now we can state and prove the first main result of this paper.

Theorem 4. *Assume that the following conditions are satisfied:*

(1) $f \in C\left([0,1]^3, \mathbb{R}\right)$;

(2) There exist the nonnegative constants L_1, L_2 *such that*

$$|f(t,x,y) - f(t,u,v)| \leq L_1|x-u| + L_2|y-v|, \tag{19}$$

for any $(t,x,y),(t,u,v) \in [0,1]^3$.

(3) $L_1 + L_2 < 8$.

Then, the problem (10)+(11) *has a unique solution* $\overline{x}$ *in* $C[0,1]$ *and, for any* $x_0 \in C[0,1]$ *satisfying* (11)*, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}(t) = t - \int_0^1 G(t,s) f\left(s, x_n(s), x_n^{[2]}(s)\right) ds,\ 0 \leq t \leq 1,\ n \geq 0, \tag{20}$$

converges uniformly to $\overline{x}$ *as* $n \to \infty$.

Proof. Let us take X to be the space of continuous functions on $[0,1]$ with norm

$$\|x\| = \max_{t \in [0,1]} |x(t)|.$$

The space X equipped with this norm is a Banach space, convergence in norm being simply uniform convergence.

The set $K = \{x \in C[0,1] : x([0,1]) \subset [0,1]\}$ is a nonempty closed and convex subset of X. Define the operator T by

$$(Tx)(t) = t - \int_0^1 G(t,s) f\left(s, x(s), x^{[2]}(s)\right) ds,\ 0 \leq t \leq 1.$$

Obviously, $(Tx)(t)$ is continuous if $x(t)$ is, which means that T maps X into X. In particular, T maps K into itself. By Lemma 1, problem (10)+(11) is equivalent to the fixed point problem $x = Tx$. For $x, y \in K$, we have

$$|(Tx)(t) - (Ty)(t)|$$

$$= \left| \int_0^1 G(t,s) \left[f\left(s, x(s), x^{[2]}(s)\right) - f\left(s, y(s), y^{[2]}(s)\right) \right] ds \right|$$

$$\leq \int_0^1 G(t,s) \left| f\left(s, x(s), x^{[2]}(s)\right) - f\left(s, y(s), y^{[2]}(s)\right) \right| ds.$$

(we used property (i) in Lemma 2)

By the Lipschitzian condition (2) we have

$$\left| f\left(s, x(s), x^{[2]}(s)\right) - f\left(s, y(s), y^{[2]}(s)\right) \right|$$

$$\leq L_1 \cdot |x(s) - y(s)| + L_2 \cdot |x^{[2]}(s) - y^{[2]}(s)|$$
$$\leq L_1 \cdot \|x - y\| + L_2 \cdot \|x - y\| = (L_1 + L_2)\|x - y\|, \forall s \in [0, 1].$$

Therefore

$$|(Tx)(t) - (Ty)(t)| \leq (L_1 + L_2)\|x - y\| \cdot \int_0^1 G(t, s)ds, \forall t \in [0, 1], \quad (21)$$

and using property (ii) in Lemma 2 and taking max with respect to $t \in [0, 1]$ in (21) we obtain

$$\|Tx - Ty\| \leq \frac{L_1 + L_2}{8} \cdot \|x - y\|, \forall x, y \in K,$$

which, by assumption (3), proves that $T : K \to K$ is a contraction.

We apply the contraction mapping principle to get the conclusion. □

Remark 3. *We note that Theorem 4 significantly improves Theorem 3.4 in Kaufmann [21]. Indeed, the corresponding condition in Theorem 3.4, i.e.,*

$$\frac{1}{6}(M + N)(b - a)^2 < 1,$$

would be in our case and with our notations ($a = 0$, $b = 1$, $M := L_1$, $N := L_2$): $L_1 + L_2 < 6$, which is more restrictive than our condition $L_1 + L_2 < 8$.

In case we are working on a interval $[a, b]$ rather than on $[0, 1]$, then our condition will read as follows

$$\frac{1}{8}(L_1 + L_2)(b - a)^2 < 1, \quad (22)$$

From the point of view of applications, this means that Theorem 4 can be applied in the cases where Theorem 3.4 in Kaufmann [21] does not apply, as shown by Example 3.

The corresponding result for the problem (10)+(12) can be stated and proven similarly.

In the special case $L_1 + L_2 = 8$, the conclusion of Theorem 4 does not hold. In such a case, we should apply appropriately Corollaries 1 or 2.

In the following we state an extension of Theorem 4.

Theorem 5. *Assume that conditions (1) and (2) in Theorem 4 and (3′) $L_1 + L_2 \leq 8$ are satisfied.*

For any $x_0 \in K = \{x \in C[0,1] : x([0,1]) \subset [0,1]\}$, *consider the sequence* $\{x_n\}$ *defined for* $t \in [0,1]$ *and* $n \geq 0$ *by*

$$x_{n+1}(t) = (1-\lambda_n)x_n(t) + \lambda_n \left(t - \int_0^1 G(t,s) f\left(s, x_n(s), x_n^{[2]}(s)\right) ds \right), \quad (23)$$

where $\{\lambda_n\}$ *is a sequence in* $[0,1]$ *satisfying:*

(b) $0 \leq \lambda_n \leq b < 1$ *for all positive integers* n; *(c)* $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then, the problem (10)+(11) *has solutions in* K *and the sequence* $\{x_n\}$ *converges uniformly to a solution in* K *of* (10)+(11) *as* $n \to \infty$.

Proof. If in condition $(3')$ strict inequality holds, then we are in the case of Theorem 4.

If $L_1 + L_2 = 8$, then, with the same notations as in the proof of Theorem 4, X with norm

$$\|x\| = \max_{t \in [0,1]} |x(t)|,$$

is a Banach space and the set $K = \{x \in C[0,1] : x([0,1]) \subset [0,1]\}$ is a nonempty bounded closed and convex subset of X. Define the operator T by

$$(Tx)(t) = t - \int_0^1 G(t,s) f\left(s, x(s), x^{[2]}(s)\right) ds, \ 0 \leq t \leq 1.$$

Obviously, $(Tx)(t)$ is continuous if $x(t)$ is, which means that T maps X into X and in particular, T maps K into itself and is nonexpansive. To reach the conclusion, we apply Corollary 2. □

Theorem 6. *Assume that conditions (1) and (2) and (3') in Theorem 5 are satisfied. For any* x_0 *in the set*

$$K = \{x \in C[0,1] : x([0,1]) \subset [0,1], |x(t_1) - x(t_2)| \leq |t_1 - t_2|, \forall t_1, t_1 \in [0,1]\},$$

consider the sequence $\{x_n\}$ *defined by* (23).

Then, the problem (10)+(11) *has solutions in* K *and the sequence* $\{x_n\}$ *converges uniformly to a solution in* K *of* (10)+(11) *as* $n \to \infty$.

Proof. The set K is compact by Arzelá-Ascoli theorem. We prove that $I - T$ maps closed bounded subsets of $X = C[0,1]$ into closed subsets of X. The proof is essentially based on the fact that typical integral operators act as compact operators on standard Banach spaces (such as $C[0,1]$ in our case).

Let $A \subset X$ be a closed, bounded subset of X. We want to show that the image set $(I-T)(A)$ is closed in X. To this end, take an arbitrary sequence

$(y_n) \in (I - T)(A)$ that converges to some element $y \in X$. We have to prove that $y \in (I - T)(A)$.

Since $(y_n) \in (I - T)(A)$, there exists a corresponding sequence (x_n) in the domain subset A such that: $(I - T)x_n = y_n \implies x_n - Tx_n = y_n$. By our assumption, $y_n \to y$ as $n \to \infty$.

Because A is a bounded subset of X and T is a compact integral operator, the image set $T(A)$ is relatively compact, that is, its closure is compact. By the definition of relative compactness, the sequence (Tx_n) must contain a sub-sequence (Tx_{n_k}) that converges to some limit $v \in X$, that is,

$$\lim_{k\to\infty} Tx_{n_k} = v.$$

Now, we restrict ourselves to our chosen convergent subsequence indices (n_k) and hence work with the equation $x_{n_k} = y_{n_k} + Tx_{n_k}$.

Take the limit of both sides as $k \to \infty$. As $y_n \to y$ it follows that $y_{n_k} \to y$.

Using the fact that $Tx_{n_k} \to v$ and that the sum of two convergent sequences is convergent, it follows that the subsequence (x_{n_k}) converges to a limit $x \in X$:

$$\lim_{k\to\infty} x_{n_k} = y + v =: x.$$

Since the subset A is closed and the subsequence $((x_{n_k}) \subset A$ converges to x, the limit point x must belong to A, that is, $(x \in A)$.

Therefore, because T is a continuous (bounded) operator, we can pass the limit inside the operator:

$$\lim_{k\to\infty} Tx_{n_k} = Tx.$$

This implies that our earlier limit v is exactly equal to Tx. Substituting this back into our limit equation yields: $x = y + Tx \implies x - Tx = y \implies (I - T)x = y$. Since $x \in A$, we have successfully shown that $y \in (I - T)(A)$. Therefore, $(I - T)(A)$ contains all its limit points and hence it is a closed subset.

The conclusion of our Theorem follows now by applying Corollary 1.

□

Remark 4. *We note that Theorems 5 and 6 ensure only existence and approximation of the solution for the problem* (10)+(11)*, while Theorem 4 ensures existence,* uniqueness *and approximation of the solution.*

4 Numerical examples

Example 3. *Consider the nested (iterative) differential equation*

$$x''(t) = t + 3x(t) + 4x^{[2]}(t),\ 0 \le t \le 1, \tag{24}$$

with the boundary value conditions

$$\begin{cases} x(0) = 0 \\ x(1) = 1. \end{cases} \tag{25}$$

In this case we have $f(t,y,z) = t+3y+4z$ which satisfies the Lipschitzian condition (19)

$$|f(t,y,z) - f(t,u,v)| \le L_1|y-u| + L_2|z-v|,$$

with $L_1 = 3$ and $L_2 = 4$, for any $(t,y,z),(t,u,v) \in [0,1]^3$.

Since $L_1 + L_2 = 7 > 6$, one cannot use Theorem 3.4 in Kaufmann [21] to decide about the existence (and uniqueness) of the solution of the two point boundary value problem (24)+(25)*. But, based on the fact that we have $L_1 + L_2 = 7 < 8$, one can apply Theorem 4 in our paper. Therefore, according to our result, the two point boundary value problem* (24)+(25) *has a unique solution $\overline{x}$ in $C[0,1]$ and, additionally, for any initial value $x_0(t)$ in $C[0,1]$ satisfying* (25)*, the sequence of successive approximations*

$$x_n(t) = t - \int_0^1 g(t,s)\Big(s + 3x_{n-1}(s) + 4x_{n-1}\big(x_{n-1}(s)\big)\Big)ds,\ n \ge 1 \tag{26}$$

converges uniformly to $\overline{x}$, as $n \to \infty$.

In order to compute simpler the iterates in (26)*, we can write*

$$x_n(t) = t - (1-t)\int_0^t s\Big(s + 3x_{n-1}(s) + 4x_{n-1}\big(x_{n-1}(s)\big)\Big)ds +$$

$$t\int_t^1 (1-s)\Big(s + 3x_{n-1}(s) + 4x_{n-1}\big(x_{n-1}(s)\big)\Big)ds \tag{27}$$

and then proceed with the computation of the integrals.

For numerical experiments, we consider the nested equation in Example 3. To approximate its solution, we take as initial guess

$$x_0(t) = t,\ t \in [0,1].$$

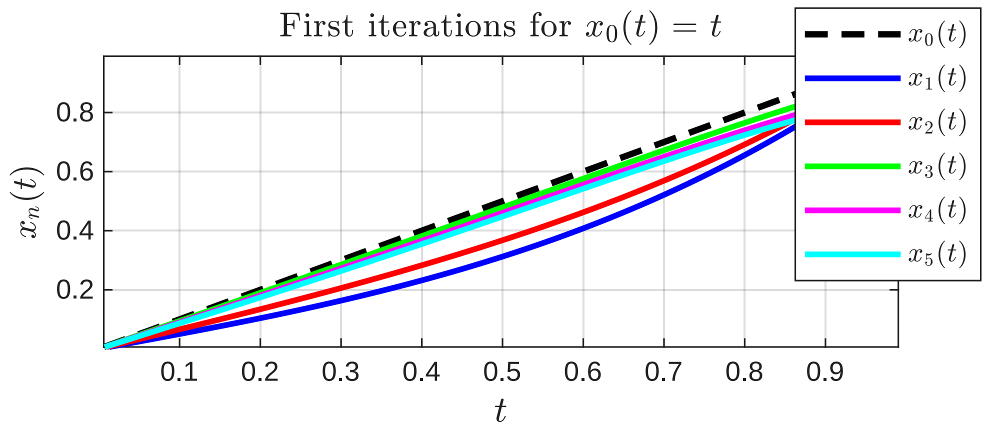

Figure 1: Graph of first iterations starting with $x_0(t) = t$

After performing symbolic computations with Matlab we obtained successively

$$x_1(t) = \frac{4}{3}t^3 - \frac{1}{3}t,$$

$$x_2(t) = \frac{512}{4455}t^{11} - \frac{32}{243}t^9 + \frac{32}{567}t^7 + \frac{41}{405}t^5 + \frac{2}{27}t^3 + \frac{73442}{93555}t,$$

and, because the coefficients are decreasing, we considered for the next iteration a truncated expression:

$$x_3(t) \approx 0.476271t - 1.940010t^3 - 1.574639t^5 - 0.211533t^7 - 0.153887t^9 + \mathcal{O}(t^{11}), ..$$

To illustrate the convergence of the Picard iteration (26) we graph in Figure 1 the first five iterations computed.

We note that, in Figure 1, due to the approximation by truncating the polynomial expressions of $x_3(t)$, $x_4(t)$, $x_5(t)$, their values at $x = 1$ are less than 1.

Example 4. *Consider the nested (iterative) differential equation*

$$x''(t) = t + 4x(t) + 4x^{[2]}(t),\ 0 \le t \le 1, \tag{28}$$

with the boundary value conditions

$$\begin{cases} x(0) = 0 \\ x(1) = 1. \end{cases} \tag{29}$$

In this case we have $f(t,y,z) = t+4y+4z$ *which satisfies the Lipschitzian condition* (19)

$$|f(t,y,z) - f(t,u,v)| \leq L_1|y-u| + L_2|z-v|,$$

with $L_1 = 4$ *and* $L_2 = 4$*, for any* $(t,y,z),(t,u,v) \in [0,1]^3$.

Since $L_1+L_2 = 8$*, one cannot use neither Theorem 3.4 in Kaufmann [21] nor our Theorem 4 to decide about the existence of the solutions of the two point boundary value problem* (28)+(29).

But, since condition $(3')$ *is satisfied, one can apply Theorem 6. We conclude that the two point boundary value problem* (28)+(29) *has solutions in the set*

$$K = \{x \in C[0,1] : x([0,1]) \subset [0,1], |x(t_1)-x(t_2)| \leq |t_1-t_2|, \forall t_1, t_1 \in [0,1]\},$$

and, additionally, for any initial value $x_0(t)$ *in* K*, the Mann iteration converges to such a solution.*

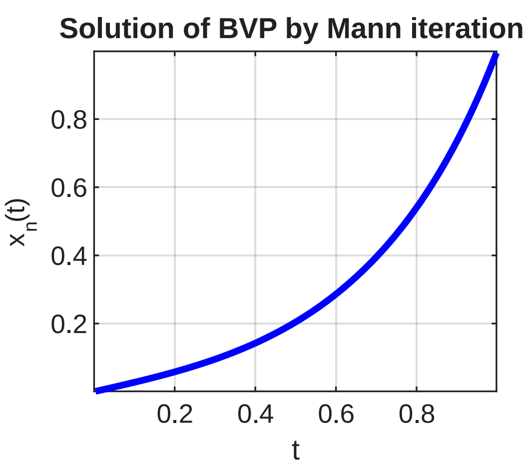

Figure 2: Graph of the solution $x_{14}(t)$ obtained by Mann iteration process starting with $x_0(t) = t$ and $\lambda_n = \dfrac{1}{2}$, $n = 0,1,2,\ldots$

We performed numerical tests in Matlab, using Mann iteration for $\lambda_n = \dfrac{1}{2}$, $n = 0,1,2,\ldots$, *starting from* $x_0(t) = t$, $t \in [0,1]$, *and with grid size* 0.01.

The Mann iteration converges to the solution of (28)+(29) *in 14 iterations. The graph of the obtained solution is plotted in Figure 2.*

5 Conclusions

(1) We studed the existence and approximation of solutions to the second-order iterative boundary-value problem

$$x''(t) = f\left(t, x(t), x^{[2]}(t)\right), \ 0 \le t \le 1, \tag{30}$$

with the boundary conditions $x(0) = 0$, $x(1) = 1$.

(2) Our approach was essentially based on the transformation of the original BVP into an equivalent fixed point problem $x = Tx$, with T a Fredholm type operator, to which we applied the contraction mapping principle and the principle of nonexpansive mappings, respectively.

(3) Our first main result (Theorem 4) significantly enlarges the area of application of Theorem 3.4 in Kaufmann [21] by imposing the weaker contraction condition $L_1 + L_2 < 8$ instead of $L_1 + L_2 < 6$, where L_1 and L_2 are the Lipschitz constants of the function f appearing in the righthand side of the nested equation (30).

(4) Our existence and approximation result can be similarly formulated and proved for the problem (30) and the dual boundary conditions $x(0) = 1$, $x(1) = 0$.

(5) It appears that our Theorem 4 is the first one in literature to provide not only existence (and uniqueness) but also *approximation* results for boundary value problems associated to iterative (nested) second order differential equations.

(6) In order to illustrate the superiority of our findings, we considered and example of a boundary value problem associated to a nested second order differential equation (Example 3), for which Theorem 3.4 in Kaufmann [21]) cannot be applied but our main result (Theorem 4) applies. We also considered the corresponding case of nonexpansive mappings in Example 4.

(7) We performed complex numerical tests in Matlab and computed the first 5 iterates of the Picard iteration (26). Figure 1 plots these iterations. For the nonexpansive case, in Figure 2 it is plotted the solution obtained after $n = 14$ iterations by the Mann iterative process, starting with $x_0(t) = t$ and $\lambda_n = \dfrac{1}{2}$, $n = 0, 1, 2, \ldots$.

(8) As differential equations with deviating arguments have very important applications in solving real world mathematical models from science and technology, we suggest the readers to consider and study such kind of concrete applications.

(9) We also suggest readers to consider some other approaches to study the existence / non existence of nested differential equations, as those studied

in Anderson [2], Bacoţiu [3], Berinde [7], Brestovanská et al. [8], Guerfi and Ardjouni [19], Karek and Ardjouni [22], Khemis et al. [23], Dunkel [11], Staněk [35]- [38], Zhang and Song [42], Zhao and Fečkan [43] etc.

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