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Ph.D. Thesis

FIXED POINT THEOREMS IN METRIC SPACES ENDOWED WITH A BINARY RELATION

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CHAPTER 1

Introduction

The fixed point theory is one of the most productive and dynamic sub-domains of nonlinear analysis. Even since Banach stated in 1922 his famous *Contraction Principle* [**17**], considered the cornerstone of this field, mathematicians all over the world were concerned to extend, generalize, improve this fundamental result. In the last decades many important results starting from the contraction principle were obtained, as we can see in [**147**],[**3**], [**5**], [**6**], [**8**], [**10**], [**14**], [**25**], [**19**], [**28**], [**30**], [**35**], [**33**], [**39**], [**37**], [**54**], [**62**], [**65**], [**70**], [**85**], [**95**], [**109**], [**113**],[**101**], [**121**], [**120**], [**126**], [**133**], [**135**], [**66**].

Studying the literature, it can be observed that fixed point theory has developed following three major directions, producing:

(1) *metric fixed point theorems*, based on the famous contraction principle of Banach [**17**], among which we mention Kannan [**79**] , Chatterjea [**48**], Zamfirescu [**150**], Reich-Rus [**114**]-[**119**], Edelstein [**62**], Caristi [**46**];
(2) *topological fixed point theorems*, having their starting point in Brower's fixed point result, followed by Schauder [**130**], Tychonoff [**142**], Ky Fan [**64**], Krasnoselskii [**87**] and many others;
(3) *fixed point theorems in ordered spaces* (Tarski [**137**], Bourbaki [**44**] and others.)

The development of metric fixed point theory, in turn, followed various directions in extending Banach's contraction principle, such as :

- weakening the contraction condition (see, for example, Kannan [**79**], Berinde [**25**], Altun and Acar [**5**] Choudhury, Metiya and Postolache [**50**]);
- extending the concept of metric (see, for example, Perov [**121**], Petruşel [**108**], Rus [**122**], Shatanawi, Abbas and Samet [**134**]);
- extending the space and endowing it with different relations (see Ben-El-Mechaiekh [**19**], Turinici and Samet [**126**], Asgari and Mousavi [**9**], Beg and Butt [**18**]);
- combination of the above directions (see Maia [**91**], Berinde [**20**], Kasahara [**85**], Rus [**124**]).

The fixed point theory evolved under the impulse of its applications, and, later on, it has developed as a stand alone discipline. From the applications of the fixed point results, strictly related to the topic of this thesis, it is important to mention:

- integral equations (see Krasnoselskii [**87**], Aghajani [**3**], Berzig and Samet [**35**], Shatanawi, Samet and Abbas [**134**], Sintunavarat, Kumam and Cho [**135**], Avramescu [**147**], Burton [**45**], Rus [**121**]);
- numerical analysis (see Timiş [**141**], Rhoades [**115**],[**116**], Păcurar [**101**]);
- initial value problems for ODE (see Amini-Harandi [**8**], Nieto and Rodriguez-Rodríguez-López [**96**], [**95**], Samet, Vetro and Vetro [**128**], Guo, Cho and Zhu [**69**]);
- periodic boundary value problems (see Bhaskar and Lakshmikantham[**36**], Berinde [**27**], Bernfeld and Lakshmikantham [**31**], Petruşel, Petruşel, Samet and Yao[**107**], Urs [**143**]);
- systems of differential and integral equations and inclusions (see Urs [**145**], Opoǐcev [**97**], Petruşel, Petruşel and Yao[**106**], Bota, Petruşel, Petruşel and Samet in [**43**], M.D. Rus [**125**], Eshi, Das and Debnath [**63**], Urs, Petruşel and Petruşel [**108**]);
- nonlinear matrix equations (see Ran and Reurings [**113**], Long, Hu and Zhang [**90**], Berzig and Samet [**34**], [**32**], Asgari and Mousavi [**10**]).

One of the most important contributions in the evolution of fixed point theory is the fixed point theorem of Ran and Reurings [**113**], who combined metric fixed point theorems and fixed point theorems in ordered sets. They endowed the metric space with a relation of partial order and assumed that the contraction condition holds only for comparable elements in X (i.e., $x \geq y$). They also added to the hypotheses a condition of monotony for the operator F involved (i.e., it should be either order-preserving or order-reversing) obtaining results regarding the existence and uniqueness of the fixed point. The importance of their research is emphasized by its applicability in solving linear matrix equations of the type $X - A_1^* X A_1 - ... - A_m^* X A_m = Q$ where Q is a positive definite matrix and $A_1, ...A_m$ are arbitrary matrices in $\mathcal{M}(n)$.

Their idea inspired many other mathematicians and lead to various interesting extensions and generalizations. Ran and Reurings fixed point theorem is based essentially on the continuity of the operator F. Nieto and Lopez have the merit to remove the continuity of the operator F, for F nonincreasing or F nondecreasing, respectively in [**96**], [**95**], [**94**]. Bhaskar and Lakshmikantham[**36**] combined the two results of Nieto and Lopez (i.e., for increasing and decreasing mappings), obtaining existence and uniqueness results for **coupled fixed points** in the context of partially ordered metric spaces, for mixed monotone mappings, using a weak contractive condition:

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \forall x \geq u, y \leq v \tag{1.1}$$

As an application, they studied the existence of a unique solution to a periodic boundary value problem, broadening in this way the sphere of problems solvable using fixed point tools.
Coupled fixed points were studied even earlier by Opoǐcev in [**97**], [**99**], [**98**], in 1974-1975 in the context of non-expansive heterogeneous operators. Then, independently, Guo and Lakshmikantham introduce coupled fixed points in [**70**], in 1987, in the context of monotone operators defined on ordered sets, followed by Bhaskar and Lakshmikantham in [**36**] who added a metric and the classical contractive condition to the fixed point theorem in the context of coupled fixed points, result received with great interest.

Using this concept, many researchers obtained new results, by weakening the contractive condition, adding assumptions regarding the operator involved to the hypotheses or by using two operators instead of one, as we can see in [**11**], [**25**], [**30**], [**32**], [**35**], [**51**], [**65**], [**81**], [**89**], [**108**], [**111**], [**112**], [**133**], [**135**], [**136**], [**143**], [**144**], [**146**]. Also, related to this concept, in very short time, coupled common and coincidence points were introduced by Ćirić and Lakshmikantham in [**53**] and Jungck and Rhoades in [**75**]. These too were widely discussed and extended in many directions in [**77**], [**76**], [**149**], [**50**], [**132**], [**26**], etc.

Despite the variety of the new results, Theorems 3.1.33-3.1.44 and their extensions were not able to solve systems of the form given in Example 6.128 from [**40**].

This was the main reason for Berinde and Borcut in [**28**] to introduce a new concept, the **tripled fixed point** of a mapping. Thus, the existence and uniqueness of a tripled fixed point of a mapping was and still is studied in various contexts, for continuous, mixed-monotone mapping [**42**],[**110**], for monotone mappings that are not necessarily continuous [**39**] and so on. Along with this new concept, following the same direction in research, tripled coincidence point results were obtained by Borcut in [**37**], [**38**].
The results presented by Berinde and Borcut in [**28**] were some of the most appreciated, cited and studied results from the last decade, representing an important moment in the evolution of fixed point theory as we can see in [**102**], [**12**], [**14**], [**49**], [**57**], [**74**], [**78**], [**82**], [**83**], [**4**], [**117**], [**141**], [**148**], due to the great number of applications in solving different types of problems:

- integral equations (see Berzig and Samet [**35**], Musatafa, Roshan and Parvaneh [**93**], Borcut, Păcurar and Berinde [**42**], Amini-Harandi [**7**], Aydi, Karapinar and Shatanawi [**14**], etc.);
- various systems of equations (see Roldán, Martínez-Moreno and Roldán [**118**], Parvaneh, Roshan and Radenović [**103**] etc.);
- matrix equations (see Berzig and Samet [**35**], Dobrican [**60**], etc.) .

It is important to remark the fact that most of the results regarding the fixed points of a mapping are obtained in partially (ordered) metric spaces even though not always all of the properties of the relation of (partial) order are being used. For example, in [**113**], [**36**] the antisymmetry of "$\leq$" is not essential in the proofs the authors present. Thus, Samet and Turinici endow the metric space with an amorphous binary relation in [**126**] extending and generalizing many existing fixed point results in ordered metric spaces. Asgari and Mousavi endow in [**10**], [**9**] the metric space with a reflexive relation R obtaining new, original results regarding coupled fixed points with various applications in solving nonlinear matrix equations. Similar results were obtained in [**19**], [**57**], [**65**], [**136**] and many others.

The purpose of this thesis is to perform a systematic study of coupled fixed points, tripled fixed points, coupled coincidence points and tripled coincidence points, in the setting of a metric space endowed with a reflexive binary relation. In this way, we unify, generalize and extend various results in the field. The material is organized in six chapters, excluding the introduction and the list of bibliographical resources as follows:

Chapter 2, **Preliminaries**, is a brief presentation of the basic notions and most important results of metric fixed point theory from the background of the results presented in Chapters 3, 4, 5, 6. This chapter consists of three paragraphs: *Basic notations of fixed point theory, Tripled fixed points of operators in ordered metric spaces, Fixed points of operators in metric spaces endowed with a binary relation*

The remaining chapters (3-6) present the original contribution of the author.

Chapter 3, **Coupled fixed point theorems in metric spaces endowed with a binary relation**, consists of three paragraphs,*Preliminaries*, *Existence and uniqueness theorems* and *Examples and applications*. In this chapter, we first present the backgound of coupled fixed points, then extend and generalize some of the results of Asgari and Mousavi in [**10**] and [**9**] by weakening and then symmetrizing the contractive condition in Theorem 3.1.49. Thus, we define the orbital continuous operatorial pair (f_1, f_2) (Definition 3.2.52), we obtain results regarding the existence (Theorems 3.3.54, 3.3.58, 3.3.53) and uniqueness (Theorems 3.3.56, 3.3.59) of coupled fixed points. Further on, we prove some existence (Theorem 3.3.61, Theorem 3.3.54, Theorem 3.3.58, Theorem 3.3.53, Theorem 3.3.55) and uniqueness results (Theorem 3.3.62, Theorem 3.3.56, Theorem 3.3.59,Theorem 3.4.70) for coupled fixed points for an operatorial pair, as presented in [**145**], [**143**] and [**108**] in the case of a metric space endowed with a reflexive relation. Some examples are also provided (Example 3.4.68, Example 3.4.69), followed by an application in the study of the solution of a first-order periodic boundary value system, following the ideas presented in [**31**].

The results in this chapter extend, unify and generalize some of the results of Berinde in [**25**], [**24**], Asgari and Mousavi in [**10**], [**9**], Bhaskar and Lakshmikantham in [**36**], Urs in [**143**], [**144**], etc.

The author's original contributions to this chapter are Definition 3.2.52, Theorem 3.3.53, Theorem 3.3.54, Theorem 3.3.56, Theorem 3.3.58, Theorem 3.3.59, Theorem 3.3.61, Theorem 3.3.62, Theorem 3.4.70, Remark 3.3.57, Remark 3.3.57, Remark 3.3.63, Remark 3.3.64, Remark 3.3.65, Remark 3.3.66, Remark 3.3.67, Remark 3.4.71, Remark 3.4.72, Example 3.4.68, Example 3.4.69. Some of the results were published in [**57**] and [**59**] and were also presented in [**55**].

Chapter 4, **Tripled fixed point theorems in metric spaces endowed with a reflexive relation**, consists of four paragraphs, *Preliminaries, Tripled fixed points of mixed-monotone operators, Tripled fixed points of monotone operators* and *Examples and applications*. In this chapter, we extend the results of Asgari and Mousavi from [**10**] and [**9**] in the case of tripled fixed points introduced by Berinde and Borcut in [**28**], then extended by Borcut et. al. in [**39**], [**40**], [**37**], [**38**], [**41**]. Thus, we obtain existence (Theorems 4.2.92, 4.2.96, 4.2.99, 4.2.100,Theorems 4.3.115, 4.3.118, 4.3.121, 4.3.122, Theorem 4.2.95) and uniqueness results (Theorems 4.2.94, 4.2.97, 4.2.102, 4.2.103, Theorems 4.3.117, 4.3.119, 4.3.124, 4.3.125), for tripled fixed points of a mixed-R-monotone operator, and, respectively, $R-$monotone operator. An application to nonlinear matrix equations is also provided. The novelty brought by these results consists in the fact that the metric space is endowed with a reflexive relation, whereas most of the results in the field use a relation of (partial) order (see [**129**], [**2**], [**7**], [**18**], [**24**], [**21**], [**30**], [**52**], [**67**], [**68**], [**73**], [**74**], [**82**], [**86**], [**89**], [**100**], [**101**], [**102**], [**104**], [**115**], [**116**], [**124**], [**123**], [**146**], [**148**], [**150**] and others).

The author's contributions to this chapter are Definition 4.2.90, Definition 4.2.91, Definition 4.2.87, Definition 4.2.88, Definition 4.2.89, Definition 4.3.112, Definitions 4.3.113, Definition 4.3.110, Definition 4.3.111, Notation 2, Notation 3, Theorem 4.2.92, Theorem 4.2.96, Remark 4.2.93, Theorem 4.2.94, Theorem 4.2.97, Theorem 4.2.99, Theorem 4.2.100, Remark 4.2.101, Theorem 4.2.102, Theorem 4.2.103, Theorem 4.2.95, Theorem 4.3.115, Theorem 4.3.117, Theorem 4.3.118, Theorem 4.3.119, Theorem 4.3.124, Theorem 4.3.125, Theorem 4.3.121, Theorem 4.3.122, Theorem 4.4.133, Theorem 4.4.134, Remark 4.2.104, Remark 4.2.105, Remark 4.2.106, Remark 4.2.107, Remark 4.2.108, Remark 4.3.116, Remark 4.3.120, Remarko46, Remark 4.3.109, Remark 4.2.98, Remark 4.3.123, Remark 4.3.126, Remark 4.3.127, Remark 4.3.128, Example 4.4.129, Example 4.4.130. Some of these results were published in [**60**] and [**57**] and were also presented in [**55**].

In the fifth chapter, **Coupled coincidence point theorems in metric spaces endowed with a reflexive relation**, we study the existence and uniqueness of coupled coincidence points in metric spaces endowed with a reflexive relation and redefine some related notions to them in this particular context. Coupled coincidence points were discussed by Lakshmikantham and Ćirić in [**88**], then extended and intensively studied in various contexts (graphs, in [**139**], [**140**], generalized metric spaces, in [**133**], L-fuzzy metric spaces, in [**132**] and so on).

This chapter consists of four paragraphs, *Coupled coincidence points of operators in partially ordered metric spaces-Preliminaries, Definitions, Existence and uniqueness theorems* and *Examples and applications*. In the first paragraph we recall some of the important results in the field. In the second section we present the definitions of a lower-$R-$coupled coincidence point, mixed $g-R$-monotone property of a mapping f, orbital $g-$continuity of a mapping f. In the third paragraph we give some existence and uniqueness results for coupled coincidence points followed by illustrative examples in the last paragraph of the chapter. These results extend, unify and generalize various previous results, see [**26**], [**51**], [**50**], [**53**], [**27**], etc.
To prove the effectiveness of our results, in the last part of this chapter, we provide an application to nonlinear matrix equations.
The author's original contributions to this chapter are Definition 5.2.154, Definition 5.2.155, Definition 5.2.156, Notation 4, Theorem 5.3.157, Corollary 5.3.158, Theorem 5.3.159, Theorem 5.3.160, Theorem 5.3.161, Theorem 5.3.162, Theorem 5.3.163, Theorem 5.3.164, Theorem 5.3.165, Theorem 5.4.172, Theorem 5.4.173, Remark 5.3.166, Remark 5.3.167, Remark 5.3.168, Remark 5.4.174, Example 5.4.169, Example 5.4.170, Example 5.4.171. Some of the results can be found in [**58**]

In Chapter 6, **Tripled coincidence point theorems in metric spaces endowed with a reflexive relation**, we extend the results obtained in Chapter 5, in the case of tripled coincidence points introduced by Borcut in [**40**], [**37**], [**38**]. In the first paragraph we present results regarding tripled coincidence points of mixed-$g-R$-monotone mappings. The second paragraph consists of definitions and theorems regarding tripled coincidence points of $g-R$-monotone mappings. In the last paragraph of this chapter, there are presented some illustrative examples. These results extend, unify and generalize various previous results, see [**40**], [**37**], [**38**], [**49**], [**12**], etc. We also provide an application to integral equations systems, motivated by the work of Eshi, Das and Debnath in [**63**].
The author's original contributions to this chapter are Definition 6.1.175, Definition

6.1.176, Definition 6.1.177, Definition 6.2.190, Definition 6.2.191, Notation 5, Notation 6, Theorem 6.1.178, Corollary 6.1.179, Theorem 6.1.180, Theorem 6.1.181, Theorem 6.1.182, Theorem 6.1.183, Theorem 6.1.184, Theorem 6.1.185, Theorem 6.1.186, Theorem 6.2.193, Corollary 6.2.194, Theorem 6.2.195, Theorem 6.2.196, Theorem 6.2.197, Theorem 6.2.198, Theorem 6.2.199, Theorem 6.2.200, Theorem 6.2.201, Theorem 6.3.206, Remark 6.2.192, Remark 6.2.189, Remark 6.1.187, Remark 6.1.188, Remark 6.2.202, Remark 6.3.207, Example 6.3.203, Example 6.3.204, Example 6.3.205. Some of the results can be found in [**61**]

In the last chapter, **Conclusions**, we resumed the original contributions from this thesis, also mentioning some future research directions starting from our results.

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CHAPTER 2

Preliminaries

In this chapter we present the definitions of the basic concepts and notations used in this paper and the most important fixed point theorems that represent the starting point in obtaining the results presented in Chapters 3, 4, 5 and 6. Most of the results presented in this chapter were taken from "Iterative Approximation of Fixed Points", [**20**] and "Principles and Applications of the Fixed Point Theory"[**121**]. The following bibliographical references were also used: [**24**], [**22**], [**29**], [**36**], [**70**], [**9**], [**19**], [**39**], [**40**], [**108**], [**109**], [**143**], [**53**], [**124**], [**109**], [**96**], [**95**].

1. Basic notions of fixed point theory

We will start by recalling the definition of a metric space. We assume that all notions in metric space, like sequence, convergent sequence, fundamental sequence, etc. are known.

Definition 2.1.1. [**24**] *Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called metric or distance on X provided that:*

(1) $d(x, y) = 0 \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x), \forall x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$ ("the triangle inequality").

A set X endowed with a metric d is called metric space and we denote it by (X, d).

Definition 2.1.2. [**24**] *A metric space (X, d) is called complete if any fundamental (Cauchy) sequence in X is convergent.*

Definition 2.1.3. [**138**] *Let A and B be two sets. An ordered triple $r = (A, B, R)$ is called a binary relation, where R is a subset of the cartesian product $A \times B$. The set A is called the domain of the relation and B, the codomain of the relation.*
If $r = (A, B, R)$ is a relation, we say that $x \in A$ is related to $y \in B$ by R, i.e. $(x, y) \in R$, also written as xRy.

Most of the results in the field are obtained in partially ordered metric spaces. We will now recall the definition of a partially ordered set:

Definition 2.1.4. [**20**] *A binary relation R on a set X is a partial order if and only if it is:*

(1) *reflexive, that is,* $xRx, \forall x \in X$*;*
(2) *antisymmetric, that is, if* xRy *and* yRx*, then* $x = y, x, y \in X$*;*
(3) *transitive, that is, if* xRy *and* yRz*, then* $xRz, x, y, z \in X$*.*

The ordered pair (X, R) is called partially ordered set (poset) when R is a partial order.

If, in addition to properties (1)-(3), for any pair (x, y), where $x, y \in X$, we have xRy or yRx, then R is called total (linear) order.

EXAMPLE 2.1.5.
(1) *If* $X = \mathbb{R}$ *and* $R :=''\leq''$*, we obtain the (totally) ordered set* $(\mathbb{R}, \leq)$*.*
(2) *Let* $X = \mathbb{Z}$ *and* $xRy \Leftrightarrow [((x \geq 0) \wedge (y \geq 0)) \vee ((x \leq 0) \wedge (y \leq 0))]$*. This relation is reflexive, because,* $\forall x \in X, (x, x) \in R$*, but it is neither transitive(because, for example,* $(1, 0) \in R$ *and* $(0, -1) \in R$*, but* $(1, -1) \notin R$*), nor antisymmetric (for example,* $(1, 0) \in R$ *and* $(0, 1) \in R$*, but* $0 \neq 1$*). Thus,* R *is a reflexive-only relation;*
(3) $X = \mathbb{R}^2, R :=''\leq'', (x_1, y_1)R(x_2, y_2)$ *if* $x_1 \leq x_2$ *and* $y_1 \leq y_2$*;* $(X, \leq)$ *is a partially ordered set.*

DEFINITION 2.1.6. [**24**] *Let* X *be a nonempty set and* $T : X \to X$ *a self-map. We say that* $x \in X$ *is a fixed point of* T *if*

$$T(x) = x.$$

We denote by F_T *(or* $Fix(T)$*) the set of fixed points of* T *and, in order to simplify the notations, we will use* Tx *instead of* $T(x)$*.*

In this context we can define $T^n(x)$ (the n^{th} iterate of x under T) inductively: $T^0(x) = x$, $T^1(x) = T(T^0(x))$,..., $T^{n+1}(x) = T(T^n(x))$.

EXAMPLE 2.1.7.
(1) *If* $X = \mathbb{R}$ *and* $T(x) = x^2 + 3x + 1$*, then* $F_T = \{-1\}$*;*
(2) *If* $X = \mathbb{R}$ *and* $T(x) = x^2 + 4x + 4$*, then* $F_T = \emptyset$*;*
(3) *If* $X = \mathbb{R}$ *and* $T(x) = x$*, then* $F_T = \mathbb{R}$*.*

DEFINITION 2.1.8. [**53**] *Let* X, Y *be two nonempty sets and* $F, G : X \to Y$ *two operators. An element* $x \in X$ *is called coincidence point of* F *and* G *if* $Fx = Gx$*. We will denote by*

$$C(F, G) = \{x \in X \,|Fx = Gx\}$$

the set of all coincidence points of F *and* G*.*

EXAMPLE 2.1.9.

Let $f, g : \mathbb{R} \to \mathbb{R}, f(x) = 2x, g(x) = -4x$. *It is easy to remark that* $x = 0$ *is a coincidence point of* f *and* g*, that is,[* $C(f, g) = \{0\}$*;*

DEFINITION 2.1.10. [**24**] *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is called:*

(1) Lipschitzian (or L-Lipschitzian) if there exists $L > 0$ *such that*

$$d(Tx, Ty) \leq L \cdot d(x, y), \forall x, y \in X;$$

(2) (strict) contraction if there exists a constant $a \in (0, 1]$ *such that* T *is* $a-$*Lipshchitzian;*
(3) nonexpansive if T *is* $1-$*Lipschitzian;*
(4) contractive is $d(Tx, Ty) < d(x, y), \forall x, y \in X, x \neq y$*;*
(5) isometry if $d(Tx, Ty) = d(x, y), \forall x, y \in X$*.*

EXAMPLE 2.1.11.
(1) $T : \mathbb{R} \to \mathbb{R}, T(x) = 4 - \dfrac{x}{4}, x \in \mathbb{R}$ *is a strict contraction and* $F_T = \left\{\dfrac{16}{5}\right\}$*;*
(2) [**24**] *the mapping* $T : \mathbb{R} \to \mathbb{R}, T(x) = \ln(1 + e^x)$ *is contractive and* $F_T = \emptyset$*;*

Next, we will present the fundamental result of the metrical fixed point theory, called the Contraction mapping principle, also known as Banach's fixed point theorem or Banach-Picard-Caccioppoli Contraction Principle.

THEOREM 2.1.12. [**17**] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping satisfying:*

$$d(Tx, Ty) \leq a \cdot d(x, y), \forall x, y \in X$$

with $a \in [0, 1)$*. Then*

i) T *has a unique fixed point,* $F_T = \{x^*\}$*;*
ii) The Picard iteration associated to T *converges to* x^**, for any initial guess* $x_0 \in X$*;*
iii) The a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{a^n}{1-a} \cdot d(x_0, x_1), n = 0, 1, 2, ...$$

$$d(x_n, x^*) \leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n), n = 0, 1, 2, ...$$

hold.
iv) The rate of convergence is given by

$$d(x_n, x^*) \leq a \cdot d(x_{n-1}, x^*) \leq a^n \cdot d(x_0, x^*), n = 0, 1, 2, ...$$

One of the first extensions of this principle was given by Edelstein in [**62**], by establishing that every uniform local contraction $f : X \to X$ of an ε-chainable complete metric space (X, d) has a unique fixed point. Let us recall the definitions of an ε-chainable sequence and uniform local contraction:

Definition 2.1.13. [**62**] *We say that a metric space (X, d) is ε-chainable, for $\varepsilon > 0$, if $\forall x, y \in X, \exists \{u_i\}_m^{i=0}$ a finite sequence in X such that:*

$$x = u_0, u_m = y, d(u_{i-1}, u_i) < \varepsilon, \forall i = 1, ..., m.$$

Definition 2.1.14. [**62**] *A mapping $f : X \to X$ is called a local contraction (l.c.) if there exist real-valued functions $\mu(x), \lambda(x)$, with $\mu(x) > 0$ and $0 < \lambda(x) < 1$, such that whenever y, z are in the sphere*

$$S(x, \mu(x)) = \{u : d(x, u) \leq \mu(x)\}$$

it follows that

$$d(f(y), f(z)) \leq \lambda(x)d(y, z).$$

If $\mu(x)$ (resp. $\lambda(x)$) is constant, we have a $\mu-$(resp. $\lambda-$) uniform local contraction (u.l.c.).

Definition 2.1.15. [**62**] *A mapping f of X to itself is said to be (ε, λ)-uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.*

The extended contraction principle of Edelstein presented in [**62**] is stated as follows:

Theorem 2.1.16. [**62**] *Let X be a complete metric $\varepsilon-$chainable space, f a mapping on X into itself, which is (ε, λ)-uniformly locally contractive. Then there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.*

In the last decades, many generalizations of Theorem 2.1.16 have been obtained. Some of them, weakening the contractive properties of the mapping (see [**25**], [**36**], [**54**], [**62**], [**96**], [**95**], [**113**], etc.), others, by altering the structure of the space involved or by combining the two methods described and, eventually, endowing the mapping or the ambient space with supplementary properties (see [**68**], [**122**], [**86**], [**10**], [**9**], [**108**], etc.)

The following theorem is an analogue of the *Contraction Principle* in partially ordered sets:

Theorem 2.1.17 ([**113**]). *Let X be a partial ordered set such that every pair $x, y \in X$ has an upper bound and lower bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. If $T : X \to X$ is a continuous, monotone mapping such that:*

$$\exists \quad 0 < \alpha < 1 : d(Tx, Ty) \leq \alpha \cdot d(x, y), \forall x \geq y \tag{2.2}$$

$$\exists \quad x_0 \in X : x_0 \leq Tx_0 \quad or \quad x_0 \geq Tx_0 \tag{2.3}$$

then T has a unique fixed point x^*. Moreover, for every $x \in X$

$$\lim_{n\to\infty} T^n x = x^*$$

In [**109**], Rus and Petruşel emphasize the fact that, when working on a set that is endowed simultaneously with a metric and a binary relation (an order structure in [**109**]), we should add an additional assumption, namely the compatibility between the two structures:
if $\{x_n\}_{n\in\mathbb{N}} \to x$, $\{y_n\}_{n\in\mathbb{N}} \to y$ and $x_n \le y_n$ then $x \le y$, $\forall n \in \mathbb{N}$.

In [**96**], [**95**] Nieto and Rodríguez-López improve Theorem 2.1.17 by removing the continuity of the operator T in the case of nonincreasing and nondecreasing, respectively, operators, obtaining more general results:

THEOREM 2.1.18. [**95**] *Let $(X, \le)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space.*
Let f be a monotone nondecreasing mapping such that there exists $k \in [0, 1)$ with

$$d(f(x), f(y)) \le k \cdot d(x, y), \forall x \ge y.$$

Suppose that either f is continuous or X is such that

$$\text{if a nondecreasing sequence } \{x_n\}_{n\in\mathbb{N}} \to x \in X, \text{ then } x_n \le x, \forall n \in \mathbb{N}. \tag{2.4}$$

If there exists $x_0 \in X$ such that $x_0 \le f(x_0)$, then f has a fixed point.

THEOREM 2.1.19. [**95**] *Let $(X, \le)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space.*
Let f be a monotone nondecreasing mapping such that there exists $k \in [0, 1)$ with

$$d(f(x), f(y)) \le k \cdot d(x, y), \forall x \ge y.$$

Suppose that either f is continuous or X is such that

$$\text{if a nonincreasing sequence } \{x_n\}_{n\in\mathbb{N}} \to x \in X, \text{ then } x_n \le x, \forall n \in N. \tag{2.5}$$

If there exists $x_0 \in X$ such that $x_0 \ge f(x_0)$, then f has a fixed point.

THEOREM 2.1.20. [**96**] *Let $(X, \le)$ be a partially ordered set such that every pair of elements of X has an upper bound and a lower bound and suppose there exists a metric d on X such that (X, d) is a complete metric space.*
Let f be a monotone nonincreasing function such that there exists $k \in [0, 1)$ with

$$d(f(x), f(y)) \le k \cdot d(x, y), \forall x \ge y.$$

Suppose that either f is continuous or X is such that
(2.6)

if $\{x_n\}_{n\in\mathbb{N}} \to x$, is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{nk}\}_{k\in\mathbb{N}}$ such that every term is comparable to the limit x.
If there exists $x_0 \in X$ such that $x_0 \ge f(x_0)$ or $x_0 \le f(x_0)$, then f has a fixed point.

THEOREM 2.1.21. [**96**] *Let $(X, \leq)$ be a partially ordered set such that every pair of elements of X has an upper bound and a lower bound and suppose there exists a metric d on X such that (X, d) is a complete metric space.*
Let $f : X \to X$ be such that it maps comparable elements into comparable elements, that is $x, y \in X, x \leq y$ implies

$$\begin{cases} f(x) \leq f(y) \\ or \\ f(x) \geq f(y) \end{cases}$$

and such that there exists $k \in [0, 1)$ with

$$d(f(x), f(y)) \leq k \cdot d(x, y), \forall x \geq y.$$

Suppose that either f is continuous or X is such that condition (2.6) holds. If there exists $x_0 \in X$ x_0 is comparable to $f(x_0)$, then f has a fixed point, x^*. Moreover, $\forall x \in X, \lim\limits_{n\to\infty} f^n(x) = x^*$.

2. Fixed points of operators defined on metric spaces endowed with a binary relation

Despite the general tendency to study the existence and uniqueness of fixed points in partially ordered metric spaces, there are researchers who obtained results regarding their existence and uniqueness, by replacing the partial order with amorphous relations (see [**126**]), transitive relations (see [**19**]) or reflexive relations ([**10**], [**9**], [**57**]). Next, we will present some of the results obtained using these relations.

2.1. Fixed points of operators defined on metric spaces endowed with an amorphous relation

Turinici and Samet extend and generalize in [**126**] many existing results in the field. They consider a metric space (X, d) and $\mathcal{R}$ a binary relation over X. They denote by $\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$, that is the symmetric relation attached to $\mathcal{R}$. Clearly,
$x, y \in X, x\mathcal{S}y \Leftrightarrow x\mathcal{R}y$ or $y\mathcal{R}x$.

Using this notation, they present the definition of an $\mathcal{S}-$directed subset D of X, of comparative mappings and other related notions in order to obtain their main results:

DEFINITION 2.2.22. [**126**] *We say that the subset D of X is $\mathcal{S}-$directed if, for every $x, y \in D$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$.*

DEFINITION 2.2.23. [**126**] *We say that $(X, d, \mathcal{S})$ is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that*

$$x_n\mathcal{S}x_{n+1}, \forall n \text{ and } \lim_{n\to\infty} d(x_n, x) = 0, \tag{2.7}$$

then there exists a subsequence $\left\{x_{n(p)}\right\}$ of $\{x_n\}$ such that $x_{n(p)}\mathcal{S}x, \forall p$.

Definition 2.2.24. [**126**] *We say that $T : X \to X$ is a comparative mapping if T maps comparable elements into comparable elements, that is,*

$$x, y \in X, x\mathcal{S}y \Rightarrow Tx\mathcal{S}Ty.$$

In order to obtain their main results, they use the following notations:

(1) Let Φ be the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying :
   - φ is nondecreasing;
   - $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where φ^n is the nth iterate of φ;

(2) $M_T(x, y) = \max \left\{ d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx] \right\}$, where $T : X \times X$ is a mapping.

Theorem 2.2.25. [**126**] *Assume that T is a comparative map, and*

$$x, y \in X, x\mathcal{S}y \Rightarrow d(Tx, Ty) \leq \varphi(M_T(x, y)), \tag{2.8}$$

where $\varphi \in \Phi$. Suppose also that the following conditions hold:

i.) there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;

ii.) $(X, d, \mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if, in addition, $D := F_T$ is $\mathcal{S}-$directed, then x^* is the unique fixed point of T in X.

Turinici and Samet also present 7 corollaries of this result, by replacing condition (2.8) with some other contractivity conditions. They also prove that, by customizing the relation involved (for example, letting $\mathcal{R}$ be a relation of order), they obtain important results in the literature.

2.2. Fixed points of operators defined on metric spaces endowed with a transitive relation

The results of Ran and Reurings also inspired the mathematician Hichem Ben-El-Mechaiekh [**19**]. In 2014, he replaces the order relation on $x, y \in X$, from the original result, with a transitive binary relation. The purpose of this is to increase the applicability of fixed point theorems. He starts from the extension that Edelstein brings to Banach's contraction principle in [**62**], by establishing that every uniform local contraction $f : X \to X$ of an ε-chainable complete metric space (X, d) has a unique fixed point. Recall that a metric space (X, d) is ε-chainable for some $\varepsilon > 0$, if $\forall x, y \in X, \exists \{u_i\}_m^{i=0}$ a finite sequence in X such that:

$$x = u_0, u_m = y, d(u_{i-1}, u_i) < \varepsilon, \forall i = 1, ..., m.$$

Let us recall the following concepts, used in his results in [**19**]:

Definition 2.2.26. [**19**] *Two elements $x, y \in X$ are **joined by an $\varepsilon-$monotone chain** for some $\varepsilon > 0$ if there exists a monotone sequence $\{u_i\}_m^{i=0}$ in X such that:*

$$x = u_0, u_m = y, d(u_{i-1}, u_i) < \varepsilon, \forall i = 1, ..., m.$$

DEFINITION 2.2.27. [**19**] *The space $(X, \leq, d)$ is said to be $\varepsilon-$**monotone chainable** for some $\varepsilon > 0$ if any two comparable elements $x, y \in X$ are joined by a $\varepsilon-$monotone chain.*

DEFINITION 2.2.28. [**19**] *The metric d is **monotone complete** if and only if every monotone Cauchy sequence converges in X.*

The most important results listed in his paper are:

THEOREM 2.2.29. [**19**] *Let $(X, \leq, d)$ be a triple consisting of a metric space (X, d) and a transitive binary relation $\leq$ on X, let $f : X \to X$ be a mapping and $\varepsilon > 0$ be such that:*

(1) $\exists x_0 \in X$ such that x_0 and $f(x_0)$ are joined by a $\varepsilon-$monotone chain;
(2) f is monotone;
(3) if $\lim_{n\to\infty} f^n(x_0) = x^* \in X$ then $f^n(x_0)$ and x^* are comparable, $\forall n$;
(4) $\exists 0 < k < 1$ such that for any comparable elements x and y in X, $d(x, y) \leq \varepsilon$ implies $d(f(x), f(y)) \leq kd(x, y)$.

Then, f has a unique fixed point $x^* = \lim\limits_{n\to\infty} f^n(x_0)$ provided the metric d is monotone complete.

THEOREM 2.2.30. [**19**] *Let $(X, \leq, d)$ a triple, where $\leq$ is a transitive relation and d is a metric, is $\varepsilon-$monotone chainable for some $\varepsilon > 0$ and $f : X \to X$ is a mapping satisfying:*

(1) $\exists x_0 \in X$ such that x_0 and $f(x_0)$ are comparable;
(2) f is monotone;
(3) if $\lim_{n\to\infty} f^n(x_0) = x^* \in X$ then $f^n(x_0)$ and x^* are comparable, $\forall n$;
(4) $\exists 0 < k < 1$ such that if x and $y \in X$ are comparable, $d(x, y) \leq \varepsilon$ implies $d(f(x), f(y)) \leq kd(x, y)$.
(5) every pair of elements of X admits a third element similarly comparable to both.

Then, f has a unique fixed point $x^* = \lim\limits_{n\to\infty} f^n(x)$, for any initial point $x \in X$ provided the metric d is monotone complete.

Similar results were obtained by Shahzad et. al. in [**131**].

CHAPTER 3

Coupled fixed point theorems in metric spaces endowed with a reflexive relation

In this chapter we will present some coupled fixed point results, that unify, extend and generalize results from [**10**], [**9**], Berinde [**25**], [**20**], [**23**], Urs [**145**], [**143**], [**144**] and others, that we summarize in the following:

1. Preliminaries

1.1. Coupled fixed points of operators in ordered metric spaces

In this paragraph we will present the concept of coupled fixed point and basic notions related to it. This concept was first studied by Opoitsev in [**97**], [**99**], [**98**] then studied and presented by Guo, Bhaskar and Lakshmikantham in [**36**], [**70**]. Their work is fundamental for all the results obtained so far for coupled fixed points, presenting a new perspective in the study of this theory.

Next, we present the definition of a coupled fixed point of a mapping as presented in [**70**] and [**36**]:

Definition 3.1.31. [**36**],[**70**] *We call an element $(x, y) \in X^2$ a* ***coupled fixed point*** *of the mapping F, $F : X^2 \to X$, if $F(x, y) = x$ and $F(y, x) = y$.*
If $x = y$ and, in consequence, $F(x, x) = x$, then $x \in X$ is a fixed point of F.

Definition 3.1.32. [**36**] *Let $(X, \leq)$ be a set endowed with a relation of partial order and $F : X^2 \to X$. We say that F has the* ***mixed monotone*** *property on X if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X, x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$.*

Note that Opoǐcev [**97**], [**99**], [**98**] used the term "heterogenous" for a mixed monotone mapping $F : X^2 \to X$.

The main result obtained by Bhaskar and Lakshmikantham is the following theorem:

Theorem 3.1.33. [**36**] *Let $F : X^2 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that $\exists k \in [0, 1)$ with*

$$d(F(x,y),F(u,v)) \le \frac{k}{2}[d(x,u)+d(y,v)], \forall x \ge u, y \le v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \le F(x_0,y_0) \text{ and } y_0 \ge F(y_0,x_0),$$

then $\exists x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

The following results complete Theorem 3.1.33, by establishing, respectively, the uniqueness of the coupled fixed point and the identity of the components in the pair (x,y):

THEOREM 3.1.34. [**36**] *Adding the condition that for every $(x,y),(x^*,y^*) \in X^2$, $\exists(u,v) \in X^2$ that is comparable to both (x,y) and (x^*,y^*) to the hypotheses of Theorem 3.1.33, we obtain the uniqueness of the coupled fixed point of F.*

THEOREM 3.1.35. [**36**] *In addition to the hypothesis of Theorem 3.1.33, suppose that every pair of elements of X has an upper bound or a lower bound in X. Then $x = y$.*

THEOREM 3.1.36. [**36**] *In addition to the hypothesis of Theorem 3.1.33, suppose that $x_0, y_0 \in X$ are comparable. Then $x = y$.*

Berinde extends their result in [**25**], for mappings having the mixed monotone property, but not necessarily continuous, obtaining results regarding the existence, uniqueness of the coupled fixed point, but also the equality of the two components of the coupled fixed point. The contraction conditions is also weakened by using a symmetric one:

THEOREM 3.1.37. [**25**] *Let $(X, \le)$ be a partially ordered set and suppose there is a metric d on X such that the metric space (X,d) is complete. Let $F : X^2 \to X$ a continuous mapping having the mixed monotone property on X, for which there exists a constant $k \in [0,1), \forall x \ge u, y \le v$, with*

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \le k[d(x,u)+d(y,v)].$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \le F(x_0,y_0) \text{ and } y_0 \ge F(y_0,x_0),$$

then $\exists \bar{x}, \bar{y} \in X$ such that $\bar{x} = F(\bar{x},\bar{y})$ and $\bar{y} = F(\bar{y},\bar{x})$.
The error estimate for the described method is:

$$d((x_n,y_n),(\bar{x},\bar{y})) \le \frac{k^n}{1-k} d((x_1,y_1),(x_0,y_0)), n \ge 0$$

THEOREM 3.1.38. [**25**] *Adding the condition that $\exists(z_1,z_2) \in X^2$, comparable to (x,y) and $(\overline{x},\overline{y})$ to the hypotheses of Theorem 3.1.37, we obtain the uniqueness of the coupled fixed point.*

Similarly to [**36**], it is obtained a result regarding the equality of the two components of the pair $(\overline{x}, \overline{y})$:

THEOREM 3.1.39. [**25**] *In addition to the hypothesis of Theorem 3.1.37, suppose that every pair of elements of X has an upper bound or a lower bound in X. Then for the coupled fixed point we have $\overline{x} = \overline{y}$, that is, F has a fixed point.*

The same conclusion is obtained under the following assumption:

THEOREM 3.1.40. [**25**] *In addition to the hypothesis of Theorem 3.1.37, suppose that $x_0, y_0 \in X$ are comparable.Then for the coupled fixed point we have $\overline{x} = \overline{y}$, that is, F has a fixed point.*

In [**108**] Urs, Petruşel and Petruşel present a new approach on coupled fixed points, by using two mappings instead of one:

DEFINITION 3.1.41. [**108**] *Let X be a nonempty set and $T : X \times X \to X \times X$ be an operator defined by*

$$T(x, y) := \begin{pmatrix} T_1(x, y) \\ T_2(x, y) \end{pmatrix},$$

where $T_1, T_2 : X^2 \to X$.

- *By definition, a solution (x, y) for the system*

$$\begin{cases} T_1(x, y) = x \\ T_2(x, y) = y \end{cases}$$

  is called a fixed point for the operator T, respectively, a coupled fixed point for the pair of singlevalued operators (T_1, T_2).
- *The cartesian product of T and T is denoted by $T \times T$ and it is defined in the following way: $T \times T : Z \times Z \to Z \times Z$, $\quad (T \times T)(z, w) := (T(z), T(w))$, where $Z := X^2$ and $z := (x, y), w := (u, v)$ are two arbitrary elements in Z.*

REMARK 3.1.42. *In the definition above, if $T_1(x, y) = F(x, y)$ and $T_2(x, y) = F(y, x)$, we obtain the classical definition of the coupled fixed point of an operator F.*

The following result is one of the main results in [**108**] and establishes the existence of a unique coupled fixed point for the pair of mappings considered.

THEOREM 3.1.43. [**108**] *Let $(X, d, \leq)$ be an ordered complete metric space and let $T_1, T_2 : X^2 \to X$ be two operators. We suppose:*

(1) *for each* $z = (x, y)$, $w = (u, v) \in X \times X$ *which are not comparable with respect to the partial ordering* $\leq$ *on* $X \times X$, *there exists* $t := (t_1, t_2) \in X \times X$ *such that* t *is comparable with both* z *and* w, *that is,*

$$((x \geq t_1 \quad and \quad y \leq t_2) \quad or \quad (x \leq t_1 \quad and \quad y \geq t_2)) \quad and$$

$$((u \geq t_1 \quad and \quad v \leq t_2) \quad or \quad (u \leq t_1 \quad and \quad v \geq t_2));$$

(2) *for all* $((x \geq u \quad and \quad y \leq v)$ *or* $(u \geq x \quad and \quad v \leq y)$ *we have*

$$\begin{cases} T_1(x,y) \geq T_1(u,v) \\ T_2(x,y) \leq T_2(u,v) \end{cases}$$

or

$$\begin{cases} T_1(u,v) \geq T_1(x,y) \\ T_2(u,v) \leq T_2(x,y) \end{cases}$$

(3) $T_1, T_2 : X^2 \to X$ *are continuous;*

(4) *there exists* $z_0 := (z_0^1, z_0^2) \in X \times X$ *such that*

$$\begin{cases} z_0^1 \geq T_1(z_0^1, z_0^2) \\ z_0^2 \leq T_2(z_0^1, z_0^2) \end{cases}$$

or

$$\begin{cases} T_1(z_0^1, z_0^2) \geq z_0^1 \\ T_2(z_0^1, z_0^2) \leq z_0^2 \end{cases}$$

(5) *there exists a matrix* $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in M_2(\mathbb{R}_+)$ *convergent toward zero such that*

$$d(T_1(x,y), T_1(u,v)) \leq k_1 d(x,u) + k_2 d(y,v)$$

$$d(T_2(x,y), T_2(u,v)) \leq k_3 d(x,u) + k_4 d(y,v)$$

for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$.

Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that*

$$x^* = T_1(x^*, y^*) \quad and \quad y^* = T_2(x^*, y^*)$$

and the sequence of the successive approximations $(T_1^n(w_0^1, w_0^2), T_2^n(w_0^1, w_0^2))$ *converges to* (x^*, y^*) *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$

For the particular case of classical coupled fixed points (that is, $T_1(x,y) = F(x,y)$ and $T_2(x,y) = F(y,x)$, where $F : X^2 \to X$ is a given operator), the authors present a generalization of Bhaskar and Lakshmikantham's fixed point theorem in [**36**]:

Theorem 3.1.44. [**36**] *Let* $(X, d, \leq)$ *be an ordered complete metric space and let* $F : X^2 \to X$ *be two operators. We suppose that:*

i). for each $x = (x, y), w = (u, v) \in X^2$, which are not comparable with respect to the partial ordering "$\leq$" on X^2, there exists $t = (t_1, t_2) \in X^2$ such that t is comparable with both z and w;

ii). for all ($x \geq u$ and $y \leq v$) or ($x \leq u$ and $y \geq v$), we have

$$\begin{cases} F(x, y) \geq F(u, v) \\ F(y, x) \leq F(v, u) \end{cases}$$

or

$$\begin{cases} F(u, v) \geq F(x, y) \\ F(v, u) \leq F(y, x); \end{cases}$$

iii). $F : X^2 \to X$ is continuous;

iv). there exists $z_0 = (z_0^1, z_0^2) \in X^2$ such that

$$\begin{cases} z_0^1 \geq F(z_0^1, z_0^2) \\ z_0^2 \leq F(z_0^2, z_0^1) \end{cases}$$

or

$$\begin{cases} F(z_0^1, z_0^2) \geq z_0^1 \\ F(z_0^2, z_0^1) \leq z_0^2; \end{cases}$$

v). there exist $k_1, k_2 \in \mathbb{R}_+$, with $k_1 + k_2 < 1$ such that

$$d(F(x, y), F(u, v) \leq k_1 d(x, u) + k_2 d(y, v)$$

for all ($x \geq u$ and $y \leq v$) or ($x \leq u$ and $y \geq v$).

Then there exists a unique element $(x^*, y^*) \in X^2$ such that
$x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$
and the sequence of the successive approximations $(F^n(w_0^1, w_0^2), F^n(w_0^2, w_0^1))$ converges to (x^*, y^*), as $n \to \infty$, for all $w_0 = (w_0^1, w_0^2) \in X^2$.

1.2. Coupled fixed points in metric spaces endowed with a reflexive relation

Asgari and Mousavi [**9**] present some coupled fixed point results in the case of a metric space endowed with a reflexive binary relation. The definition for R-coupled fixed point they provide is the following:

DEFINITION 3.1.45. [**9**] *Let X be a nonempty set and let R be a reflexive relation on X, $F : X^2 \to X$. An element $(x, y) \in X^2$ is called $R-$coupled fixed point of F, if $F \times F(x, y) \in X_R(x, y)$, where $X_R(x, y) = \{(z, t) \in X^2 : zRx \wedge yRt\}$, $\forall (x, y) \in X^2$.*

If $R =''\leq''$, $F \times F(x_0, y_0) \in X_R(x_0, y_0)$, which is the condition for $(x, y) \in X^2$ to be an R-coupled fixed point of F, becomes $(x_0, y_0) \leq (F(x_0, y_0), F(y_0, x_0))$, that is, $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Motivated by this fact, we will use along this thesis the term of lower R-coupled fixed point for the concept of R-coupled fixed point defined in [**9**] by Asgari and Mousavi (see Definition 3.1.45).

Asgari and Mousavi , in [**9**], also redefine the mixed monotone property of a mapping, introduced by Bhaskar and Lakshmikantham [**36**], for the case of a partial ordered metric space endowed with a reflexive relation. This is given as follows:

Definition 3.1.46. [**9**] *Let X be a nonempty set and let R be a reflexive relation on X, $F : X^2 \to X$. The mapping F has the mixed $R-$monotone property on X if $F \times F(X_R(x, y)) \subseteq X_R(F \times F(x, y))$, for all $(x, y) \in X^2$, where $F \times G(x, y) = (F(x, y), G(y, x))$.*

The definition for an $R-$monotone sequence is the following:

Definition 3.1.47. [**9**] *A sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X^2$ is called $R-$monotone sequence if $(x_n, y_n) \in X_R(x_{n-1}, y_{n-1})$ for all $n \in \mathbb{N}$.*

We will also recall the definition of the coupled attractor basin element of a mapping, orbital continuity of a mapping and Picard operators, as presented in [**9**]:

Definition 3.1.48. [**9**] *Let X be a topological space and let $F : X^2 \to X$ be a mapping.*

- *Then an element $(x, y) \in X^2$ is called a coupled attractor basin element of F with respect to $(\overline{x}, \overline{y}) \in X^2$ if $F^n(x, y) \to \overline{x}$ and $F^n(y, x) \to \overline{y}$, as $n \to \infty$ and an element $x \in X$ is called an attractor basin element of F with respect to $\overline{x} \in X$, if $F^n(x, x) \to \overline{x}$, as $n \to \infty$. The set of all coupled attractor basin elements of F with respect to $(\overline{x}, \overline{y})$ will be denoted by $A_F(\overline{x}, \overline{y})$ and the set of all attractor basin elements of F with respect to $\overline{x} \in X$, by $A_F(\overline{x})$.*
- *The mapping F is called **orbitally continuous** if $(x, y), (a, b) \in X \times X$ and $F^{n_k}(x, y) \to a, F^{n_k}(y, x) \to b$, when $k \to \infty$, implies $F^{n_k+1}(x, y) \to F(a, b)$ and $F^{n_k+1}(y, x) \to F(b, a)$, when $k \to \infty$;*
- *The mapping F is called a Picard operator, if there exists $\overline{x} \in X$ such that $F_F = \{\overline{x}\}$ and $A_F(\overline{x}) = X$.*

In [**10**] Asgari and Mousavi also prove some coupled fixed point theorem with respect to a reflexive relation. Their result for orbitally continuous mappings is the following:

Theorem 3.1.49 ([**9**]). *Let (X, d) be a metric space and R a reflexive relation on X. If $F : X^2 \to X$ is a mapping such that:*

- *F has the mixed $R-$monotone property on X.*
- *(X,d) is a complete metric space.*
- *F has an $R-$coupled fixed point, that is there exists $(x_0,y_0) \in X^2$ such that $F \times F(x_0,y_0) \in X_R(x_0,y_0)$.*
- *there exists a constant $k \in [0,1)$ such that:*
$$d(F(x,y),F(z,t)) \leq \frac{k}{2}[d(x,z)+d(y,t)], \forall (x,y) \in X_R(z,t).$$
- *f is orbitally continuous.*

Then:

- *There exist $x^*, y^* \in X$ such that $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$.*
- *The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$ converge respectively to x^* and y^*, as $n \to \infty$.*
- *The error estimate is given by :*
$$\max_{n\in\mathbb{N}}\{d(x_n,x^*), d(y_n,y^*)\} \leq \frac{k^n}{2(1-k)}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)].$$

They also prove a result where they remove the orbital continuity of the mapping F:

THEOREM 3.1.50 ([**9**]). *Let (X,d) be a metric space and R a reflexive relation on X. If $F: X^2 \to X$ is a mapping such that:*

- *F has the mixed $R-$monotone property on X.*
- *(X,d) is a complete metric space.*
- *F has an $R-$coupled fixed point, that is, there exists $(x_0,y_0) \in X^2$ such that $F \times F(x_0,y_0) \in X_R(x_0,y_0)$.*
- *there exists a constant $k \in [0,1)$ such that:*
$$d(F(x,y),F(z,t)) \leq \frac{k}{2}[d(x,z)+d(y,t)], \forall (x,y) \in X_R(z,t).$$
- *if an $R-$monotone sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}} \to (x,y)$, then $(x_n,y_n) \in X_R(x,y)$, for all $n \in \mathbb{N}$.*

Then:

- *There exist $x^*, y^* \in X$ such that $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$.*
- *The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$ converge respectively to x^* and y^*.*
- *The error estimate is given by :*
$$\max_{n\in\mathbb{N}}\{d(x_n,x^*), d(y_n,y^*)\} \leq \frac{k^n}{2(1-k)}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)].$$

REMARK 3.1.51. *The compatibility condition between the metric d and the reflexive relation R used in Theorem 3.1.49 and Theorem 3.1.50 should be :*

If $x_n R y_n$ then $\lim_{n\to\infty} x_n R \lim_{n\to\infty} y_n, \forall n \in \mathbb{N}$.

It is also important to recall some of the notations presented by Asgari and Mousavi, notations used to obtain and prove the results presented in the following section:

NOTATION 1. [**9**] *Let X be a nonempty set and let $f : X \times X \to X$ be a mapping. Then*

(1) The cartesian product of f and itself is denoted by $f \times f$ and it is defined by

$$f \times f(x, y) = (f(x, y), f(y, x)).$$

(2) We will denote $f^0(x, y) = x$ and $f^n(x, y) = f(f^{n-1}(x, y), f^{n-1}(y, x))$, for all $x, y \in X, n \in \mathbb{N}$.

2. Definitions

In order to obtain some of the results in the following section, we need to define the orbital continuity of an operatorial pair (f_1, f_2), starting from the classical definition of orbital continuity in [**9**]:

DEFINITION 3.2.52. *Let X be a topological space and $f_1, f_2 : X \times X \to X$ two mappings. We say that the pair (f_1, f_2) is orbitally continuous if, for $(x, y), (a, b) \in X^2$ such that $f_1^{n_k}(x, y) \to a$ and $f_2^{n_k}(x, y) \to b$, we have $f_1^{n_k+1}(x, y) \to f_1(a, b)$ and $f_2^{n_k+1}(x, y) \to f_2(a, b)$, when $k \to \infty$.*

3. Existence and uniqueness theorems

The following result extends the coupled fixed point Theorem 3.1.49 of Asgari and Mousavi presented in Chapter 2, that is, Theorem 3.1.49, by replacing the original contractive condition (3.1.49) with a more general, symmetrical one. This type of condition was used by Berinde [**25**] in Theorem 3.1.37 in the extension of Bhaskar and Lakshmikantham's results.

THEOREM 3.3.53. [**55**],[**57**]

Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $F : X^2 \to X$ is a mapping such that

- *F has an lower-R-coupled fixed point;*
- *F has the the mixed R-monotone property on X;*
- *F is orbitally continuous;*
- *$\exists k \in [0, 1)$ such that*

$$d(F(x, y), F(z, t)) + d(F(y, x), F(t, z)) \leq k[d(x, z) + d(y, t)], \quad \forall (x, y) \in X_R(z, t). \tag{3.9}$$

Then:

(1) *F has a coupled fixed point, that is, $\exists(\overline{x},\overline{y}) \in X^2$ such that $F(\overline{x},\overline{y}) = \overline{x}$ and $F(\overline{y},\overline{x}) = \overline{y}$.*

(2) *The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$, converge to $\overline{x}$ and $\overline{y}$, respectively.*

(3) *The error estimation that holds is:*

$$\max_{n\in\mathbb{N}}\{d(x_n,\overline{x}), d(y_n,\overline{y})\} \leq \frac{k^n}{1-k}[d(F(x_0,y_0),x_0) + d(F(y_0,x_0),y_0)].$$

Proof: Since the mapping F admits a lower-R-coupled fixed point, let $(x_0, y_0) \in X \times X$ be it, we have $F \times F(x_0, y_0) \in X_R(x_0, y_0)$. Further, using the mixed R-monotone property of F, we have $F \times F(x_0, y_0) \in X_R(F(x_0, y_0), F(y_0, x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0), F^n(y_0,x_0)) \in X_R(F^{n-1}(x_0,y_0), F^{n-1}(y_0,x_0)). \tag{3.10}$$

We define $d_2 : X^2 \times X^2 \to \mathbb{R}_+$ $d_2(Y,Z) = \frac{1}{2}[d(x,z) + d(y,t)], \forall Y = (x,y), Z = (z,t) \in X^2$.

d_2 is a metric on X^2 because:

- $d_2(Y,Z) = 0 \Leftrightarrow Y = Z$ is a simple task to check , using the definition of d_2 and the fact that d is a metric.
- $d_2(Y,Z) = d_2(Z,Y), \forall Y,Z \in X^2$ holds, because d is a metric, and the sum in d_2's definition is commutative.
- $d_2(Y,Z) \leq d_2(Y,T) + d_2(T,Z), \forall Y,T,Z \in X^2$ can also be easily checked.

Therefore the metric space (X^2, d_2) is complete.

We consider the operator:

$T : X^2 \to X^2$ defined by $T(Y) = (F(x,y), F(y,x)), \forall Y = (x,y) \in X^2$.

For $Y = (x,y), Z = (z,t) \in X^2$, considering the definition for d_2, we have:

$$d_2(T(Y),T(Z)) = \frac{d(F(x,y),F(z,t)) + d(F(y,x),F(t,z))}{2}$$

and

$$d_2(Y,Z) = \frac{d(x,z) + d(y,t)}{2}.$$

By the contractivity condition (3.9) we have

$$d_2(T(Y),T(Z)) \leq k \cdot d_2(Y,Z), \forall Y,Z \in X^2, Y \in X_R(Z). \tag{3.11}$$

Denote $Z_0 = (x_0, y_0) \in X^2$ and consider the sequence $\{Z_n\}_{n\geq 0} \subset X^2$, defined by $Z_{n+1} = T(Z_n), n \geq 1$, where $Z_n = (x_n, y_n) \in X^2, n \geq 1$.This means $Z_n = (F^n(x_0,y_0), F^n(y_0,x_0))$, Since F has the mixed R-monotone property on X, we have

$$T(X_R(Z_0)) \subset X_R(T(Z_0)).$$

But $T(Z_0) = Z_1$, so, by induction, we have $T(X_R(Z_n)) \subset X_R(Z_{n+1})$. Next, we denote $Y = Z_n \geq Z_{n-1} = V$.
We replace this in (3.11), obtaining:

$$d_2(T(Z_n), T(Z_{n-1})) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1 \Leftrightarrow$$

$$\Leftrightarrow d_2(Z_{n+1}, Z_n) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1.$$

Using the induction, we have:

$$d_2(Z_{n+1}, Z_n) \leq k^n \cdot d_2(Z_1, Z_0), n \geq 1.$$

Let $i < j$. We get:

$$d_2(Z_i, Z_j) \leq \sum_{l=i+1}^{j} d_2(Z_l, Z_{l-1}) \leq (k^i + k^{i+1} + ... + k^{j-i-1}) \cdot d_2(Z_1, Z_0) \leq$$

$$\leq k^i \frac{1 - k^{j-i-1}}{1-k} \cdot d_2(Z_1, Z_0) \tag{3.12}$$

$\Rightarrow \{Z_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space $(X^2, d_2) \Rightarrow$

$$\Rightarrow \lim_{n \to \infty} Z^n = \overline{Z}.$$

So, $T(\overline{Z}) = \overline{Z} \Leftrightarrow (F(\overline{x}, \overline{y}), F(\overline{y}, \overline{x})) = (\overline{x}, \overline{y}) \Leftrightarrow F(\overline{x}, \overline{y}) = \overline{x}, F(\overline{y}, \overline{x}) = \overline{y} \Leftrightarrow (\overline{x}, \overline{y})$ is the coupled fixed point for F.
Since the considered metric space is complete , $\exists \quad \overline{x}, \overline{y} \in X$ such that $F^n(x_0, y_0) \to \overline{x}$, $F^n(y_0, x_0) \to \overline{y}, \quad n \to \infty$. Using the fact that F is orbitally continuous, we have:

$$\{x_n\}_{n \in \mathbb{N}} \to \overline{x}, x_{n+1} = F(x_n, y_n)$$

$$\{y_n\}_{n \in \mathbb{N}} \to \overline{y}, y_{n+1} = F(y_n, x_n)$$

So, by (3.12) we have:

$$d_2((x_n, y_n), (\overline{x}, \overline{y})) \leq \frac{k^n}{1-k} \cdot d_2((x_1, y_1), (x_0, y_0)), n \geq 0.$$

We return to the original metric d:

$$\frac{d(x_n, \overline{x}) + d(y_n, \overline{y})}{2} \leq \frac{k^n}{1-k} \cdot \frac{d(x_1, x_0) + d(y_1, y_0)}{2} \Leftrightarrow$$

$$\Leftrightarrow d(x_n, \overline{x}) + d(y_n, \overline{y}) \leq \max_{n \in \mathbb{N}}\{d(x_n, \overline{x}), d(y_n, \overline{y})\} \leq \frac{k^n}{1-k} \cdot [d(x_1, x_0) + d(y_1, y_0)].$$

But $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. We get:

$$\max_{n \in \mathbb{N}}\{d(x_n, \overline{x}), d(y_n, \overline{y})\} \leq \frac{k^n}{1-k} \cdot [d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)].$$

□

Next, let us recall the definition of a mapping γ, called comparison function, from [**122**]: Let $\gamma : [0, \infty) \to [0, \infty)$ satisfying :

i) γ is monotone increasing;

ii) $\lim_{n\to+\infty} \gamma^n(t) = 0, \forall t \in [0,\infty)$;

Introducing in the right hand of the contractive condition of Theorem 2.6 in [**9**] this mapping instead of constant k, we obtain the following result, with a more general contractive condition:

THEOREM 3.3.54. *Let (X,d) be a complete metric space, γ a comparison function, R be a binary reflexive relation on X such that R and d are compatible. If $F : X^2 \to X$ is a mapping such that*

(i) F has the mixed R-monotone property;

(ii) F is orbitally continuous;

(iii)

(3.13)

$$d(F(x,y),F(z,t)) \leq \gamma\left(\frac{d(x,z)+d(y,t)}{2}\right), \forall (x,y) \in X_R(z,t), \text{ where } \gamma \text{ is a comparison function};$$

(iv) F has a lower-R-coupled fixed point;

Then F has a coupled fixed point, that is, there exists $(x,y) \in X^2$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof: Since the mapping F admits a lower-R-coupled fixed point, let $(x_0,y_0) \in X \times X$ be it, we have $F \times F(x_0,y_0) \in X_R(x_0,y_0)$. Further, using the mixed R-monotone property of F, we have $F \times F(x_0,y_0) \in X_R(F(x_0,y_0),F(y_0,x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0),F^n(y_0,x_0)) \in X_R(F^{n-1}(x_0,y_0),F^{n-1}(y_0,x_0)). \tag{3.14}$$

Now, we claim that for $n \in \mathbb{N}$, we have

$$d(F^{n+1}(x_0,y_0),F^n(x_0,y_0)) \leq \gamma^n\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right) \tag{3.15}$$

(3.16) and

$$d(F^{n+1}(y_0,x_0),F^n(y_0,x_0)) \leq \gamma^n\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right)$$

For $n = 1$, we get

$$d(F^2(x_0,y_0),F(x_0,y_0)) \leq \gamma\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right)$$

and

$$d(F^2(y_0,x_0),F(y_0,x_0)) \leq \gamma\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right).$$

Next, we assume (3.15) holds. Using (3.13), we have

$$d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0))$$

$$= d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^n(x_0,y_0),F^n(y_0,x_0)))$$

$$\leq \gamma\left(\frac{d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))+d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))}{2}\right)$$

$$\gamma^{n+1}\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right)$$

Similarly, we can prove that

$$d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)) \leq$$

$$\gamma^{n+1}\left(\frac{d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)}{2}\right)$$

But condition ii holds (i.e. $\lim_{n\to+\infty}\gamma^n(t) = 0, \forall t \in [0,\infty)$), so $\{F^n(x_0,y_0)\}_{n\in\mathbb{N}}$ and $\{F^n(y_0,x_0)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X,d).
Since (X,d) is complete, there exist $x,y\in X$ such that $F^n(x_0,y_0)\to x$ and $F^n(y_0,x_0)\to y$, as $n\to\infty$. Since F is orbitally continuous, we have that $F^{n+1}(x_0,y_0)\to F(x,y)$ and $F^{n+1}(y_0,x_0)\to F(y,x)$, so the proof of the theorem is complete. □

Next, let us replace the γ function with a more general one, the function φ introduced in [**53**] by Ćirić and Lakshmikantham: Let $\varphi : [0,\infty)\to[0,\infty)$ satisfying :

i) $\varphi(t) < t, \forall t \in (0,\infty)$;
ii) $\lim_{r\to t_+}\varphi(r) < t, \forall t\in(0,\infty)$;

The set of all these mappings φ is denoted by Φ.

THEOREM 3.3.55. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $F : X^2\to X$ is a mapping such that*

(i) F has the mixed R-monotone property;
(ii) F is orbitally continuous;
(iii)

$$d(F(x,y),F(z,t)) \leq \varphi\left(\frac{d(x,z)+d(y,t)}{2}\right), \forall (x,y)\in X_R(z,t), \tag{3.17}$$

where $\varphi\in\Phi$;
(iv) F has a lower-R-coupled fixed point;

Then F has a coupled fixed point, that is, there exists $(x,y)\in X^2$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof: Since F has a lower-R-coupled fixed point, let (x_0, y_0) be it. Thus, $(F \times F)(x_0, y_0) \in X_R(x_0, y_0)$.
From (i) we have that $(F \times F)(X_R(x_0, y_0)) \subseteq X_R((F \times F)(x_0, y_0))$.
Further, it can easily be checked that

$$(F^n(x_0, y_0), F^n(y_0, x_0)) \in X_R(F^{n-1}(x_0, y_0), F^{n-1}(y_0, x_0)). \tag{3.18}$$

Next, let $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n) \tag{3.19}$$

Let's consider the nonnegative sequence $\{z_n\}_{n \in \mathbb{N}^*}$ such that $z_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n), n \in \mathbb{N}^*$.
Now, using (3.17), (3.18) and letting $x := x_n$ and $y := y_n$, $z := x_{n-1}$ and $t := y_{n-1}$, we obtain

$$d(x_{n+1}, x_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq \varphi\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\right) =$$
$$\varphi\left(\frac{z_{n-1}}{2}\right)$$

and

$$d(y_{n+1}, y_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \leq \varphi\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\right) =$$
$$\varphi\left(\frac{z_{n-1}}{2}\right).$$

By summing up the last two relations, we get that

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) = z_n \leq 2 \cdot \varphi\left(\frac{z_{n-1}}{2}\right).$$

Now, using (i) from the definition of φ, we have that

$$z_n \leq 2 \cdot \varphi\left(\frac{z_{n-1}}{2}\right) < 2 \cdot \frac{z_{n-1}}{2} = z_{n-1}. \tag{3.20}$$

Thus, the sequence $\{z_n\}_{n \in \mathbb{N}^*}$ is decreasing and nonnegtive. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n \to \infty} z_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (3.20), let $n \to \infty$. Using (ii) (the second condition satisfied by φ), we have

$$\varepsilon_0 = \lim_{n \to \infty} z_n \leq 2 \cdot \lim_{n \to \infty} \varphi\left(\frac{z_{n-1}}{2}\right) = 2 \cdot \lim_{z_{n-1} \to \varepsilon_0 +} \varphi\left(\frac{z_{n-1}}{2}\right) < \varepsilon_0,$$

which is a contradiction. Thus, $\lim_{n\to\infty} z_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim\limits_{n\to\infty} d(y_{n+1}, y_n) = 0$.

Next, we will prove that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(x_{n_2(k)}, x_{n_1(k)}) + d(y_{n_2(k)}, y_{n_1(k)}), \tag{3.21}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{N}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (3.21). Then, we have

$$d(x_{n_2(k)}, x_{n_1(k)-1}) + d(y_{n_2(k)}, y_{n_1(k)-1}) < \delta. \tag{3.22}$$

Now, using the triangle inequality and the last two inequalities ((3.21) and (3.22)), we have

$$\delta \leq d(x_{n_2(k)}, x_{n_1(k)}) + d(y_{n_2(k)}, y_{n_1(k)})$$

$$\leq d(x_{n_1(k)}, x_{n_1(k)-1}) + d(y_{n_1(k)}, y_{n_1(k)-1}) + d(x_{n_2}(k), x_{n_1}(k)) + d(y_{n_2(k)}, y_{n_1(k)})$$

$$\leq d(x_{n_2(k)}, x_{n_1(k)}) + d(y_{n_2(k)}, y_{n_1(k)}) + \delta.$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(x_{n_2(k)}, x_{n_1(k)}) + d(y_{n_2(k)}, y_{n_1(k)})] = \delta.$$

Now we will show that $\delta = 0$. Supposing the contrary, we have

$$s_k = d(x_{n_2(k)}, x_{n_1(k)}) + d(y_{n_2(k)}, y_{n_1(k)})$$

$$\leq d(x_{n_1(k)}, x_{n_1(k)+1}) + d(x_{n_1(k)+1}, x_{n_2(k)}) + d(y_{n_1(k)}, y_{n_1(k)+1}) + d(y_{n_1(k)+1}, y_{n_2(k)})$$

$$= z_{n_1(k)} + d(x_{n_1(k)+1}, x_{n_2(k)}) + d(y_{n_1(k)+1}, y_{n_2(k)})$$

$$\leq z_{n_1(k)} + z_{n_2(k)} + d(x_{n_1(k)+1}, x_{n_2(k)+1}) + d(y_{n_1(k)+1}, y_{n_2(k)+1}). \tag{3.23}$$

But

$$d(x_{n_1(k)+1}, x_{n_2(k)+1}) + d(y_{n_1(k)+1}, y_{n_2(k)+1})$$

$$= d(F(x_{n_1(k)}, y_{n_1(k)}), F(x_{n_2(k)}, y_{n_2(k)})) + d(F(y_{n_1(k)}, x_{n_1(k)}), F(y_{n_2(k)}, x_{n_2(k)}))$$

$$\leq 2 \cdot \varphi \left(\frac{d(x_{n_1(k)}, x_{n_2(k)}) + d(y_{n_1(k)}, y_{n_2(k)})}{2} \right)$$

$$\leq 2 \cdot \varphi \left(\frac{s_k}{2} \right).$$

Now, returning to (3.23), we have

$$s_k \leq z_{n_1(k)} + z_{n_2(k)} + 2 \cdot \varphi \left(\frac{s_k}{2} \right).$$

Let $k \to \infty$. Using ii (the second property of φ), we obtain

$$\delta \leq 2 \cdot \lim_{k\to\infty} \varphi\left(\frac{s_k}{2}\right) = 2 \cdot \lim_{s_k\to\delta_+} \varphi\left(\frac{s_k}{2}\right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since X is complete, there exist $\overline{x}$ and $\overline{y}$ such that $x_n \to \overline{x}$ and $y_n \to \overline{y}$ as $n \to \infty$. Which means that $F^{n-1}(x_n, y_n) \to \overline{x}$ and $F^{n-1}(y_n, x_n) \to \overline{y}$, as $n \to \infty$. Using the orbital continuity of F, we get that $F^n(x_n, y_n) \to \overline{x}$ and $F^n(y_n, x_n) \to \overline{y}$, as $n \to \infty$, that is, $(\overline{x}, \overline{y})$ is a coupled fixed point for F. □

THEOREM 3.3.56. *In addition to the hypothesis of Theorem 3.3.54, if for every $(x, y), (x^*, y^*) \in X^2$, there exists $(u, v) \in X^2$ such that $(x, y), (x^*, y^*) \in X_R(u, v)$, then F has a unique coupled fixed point.*

Proof: The proof of this Theorem follows the steps of Theorem 2.9 in [**9**]. According to the proof of Theorem 3.3.54, there exists a coupled fixed point for the mapping F. Let $(x^*, y^*) \in X^2$ be it. In order to prove the uniqueness of this point, we have to show that $A_F(x^*, y^*) = X^2$, where A_F is the set of coupled attractor basin elements introduced in Definition 3.1.48.
From the hypothesis of Theorem 3.3.54, we know that F has the mixed R-monotone property on X and it admits a lower-R-coupled fixed point. Hence, we have that

$$F \times F(X_R(x, y)) \subseteq X_R(F \times F(x, y)), \forall (x, y) \in X$$

and

$$F \times F(x, y)) \in X_R(x, y)(\text{ that is } (x, y) \text{ lower} - R - \text{coupled fixed point})$$

. Next, let $(x, y) \in X^2$ be arbitrary and $x^* = x_0$ and $y^* = y_0$. Thus, there exists $(u, v) \in X^2$ such that $(x, y), (x_0, y_0) \in X_R(u, v)$.
Since (X, d) is a complete metric space, we have that

$$(F^n((x_0, y_0), F^n(y_0, x_0)) \in X_R(F^n(u, v), F^n(v, u)), n \in \mathbb{N}$$

So the pair $(F^n(x_0, y_0), F^n(y_0, x_0))$ is eligible to satisfy the contractive condition (3.13). In consequence, similarly to the proof of Theorem 3.3.54, we have:

$$d(F^n(x_0, y_0), F^n(u, v)) \leq \gamma^n \left(\frac{d(x_0, u) + d(y_0, v)}{2}\right) \tag{3.24}$$

and

$$d(F^n(y_0, x_0), F^n(v, u)) \leq \gamma^n \left(\frac{d(x_0, u) + d(y_0, v)}{2}\right). \tag{3.25}$$

From the proof of Theorem 3.3.54 we have that $F^n(x_0,y_0)\to x^*$ and $F^n(y_0,x_0)\to y^*$, when $n\to\infty$. Thus, $(x_0,y_0)\in X^2$ is a coupled attractor basin element for F, that is $(x_0,y_0)\in A_F(x^*,y^*)$. If, in addition to this, we consider the last two inequalities (3.24) and (3.25), it follows that $(u,v)\in A_F(x^*,y^*)$.
From the hypothesis, we have that $(x,y)\in X_R(u,v)$, where $(x,y)\in X^2$ arbitrary. This implies that $A_F(x^*,y^*)=X^2$, so the proof of the theorem is complete. □

REMARK 3.3.57. *Note that for $\gamma(t)=kt$ in Theorem 3.3.54, we obtain Asgari and Mousavi's fixed point Theorem 2.6 in* [**9**].

We may obtain a more general result than the one given by Theorem 3.3.55, by considering a symmetric contractive condition.

THEOREM 3.3.58. *Let (X,d) be a complete metric space, γ a comparison function, R be a binary reflexive relation on X such that R and d are compatible. If $F:X^2\to X$ is a mapping such that*

(i) F is mixed R-monotone;
(ii) F is orbitally continuous;
(iii)

$$d(F(x,y),F(z,t))+d(F(y,x),F(t,z))\le 2\cdot\gamma\left(\frac{d(x,z))+d(y,t)}{2}\right), \tag{3.26}$$

$$\forall (x,y)\in X_R(z,t), \textit{ where } \gamma \textit{ is a comparison function;}$$

(iv) F has a lower-R-coupled fixed point.

Then F has a coupled fixed point, that is, there exists $(x,y)\in X^2$ such that $F(x,y)=x$ and $F(y,x)=y$.

Proof: Since F has a lower-R-coupled fixed point, let (x_0,y_0) be it.
Thus, $(F\times g)(x_0,y_0)\in X_R(x_0,y_0)$. Further, using the mixed R-monotone property of F, we have
$F\times F(x_0,y_0)\in X_R(F(x_0,y_0),F(y_0,x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0),F^n(y_0,x_0))\in X_R(F^{n-1}(x_0,y_0),F^{n-1}(y_0,x_0)). \tag{3.27}$$

Since $F(X^2)\subseteq X$, let $x_1,y_1\in X$ such that $x_1=F(x_0,y_0)$, $y_1=F(y_0,x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1}=F(x_n,y_n), y_{n+1}=F(y_n,x_n) \tag{3.28}$$

Now, we claim that for $n \in \mathbb{N}$, we have

$$d(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \gamma^n \left(\frac{d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)}{2} \right) \tag{3.29}$$

$$d(F^{n+1}(y_0, x_0), F^n(y_0, x_0)) \leq \gamma^n \left(\frac{d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)}{2} \right)$$

For $n = 1$, we get

$$d(F^2(x_0, y_0), F(x_0, y_0)) \leq \gamma \left(\frac{d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)}{2} \right)$$

and

$$d(F^2(y_0, x_0), F(y_0, x_0)) \leq \gamma \left(\frac{d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)}{2} \right).$$

Next, we assume (3.29) holds. By summing up and using (3.26), we have

$$d(F^{n+2}(x_0, y_0), Fn+1(x_0, y_0)) + d(F^{n+2}(y_0, x_0), F^{n+1}(y_0, x_0))$$

$$\leq 2\gamma \left(\frac{d(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) + d(F^{n+1}(y_0, x_0), F^n(y_0, x_0))}{2} \right)$$

$$2\gamma^{n+1} \left(\frac{d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)}{2} \right)$$

Now, using condition (ii) (i.e. $\lim_{n \to +\infty} \gamma^n(t) = 0, \forall t \in [0, \infty))$, we have that $\{F^n(x_0, y_0)\}_{n \in \mathbb{N}}$ and $\{F^n(y_0, x_0)\}_{n \in \mathbb{N}}$ are Cauchy sequences in X.
Since (X, d) is complete, there exist $x, y \in X$ such that $F^n(x_0, y_0) \to x$ and $F^n(y_0, x_0) \to y$, as $n \to \infty$. Since F is orbitally continuous, we have that $F^{n+1}(x_0, y_0) \to F(x, y)$ and $F^{n+1}(y_0, x_0) \to F(y, x)$ and by (3.28) the proof of the theorem is complete. □

Similarly to Theorem 3.3.54, by adding one more assumption to the hypothesis of Theorem 3.3.58, we obtain the uniqueness of the coupled fixed point. The proof of the result is similar to the proof of Theorem 3.3.56.

THEOREM 3.3.59. *In addition to the hypothesis of Theorem 3.3.58, if for every* $(x, y), (x^*, y^*) \in X^2$, *there exists* $(u, v) \in X^2$ *such that* $(x, y), (x^*, y^*) \in X_R(u, v)$, *then* F *has a unique coupled fixed point.*

Proof: The proof of this result also follows the steps of Theorem 2.9 in [**9**]. According to the proof of Theorem 3.3.58, there exists a coupled fixed point of the mapping F. Let $(x^*, y^*) \in X^2$ be it. In order to prove the uniqueness of this point, we have to show that $A_F(x^*, y^*) = X^2$, where A_F is the set of coupled attractor basin elements introduced in Definition 3.1.48.

From the hypothesis of Theorem 3.3.58, we know that F has the mixed R-monotone property on X and it admits a lower-R-coupled fixed point. Hence, we have that

$$F \times F(X_R(x,y)) \subseteq X_R(F \times F(x,y)), \forall (x,y) \in X$$

and

$$F \times F(x,y)) \in X_R(x,y)(that is (x,y) \text{ lower} - R - \text{coupled fixed point})$$

. Next, let $(x,y) \in X^2$ be arbitrary and $x^* = x_0$ and $y^* = y_0$. Thus, there exists $(u,v) \in X^2$ such that $(x,y),(x_0,y_0) \in X_R(u,v)$.
Since (X,d) is a complete metric space, we have that

$$(F^n((x_0,y_0), F^n(y_0,x_0)) \in X_R(F^n(u,v), F^n(v,u)), n \in \mathbb{N}$$

Moreover, we know that the pair $(F^n(x_0,y_0), F^n(y_0,x_0))$ satisfies the contractive condition (3.26). In consequence, similarly to the proof of Theorem 3.3.58, we have:
(3.30)

$$d(F^n(x_0,y_0), F^n(u,v))+d(F^n(y_0,x_0), F^n(v,u) \leq 2\gamma^n \left(\frac{d(x_0,u)+d(y_0,v)}{2}\right) \to 0, \text{ as } n \to \infty.$$

From the proof of Theorem 3.3.58 we have that $F^n(x_0,y_0) \to x^*$ and $F^n(y_0,x_0) \to y^*$, when $n \to \infty$. Thus, $(x_0,y_0) \in X^2$ is a coupled attractor basin element for F, that is $(x_0,y_0) \in A_F(x^*,y^*)$. If, in addition to this, we consider the last inequality (3.30), it follows that $(u,v) \in A_F(x^*,y^*)$.
From the hypothesis, we have that $(x,y) \in X_R(u,v)$, where $(x,y) \in X^2$ arbitrary. This implies that $A_F(x^*,y^*) = X^2$, so the proof of the theorem is complete. □

REMARK 3.3.60. *By letting* $\gamma(t) = kt$ *in* (3.26) *from Theorem 3.3.58, we obtain Theorem 3.3.53.*

The next theorem extends the results of Urs from [**144**] (see Theorem 2.6) in the case of metric space endowed with a reflexive relation.

THEOREM 3.3.61. [**56**] *Let* (X,d) *be a metric space and* R *a reflexive relation on* X *such that* R *and* d *are compatible. If* $f_1, f_2 : X^2 \to X$ *are two mappings such that:*

(i) f_1, f_2 *have the mixed* $R-$*monotone property on* X*.*
(ii) (X,d) *is a complete metric space.*
(iii) there exists $(x_0,y_0) \in X^2$ *such that* $f_1 \times f_2(x_0,y_0) \in X_R(x_0,y_0)$*.*
(iv) there exists a constant $k \in [0,1)$ *such that:*

$$d(f_1(x,y), f_1(z,t)) + d(f_2(x,y), f_2(z,t)) \leq k \cdot [d(x,z) + d(y,t)], \forall (x,y) \in X_R(z,t).$$

(v) the pair (f_1,f_2) *is orbitally continuous.*

Then:

(i) There exist $x^*, y^* \in X$ *such that* $f_1(x^*,y^*) = x^*$ *and* $f_2(x^*,y^*) = y^*$*.*

(ii) The sequences $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *defined by* $x_{n+1} = f_1(x_n, y_n)$ *and* $y_{n+1} = f_2(x_n, y_n)$ *converge respectively to* x^* *and* y^**.*

(iii) The error estimation is given by:

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)}[d(f_1(x_0, y_0), x_0) + d(f_2(x_0, y_0), y_0)].$$

Proof: From (iii), we have that the pair (f_1, f_2) admits a lower-R-coupled fixed point; let $(x_0, y_0) \in X \times X$ be it, we have $f_1 \times f_2(x_0, y_0) \in X_R(x_0, y_0)$. Further, using the mixed R-monotone property of f_1, f_2, we have $f_1 \times f_2(x_0, y_0) \in X_R(F(x_0, y_0), F(y_0, x_0))$. Using the induction, we can easily prove that:

$$(f_1^n(x_0, y_0), f_2(x_0, y_0)) \in X_R(f_1^{n-1}(x_0, y_0), f_2^{n-1}(x_0, y_0)). \tag{3.31}$$

We define $d_2 : X^2 \times X^2 \to \mathbb{R}_+$ $d_2(Y, Z) = \frac{1}{2}[d(x, z) + d(y, t)], \forall Y = (x, y), Z = (z, t) \in X^2$.

d_2 is a metric on X^2 because:

- $d_2(Y, Z) = 0 \Leftrightarrow Y = Z$ is a simple task to check , using the definition of d_2 and the fact that d is a metric.
- $d_2(Y, Z) = d_2(Z, Y), \forall Y, Z \in X^2$ holds, because d is a metric, and the sum in d_2's definition is commutative.
- $d_2(Y, Z) \leq d_2(Y, T) + d_2(T, Z), \forall Y, T, Z \in X^2$ can also be easily checked.

Therefore (X^2, d_2) is a complete metric space.
We consider the operator:
$T : X^2 \to X^2$ defined by $T(Y) = (f_1(x, y), f_2(x, y)), \forall Y = (x, y) \in X^2$.
For $Y = (x, y), Z = (z, t) \in X^2$, considering the definition for d_2, we have:

$$d_2(T(Y), T(Z)) = \frac{d(f_1(x, y), f_1(z, t)) + d(f_2(x, y), f_2(z, t))}{2}$$

and

$$d_2(Y, Z) = \frac{d(x, z) + d(y, t)}{2}.$$

By the contractivity condition (iv) we have

$$d_2(T(Y), T(Z)) \leq k \cdot d_2(Y, Z), \forall Y, Z \in X^2, Y \geq Z. \tag{3.32}$$

Denote $Z_0 = (x_0, y_0) \in X^2$ and consider the sequence $\{Z_n\}_{n\geq 0} \subset X^2$, defined by $Z_{n+1} = T(Z_n), n \geq 1$, where $Z_n = (x_n, y_n) \in X^2, n \geq 1$.This means $Z_n = (f_1^n(x_0, y_0), f_2^n(x_0, y_0))$, Since f_1, f_2 have the mixed R-monotone property on X, we have

$$T(X_R(Z_0)) \subset X_R(T(Z_0)).$$

But $T(Z_0) = Z_1$, so, by induction, we have $T(X_R(Z_n)) \subset X_R(Z_{n+1})$. We denote $Y = Z_n \geq Z_{n-1} = V$.

We replace this in (3.32), obtaining:

$$d_2(T(Z_n), T(Z_{n-1})) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1 \Leftrightarrow$$

$$\Leftrightarrow d_2(Z_{n+1}, Z_n) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1.$$

Using the induction, we have:

$$d_2(Z_{n+1}, Z_n) \leq k^n \cdot d_2(Z_1, Z_0), n \geq 1.$$

Let $i < j$. We get:

$$d_2(Z_i, Z_j) \leq \sum_{l=i+1}^{j} d_2(Z_l, Z_{l-1}) \leq (k^i + k^{i+1} + ... + k^{j-i-1}) \cdot d_2(Z_1, Z_0) \leq$$

$$\leq k^i \frac{1 - k^{j-i-1}}{1-k} \cdot d_2(Z_1, Z_0) \tag{3.33}$$

$\Rightarrow \{Z_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space $(X^2, d_2) \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} Z^n = Z^*.$$

We now use (3.31): $T(Z^*) = Z^* \Leftrightarrow (f_1(x^*, y^*), f_2(y^*, x^*)) = (x^*, y^*) \Leftrightarrow f_1(x^*, y^*) = x^*, f_2(x^*, y^*) = y^* \Leftrightarrow (x^*, y^*)$ is the coupled fixed point for the pair (f_1, f_2).
Since (X, d) is a complete metric space, $\exists \quad x^*, y^* \in X$ such that $f_1^n(x_0, y_0) \to x^*$, $f_2^n(x_0, y_0) \to y^*, \quad n \to \infty$. Using the last assumption in the hypothesis, we have:

$$\{x_n\}_{n\in\mathbb{N}} \to x^*, x_{n+1} = f_1(x_n, y_n)$$

$$\{y_n\}_{n\in\mathbb{N}} \to y^*, y_{n+1} = f_2(x_n, y_n)$$

So, by (3.33) we have:

$$d_2((x_n, y_n), (x^*, y^*)) \leq \frac{k^n}{1-k} \cdot d_2((x_1, y_1), (x_0, y_0)), n \geq 0.$$

We return to the original metric d:

$$\frac{d(x_n, x^*) + d(y_n, y^*)}{2} \leq \frac{k^n}{1-k} \cdot \frac{d(x_1, x_0) + d(y_1, y_0)}{2} \Leftrightarrow$$

$$\Leftrightarrow d(x_n, x^*) + d(y_n, y^*) \leq \max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{1-k} \cdot [d(x_1, x_0) + d(y_1, y_0)].$$

But $x_{n+1} = f_1(x_n, y_n)$ and $y_{n+1} = f_2(x_n, y_n)$. We get:

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{1-k} \cdot [d(f_1(x_0, y_0), x_0) + d(f_2(x_0, y_0), y_0)].$$

□

The following result establishes the conditions under which we obtain the uniqueness of the coupled fixed point. It is sufficient to suppose that there exists a couple (r, s) in X^2 such that any other two arbitrary coupled in X^2 are in the reflexive relation R with (r, s).

THEOREM 3.3.62. [**56**] *In addition to the hypothesis of Theorem 3.3.61, we suppose that, for every* $(x,y),(x_0,y_0)\in X^2$*, there exists* $(r,s)\in X^2$ *such that* $(x,y),(x_0,y_0)\in X_R(r,s)$*. Then, the pair* (f_1,f_2) *admits a unique fixed point.*

Proof: From Theorem 3.3.61 it follows that there exists $x^*,y^*\in X$ such that $f_1(x^*,y^*)=x^*, f_2(x^*,y^*)=y^*$.
The next step is to show that $A_(f_1,f_2)(x^*,y^*)=X\times X$.
Let $(x,y)\in X^2$. Since f_1,f_2 have the mixed $R-$monotone property on X, then there exists $(r,s)\in X^2$ such that $(x,y),(x_0,y_0)\in X_R(r,s)$. From $(x_0,y_0)\in X_R(r,s)$ and the fact that (X,d) is a complete metric space, it follows that for $n\in\mathbb{N}$

$$(f_1^n(x_0,y_0),f_2^n(x_0,y_0))\in X_R(f_1^n(r,s),f_2^n(r,s)).$$

From the sixth assumption of Theorem 3.3.61 (that is, for $(x,y),(a,b)\in X^2$ such that $f_1^{n_k}(x,y)\to a$ and $f_2^{n_k}(x,y)\to b$, we have $f_1^{n_k+1}(x,y)\to f_1(a,b)$ and $f_2^{n_k+1}(x,y)\to f_2(a,b)$, when $k\to\infty$) we have:

$$d(f_1^n(x_0,y_0),f_1^n(r,s))\le k^n\cdot[d(x_0,r)+d(y_0,s)],$$

and

$$d(f_2^n(x_0,y_0),f_2^n(r,s))\le k^n\cdot[d(x_0,r)+d(y_0,s)].$$

Now, using the fact that $(x_0,y_0)\in A_(f_1,f_2)(x^*,y^*)$, it follows that $(r,s)\in A_(f_1,f_2)(x^*,y^*)$.
Thus, $A_(f_1,f_2)(x^*,y^*)=X^2$.
Therefore the pair(f_1,f_2) admits a unique fixed point. □

It is important to note that the results presented are extensions of important results in the field.

REMARK 3.3.63. *If, in Theorem 3.3.61 we take* $f_1(x,y)=F(x,y)$ *and* $f_2(x,y)=F(y,x)$*, we obtain Theorem 3.3.53.*

REMARK 3.3.64. *If, in Theorem 3.3.62 we take* $f_1(x,y)=F(x,y)$ *and* $f_2(x,y)=F(y,x)$*, we obtain Theorem 3.3.54.*

REMARK 3.3.65. *If, in Theorem 3.3.61, we take* $f_1(x,y)=F(x,y)$ *and* $f_2(x,y)=F(y,x)$ *and the contractive condition is replaced by*

$$d(F(x,y),F(z,t))\le\frac{k}{2}\left[d(x,z)+d(y,t)\right],\forall(x,y)\in X_R(z,t),$$

we obtain Theorem 3.1.49 of Asgari and Mousavi in [**9**].

REMARK 3.3.66. *If, in Theorems 3.3.61 and 3.3.62, we endow the metric space with a relation of partial order (instead of the reflexive relation), we obtain similar results to the ones obtained by Urs, Petruşel and Petruşel in* [**108**], [**143**] *and* [**144**].

REMARK 3.3.67. *If, in Theorems 3.3.61 and 3.3.62 we take $f_1(x,y) = F(x,y)$ and $f_2(x,y) = F(y,x)$ and R is the partial order on X we obtain Theorems 3.1.37 and 3.1.38, that is, the results in Berinde* [**25**].

4. Examples and applications

4.1. Examples

Theorem 3.3.53 is more general than the fixed point theorem of Asgari and Mousavi [**9**] because of the new contractivity condition, fact illustrated by the following example.

EXAMPLE 3.4.68. *Let $X = \mathbb{R}$, the metric $d(x,y) = |x-y|$, the relation R on X given by*

$$xRy \Leftrightarrow \frac{x^2-2y}{3} = \frac{y^2-2x}{3}.$$

Let $F : X^2 \to X$ be defined by $F(x,y) = \dfrac{x-2y}{3}$. So, $\forall (x,y) \in X^2$, F has the mixed $R-$monotone property, satisfies (3.9)*, but does not satisfy the contractivity condition in Asgari and Mousavi's fixed point theorem* [**9**]*. Indeed, assume that there exists $k \in [0,1)$ such that* (3.1.49) *holds. This means*

$$\left|\frac{x-2y}{3} - \frac{z-2t}{3}\right| \leq \frac{k}{2}[|x-z| + |y-t|], \forall (x,y) \in X_R(z,t)$$

by which, for $x = z$, we get

$$\frac{2}{3}|y-t| \leq \frac{k}{2}|y-t|$$

which, for $y \neq t$, would imply $\dfrac{4}{3} \leq k$, where $k < 1$, a contradiction. Now, we prove that (3.9) *holds. We have*

$$\left|\frac{x-2y}{3} - \frac{z-2t}{3}\right| \leq \frac{1}{2}|x-z| + \frac{1}{3}|y-t|, \forall (x,y) \in X_R(z,t)$$

and

$$\left|\frac{y-2x}{3} - \frac{t-2z}{3}\right| \leq \frac{1}{2}|y-t| + \frac{1}{3}|x-z|, \forall (x,y) \in X_R(z,t).$$

By summing up these two relations, we get (3.9) *with $k = \dfrac{5}{6} < 1$.*

On the other hand,

$$\frac{x^2-2y}{3} = \frac{y^2-2x}{3}$$
$$\Leftrightarrow (x-y)(x+y+2) = 0.$$

Thus, we have $X_R(x,y) = \{(x,y)(x,-y-2),(-x-2,y),(-x-2,-y-2)\}$*. Also,* $F\times F(0,0)\in X_R(0,0)$*. So, by Theorem 3.3.53 we obtain that* $(0,0)$ *is a coupled fixed point of* F*, and, moreover, it is unique.*

EXAMPLE 3.4.69. *Let* $X=\mathbb{R}$*, the metric* $d(x,y)=|x-y|$*, the relation* R *on* X *given by*

$$xRy \Leftrightarrow x^2+4y=y^2+4x.$$

Let $F: X^2\to X$ *be defined by* $F(x,y)=\dfrac{x-2y}{4}$*. So,* $\forall (x,y)\in X^2$*,* F *has the mixed* $R-$*monotone property, satisfies* (3.26)*, but does not satisfy the contractivity condition in Asgari and Mousavi's fixed point Theorem 2.6 in* [**9**].
Let's suppose that there exists $k\in[0,1)$ *such that* (3.1.49) *holds. This means*

$$\left|\frac{x-2y}{4}-\frac{u-2v}{4}\right| \le \frac{k}{2}\left(|x-u|+|y-v|\right), \forall (x,y)\in X_R(u,v).$$

Letting $x=u$ *in the inequality above, we have*

$$\left|\frac{y-v}{2}\right| \le \frac{k}{2}|y-v|$$

which implies

$$1\le k,\ \text{which is a contradiction}\ .$$

Next, we prove that (3.26) *holds. We have*

$$\left|\frac{x-2y}{4}-\frac{u-2v}{4}\right| \le \frac{1}{4}|x-u|+\frac{1}{2}|y-v|, \forall (x,y)\in X_R(u,v)$$

and

$$\left|\frac{y-2x}{4}-\frac{v-2u}{4}\right| \le \frac{1}{4}|y-v|+\frac{1}{2}|x-u|, \forall (x,y)\in X_R(u,v).$$

By summing up, we obtain

$$\left|\frac{x-2y}{4}-\frac{u-2v}{4}\right| + \left|\frac{y-2x}{4}-\frac{v-2u}{4}\right| \le \frac{3}{4}\left(|x-u|+|y-v|\right),$$

which is exactly (3.26) *for* $\gamma(t)=\dfrac{3t}{8}$*. Note that* $\gamma:[0,\infty)\to[0,\infty), \gamma(t)=\dfrac{3t}{8}$ *satisfies conditions i and ii, thus* γ *is, indeed, a comparison function.*

On the other hand,

$$x^2+4y=y^2+4x$$

$$\Leftrightarrow (x-y)(x+y-4)=0.$$

Thus, we have $X_R(x,y)=\{(x,y)(x,4-y),(4-x,y),(4-x,4-y)\}$*. Also,* $F\times F(0,0)\in X_R(0,0)$*. So, by Theorem 3.3.58, we obtain that* F *has a (unique) coupled fixed point* $(0,0)$*, but Theorem 3.1.49 cannot be applied to* F *in this particular example.*

4.2. Applications

In this section we will study the existence and uniqueness of the solution of a first-order periodic boundary value system, as an application to the results presented in the previous section.

In a similar context, Berinde in [**25**], Bhaskar and Lakshmikantham in [**36**], Lakshmikantham and Guo in [**70**], [**71**] and Urs in [**145**], [**143**] also studied the existence and uniqueness of solutions for a periodic boundary value problem, in the framework of a partially ordered metric space. In this case, we will endow the metric space with a reflexive relation.

Let's denote the reflexive relation by "R" on $C(I) \times C(I)$ and let there be $z := (x, y)$ and $w := (u, v)$ two arbitrary elements of $C(I) \times C(I)$. Then, by definition, $z \in X_R(w) \Leftrightarrow x \leq u$ and $y \geq v$.

It can easily be checked that $(x, x) \in X_R(x, x)$ and if $(x, y) \in X_R(u, v)(i.e., z \in X_R(w))$ and $(u, v) \in X_R(x, y)(i.e., w \in X_R(z))$, we have $z = w$, but the proprerty of transitivity (necessary for R to be a relation of order) does not hold in this case.

Let's consider the periodic boundary value system studied in [**144**]:

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)), \quad \forall t \in I := [0, T] \\ x(0) = x(T) \\ y(0) = y(T) \end{cases} \tag{3.34}$$

where $T > 0$ and $f_1, f_2 : I \times \mathbb{R}^2 \to \mathbb{R}$. We also suppose that :

C.1 there exist $\lambda, \mu_1, \mu_2, \mu_3, \mu_4 > 0$, $\dfrac{\mu_1 + \mu_2}{1 - \mu_3 - \mu_4} < 1$ such that

$$0 \leq [f_1(t, x, y) + \lambda x] - [f_1(t, u, v) + \lambda u] \leq \lambda[\mu_1(x - u) + \mu_2(y - v)] - \lambda[\mu_3(x - u) + \mu_2(y - v)] \leq [f_2(t, x, y) + \lambda x] - [f_2(t, u, v) + \lambda u] \leq 0,$$

$\forall t \in I$ and $x, y, u, v \in \mathbb{R}$, where f_1, f_2 are two continuous functions.

C.2 for each $z = (x, y)$ and $w = (u, v) \in C(I) \times C(I)$, if $z \in X_R(w)$ or $w \in X_R(z)$, we have:

$$\begin{cases} f_2(t, x, y)] \leq f_2(t, u, v) \\ f_1(t, x, y) \geq f_1(t, u, v) \end{cases}$$

or

$$\begin{cases} f_2(t, u, v) \leq f_2(t, x, y) \\ f_1(t, u, v) \geq f_1(t, x, y) \end{cases}$$

C.3 there exists $z_0 := (z_0^1, z_0^2) \in C(I) \times C(I)$ such that:

$$\begin{cases} z_0^1(t) \leq f_1(t, z_0^1(t), z_0^2(t)) \\ z_0^2(t) \geq f_2(t, z_0^1(t), z_0^2(t)) \end{cases}$$

or

$$\begin{cases} f_1(t, z_0^1(t), z_0^2(t)) \leq z_0^1(t) \\ f_2(t, z_0^1(t), z_0^2(t)) \geq z_0^2(t) \end{cases}$$

C.4 the following inequalities hold:

$$\begin{cases} (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^1(s) ds \geq z_0^1(t) \\ (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^2(s) ds \leq z_0^2(t), \forall t \in I. \end{cases}$$

We recall that the problem (see [**144**],[**36**], [**125**]),

$$\begin{cases} x'(t) = h(t) \\ x(0) = x(T), t \in I, \end{cases}$$

where $h \in C(I)$ and $x \in C^1(I)$, is equivalent, for some $\lambda \neq 0$ to

$$x(t) = \int_0^T G_\lambda(t, s)[h(s) + \lambda x(s)] ds, \forall t \in I,$$

where $G_\lambda(t, s)$ is defined (see [**144**]) by:

$$G_\lambda(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, 0 \leq s \leq t \leq T \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, 0 \leq t \leq s \leq T \end{cases}$$

Thus, we have that the system (3.34) is equivalent to the coupled fixed point problem :

$$\begin{cases} x = F_1(x, y) \\ y = F_2(x, y), \end{cases}$$

where $F_1, F_2 : X^2 \to X, X = C(I)$,

$$F_1(x,y)(t) = \int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s)) + \lambda x(s)]ds$$
$$F_2(x,y)(t) = \int_0^T G_\lambda(t,s)[f_2(s,x(s),y(s)) + \lambda y(s)]ds$$

In order to apply the results presented in the previous section, we consider the complete metric space (X,d), where $X = C(I,\mathbb{R}$ and the metric d is induced by the sup-norm on X,

$$d(u,v) = \sup_{t\in I} |u(t) - v(t)|, \forall u, v \in C(I).$$

We also have to link the problem introduced above to the theoretical results recalled and presented; consequently, if $(x,y) \in X^2$ is a coupled point of F, then we have $x(t) = F_1(x,y)(t)$ and, similarly, $y(t) = F_2(x,y)(t)$, $\forall t \in I$, where $F := (F_1, F_2)$.

Theorem 3.4.70. [**59**] *Consider the problem* (3.34) *under the assumptions (1)-(4). Then there exists a unique solution* (x^*, y^*) *of the BVP* (3.34).

Proof: In order to reach the conclusion of this results, we will apply Theorem 3.3.61.For this, we have to verify all the assumptions of this Theorem:
We have that (X,d) is a complete metric space, so (ii) from Theorem 3.3.61 is verified. From the first condition (1), $0 \leq [f_1(t,x,y) + \lambda x] - [f_1(t,u,v) + \lambda u] \leq \lambda[\mu_1(x-u) + \mu_2(y-v)] - \lambda[\mu_3(x-u) + \mu_2(y-v)] \leq [f_2(t,x,y) + \lambda x] - [f_2(t,u,v) + \lambda u] \leq 0$, we have that

$$|[F_1(x,y)(t) - F_1(u,v)(t)| =$$
$$\left| \int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s)) + \lambda x(s)]ds - \int_0^T G_\lambda(t,s)[f_2(s,u(s),v(s)) + \lambda u(s)]ds \right|$$
$$= \left| \int_0^T G_\lambda[(t,s)[f_1(s,x(s),y(s)) - f_1(s,u(s),v(s)) + \lambda x(s) - \lambda u(s)]ds \right|$$
$$\leq \lambda \int_0^T G_\lambda(t,s) \left|(\mu_1(x(s) - u(s))\right| + |\mu_2(y(s) - v(s))|)ds$$
$$\leq \mu_1 d(x,u) + \mu_2 d(y,v).$$

Applying $\sup_{t\in I}$, we get:

$$d(F_1(x,y), F_1(u,v)) \leq \mu_1 d(x,u) + \mu_2 d(y,v). \tag{3.35}$$

In a similar way, we get

$$d(F_2(x,y), F_2(u,v)) \leq \mu_3 d(x,u) + \mu_4 d(y,v). \tag{3.36}$$

Summing up relations (3.35) and (3.36), we get:

$$d(F_1(x,y), F_1(u,v)) + d(F_2(x,y), F_2(u,v)) \leq (\mu_1 + \mu_3)d(x,u) + (\mu_2 + \mu_4)d(y,v),$$

where $\mu_1 + \mu_2 + \mu_3 + \mu_4 < 1$ follows from condition (1). Consequently, (iv) from Theorem 3.3.61 is verified.

From the second condition (2) we have that $(f_1(t,x,y), f_2(t,x,y)) \in X_R(f_1(t,u,v), f_2(t,u,v)$, $\forall w \in X_R(z), z = (x,y), w = (u,v)$ which is equivalent to $f_1 \times f_2(t,x,y) \in X_R(f_1 \times f_2(t,u,v),(t,u,v))$. Thus f_1 and f_2 have the mixed $R-$monotone property on X, so (i) from Theorem 3.3.61 is also checked. In a similar way we prove the mixed $R-$monotone property of f_1 and f_2 using the other pair of assumptions in condition (2).

Since f_1, f_2 have the mixed $R-$monotone property on X, then there exists $(r,s) \in X^2$ such that $(x,y),(x_0,y_0) \in X_R(r,s)$, so the additional assumption of 3.3.62 is verified.

Now, from the third condition (3), $z_0^1(t) \leq f_1(t, z_0^1(t), z_0^2(t))$ and $z_0^2(t) \geq f_2(t, z_0^1(t), z_0^2(t))$, where $z_0 = (z_0^1, z_0^2)$ we obtain that $(f_1(t, z_0^1(t), z_0^2(t)) \in X_R(z_0^1, z_0^2) \leftrightarrow f_1 \times f_2(t, z_0^1(t), z_0^2(t)) \in X_R(z_0^1, z_0^2)$. It follows that there exists a coupled fixed point, namely $z_0 = (z_0^1, z_0^2) \in X^2$, for the pair (f_1, f_2)(the third hypothesis of Theorem 3.3.61).

Further, it can be easily checked that, for any $n \in \mathbb{N}$,

$$(f_1^n(t, z_0^1(t), z_0^2(t)), f_2^n(t, z_0^1(t), z_0^2(t))) \in X_R(f_1^{n-1}(t, z_0^1(t), z_0^2(t)), f_2^{n-1}(t, z_0^1(t), z_0^2(t))).$$

Using this and the continuity of f_1 and f_2, it can be easily proved that $\{f_1^n(t, z_0^1(t), z_0^2(t))\}_{n\in\mathbb{N}}$ and $\{f_1^n(t, z_0^1(t), z_0^2(t))\}_{n\in\mathbb{N}}$ are Cauchy sequences in X, so the last hypothesis of Theorem 3.3.61 is also checked. Thus, we get that the periodic boundary problem (3.34) has a unique solution in $C(I) \times C(I)$. □

Remark 3.4.71. *If, in Theorem 3.4.70 we take $R := \leq$, we obtain Theorem 3.2 in* [**144**].

Remark 3.4.72. *If, in Theorem 3.4.70, we take $R := \leq$ and $f_1(x,y) = F(x,y)$ and $f_2(x,y) = F(y,x)$, we obtain Theorem 3.7 in* [**36**].

CHAPTER 4

Tripled fixed point theorems in metric spaces endowed with a reflexive relation

In this chapter, we will extend the results of Asgari and Mousavi [**9**], using the tripled fixed points concept introduced by Berinde and Borcut in [**28**], which are briefly presented in the following. We will also extend the notions of coupled attractor basin element, orbital continuity, mixed $R-$monotony of a mapping, R-coupled fixed point in the case of tripled fixed points.

1. Preliminaries

1.1. Tripled fixed points of mixed-monotone operators in partially ordered metric spaces

Definition 4.1.73. [**28**] *Let $(X, \leq)$ be a partially ordered space and $F : X^3 \to X$. We say that the operator F has the mixed-monotone property on X if $F(x, y, z)$ is monotone nondecreasing in x and z and it is monotone nonincreasing in y, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z)$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

$$z_1, z_2 \in X, z_2 \leq z_1 \Rightarrow F(x, y, z_2) \geq F(x, y, z_1)$$

Definition 4.1.74. [**40**] *Let X, Y, Z be three nonempty sets and $F : X^3 \to Y$, $G : Y \times Y \times Y \to Z$. We define the symmetrical composition (or $s-$composition) of F and G, $F * G : X^3 \to Z$, by*

$$(G * F)(x, y, z) = G(F(x, y, z), F(y, x, y), F(z, y, x)), x, y, z \in X.$$

Definition 4.1.75. [**28**] *We call an element $(x, y, z) \in X \times X$ a tripled fixed point of the mixed-monotone mapping $F : X^3 \to X$, if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*
If $x = y = z$ and, in consequence, $F(x, x, x) = x$, then $x \in X$ is a fixed point of F.

Remark 4.1.76. *Note that in the definition above (y, x, y) is not a permutation of (x, y, z), like (z, y, x).*

Let (X,d) be a complete metric space. The mapping $\widetilde{d} : X^3 \to X$, given by

$$\widetilde{d}((x,y,z),(u,v,w)) = d(x,u) + d(y,v) + d(z,w)$$

defines a metric on X^3, which will be denoted, for convenience by d, too.

The following theorem is the main result in [**39**] and it establishes the existence of a tripled fixed point of a mixed monotone operator:

Theorem 4.1.77. [**28**],[**40**],[**41**] *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that the metric space (X,d) is complete. Let $T : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist $k,l,m \in [0,1), k+l+m < 1$ such that*

$$d(T(x,y,z),T(u,v,t)) \leq k \cdot d(x,u) + l \cdot d(y,v) + m \cdot d(z,t), \forall x \geq u, y \leq v, z \geq t.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq T(x_0,y_0,z_0), y_0 \geq T(y_0,x_0,z_0) \text{ and } z_0 \leq T(z_0,y_0,x_0).$$

Then there exist $x,y,z \in X$ such that $x = F(x,y,z), y = F(y,x,y)$ and $z = F(z,y,x)$.

The conclusion of Theorem 4.1.77 holds, even if the mapping is not continuous, by adding some supplementary conditions to the hypothesis, conditions first introduced by Nieto and R. Rodríguez-López in [**95**].

Theorem 4.1.78. [**40**] *Let $(X,\leq)$ be a partially ordered space and suppose there exists a metric d on X, such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mixed monotone operator. Suppose there exists a constant $k \in [0,1)$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq \frac{k}{3}[d(x,u) + d(y,v) + d(z,w)] \forall x \geq u, y \leq v, z \geq w. \tag{4.37}$$

Let's suppose that the following properties hold on X:

(i.)

$$\text{if there exists an increasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x \text{ for every } n,$$

(ii.)

$$\text{if there exists a decreasing sequence } \{y_n\} \to y, \text{ then } y_n \geq y \text{ for every } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0), y_0 \geq F(y_0,x_0,y_0) \text{ and } z_0 \leq F(z_0,y_0,x_0),$$

then there exist $x,y,z \in X$, such that

$$x = F(x,y,z), y = F(y,x,y) \text{ and } z = F(z,y,x).$$

The conclusion of the result holds, even if the contractive condition (4.37) is replaced by :

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,w\right)\text{ for all }x\geq u,y\leq v,z\geq w,$$

$$\text{where } j,k,l\in[0,1), j+k+l<1.$$

THEOREM 4.1.79. [**41**] *Let $(X,\leq)$ be a partially ordered space and suppose there exists a metric d on X, such that (X,d) is a complete metric space. Let $F:X\times X\times X\to X$ be a mixed monotone operator. Suppose there exists the constants $j,k,l\in[0,1)$, where $j+k+l<1$ such that*

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right)\leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,\right)\text{ for all }x\geq u,y\leq v,z\geq w.$$

Let's suppose that the following properties hold on X:

(i.)

$$\text{if there exists an increasing sequence }\{x_n\}\to x,\text{ then }x_n\leq x\text{ for every }n,$$

(ii.)

$$\text{if there exists a decreasing sequence }\{y_n\}\to y,\text{ then }y_n\geq y\text{ for every }n.$$

If there exist $x_0,y_0,z_0\in X$ such that

$$x_0\leq F(x_0,y_0,z_0),y_0\geq F(y_0,x_0,y_0)\text{ and }z_0\leq F(z_0,y_0,x_0),$$

then there exist $x,y,z\in X$ such that

$$x=F(x,y,z),y=F(y,x,y)\text{ and }z=F(z,y,x).$$

Several results regarding the uniqueness of the tripled fixed points of mixed monotone continuous or non-continuous mappings were obtained by Borcut, Berinde and Păcurar, in [**28**], [**42**], [**39**], [**40**], [**41**], by endowing the metric space with an additional property regarding the relation of order, by considering that every triple consisting in elements from X has an upper bound and a lower bound. We present the following one:

THEOREM 4.1.80. [**28**] *In addition to the hypothesis of Theorem 4.1.77 if we have that for $(x,y,z),(x_1,y_1,z_1)\in X\times X\times X$, there exists $(u,v,w)\in X\times X\times X$ comparable to (x,y,z) and to (x_1,y_1,z_1), then there exists a unique fixed point of F.*

1.2. Tripled fixed points of monotone operators in partially ordered metric spaces

First, we will present the definitions for monotone mappings and tripled fixed points for these mappings, concepts introduced by Borcut in [**39**].

DEFINITION 4.1.81. [**39**] *Let* $(X, \leq)$ *be a partially ordered space and the mapping* $F : X^3 \to X$. *We say that* F *is monotone, if* $F(x, y, z)$ *is monotone increasing in* x, y *and* z, *that is, for every* $x, y, z, x_1, x_2, y_1, y_2, z_1, z_2$ *we have*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \leq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

DEFINITION 4.1.82. [**39**] *An element* $(x, y, z) \in X^3$ *is a tripled fixed point of the monotone mapping* $F : X^3 \to X$, *if*

$$F(x, y, z) = x, F(y, x, z) = y, \text{ and } F(z, y, x) = z.$$

If $x = y = z$ *and, in consequence,* $F(x, x, x) = x$, *we say that* $x \in X$ *is a fixed point of* F.

REMARK 4.1.83. [**39**] *We can remark the fact that, in this context, the tripled fixed point is different than in the case of mixed-monotone mappings, that is, the triples* $(x, y, z), (y, x, z)$ *and* (z, y, x) *appearing in 4.1.82 are all permutations of* (x, y, z).

One of the results regarding the existence of these tripled fixed points presented in [**39**] is the following:

THEOREM 4.1.84. [**39**] *Let* $(X, \leq)$ *be a partially ordered space and* d *be a metric on* X, *such that the metric space* (X, d) *is complete. Let* $F : X^3 \to X$ *be a continuous, monotone mapping on* X. *Let's suppose that there exists a constant* $k \in [0, 1)$, *such that*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)] \tag{4.38}$$

$$\text{for all } x \geq u, y \leq v, z \geq w.$$

If there exist $x_0, y_0, z_0 \in X$, *such that*

$$x_0 \leq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, z_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$, *such that*

$$x = F(x, y, z), y = F(y, x, z) \text{ and } z = F(z, y, x).$$

The result regarding the uniqueness of the tripled fixed point in this case is the following:

THEOREM 4.1.85. [**39**] *In addition to the hypothesis of Theorem 4.1.84 we have: for any* $(x, y, z), (x_1, y_1, z_1) \in X \times X \times X$, *there exists* $(u, v, w) \in X \times X \times X$ *comparable to* (x, y, z) *and* (x_1, y_1, z_1), *then* F *has a unique tripled fixed point.*

In this case too, the conclusion of the theorem holds for non-continuous mappings:

THEOREM 4.1.86. [**39**] *Let $(X,\leq)$ a partially ordered space and d be a metric on X, such that (X,d) is a complete metric space. Let $F : X^3 \to X$ a monotone operator on X. Suppose that there exists a constant $k \in [0,1)$, such that*

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq \frac{k}{3}\left[d\left(x,u\right)+d\left(y,v\right)+d\left(z,w\right)\right]$$

for all $x \geq u, y \leq v, z \geq w$. Suppose that we have the following property on X:

if there exists the increasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for every n.

If there exist $x_0, y_0, z_0 \in X$, such that

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \leq F\left(y_0,x_0,z_0\right) \text{ and } z_0 \leq F\left(z_0,y_0,x_0\right),$$

then there exist $x,y,z \in X$, such that

$$x = F(x,y,z), y = F(y,x,z) \text{ and } z = F(z,y,x).$$

2. Tripled fixed points of mixed-monotone operators

2.1. Definitions

The following concepts are extensions of the notions presented in [**9**] in the case of coupled fixed points in the framework of tripled fixed points of mixed monotone mappings.

NOTATION 2. *Let X be a nonempty set and let $F : X \times X \times X \to X$ be a mapping. Then*

(1) The cartesian product of F and itself is denoted by $F \times F$ and it is defined by

$$F \times F(x,y,z) = (F(x,y,z), F(y,x,y), F(z,y,x)).$$

(2) We will denote by $F^0(x,y,z) = x$ and $F^n(x,y,z) = F(F^{n-1}(x,y,z),$ $F^{n-1}(y,x,y), F^{n-1}(z,y,x))$, for all $x,y,z \in X, n \in \mathbb{N}$.

(3) The set X_R has the same meaning as it has in X^2: $X_R(x,y,z) = \{(t,u,v) \in X \times X \times X : tRx \wedge yRu \wedge vRz\}$.

DEFINITION 4.2.87. [**57**] *Let X be a nonempty set and let R be a reflexive relation on X, $F : X^3 \to X$. The mapping F has the **mixed $R-$monotone property** on X if*

$$F \times F(X_R(x,y,z)) \subseteq X_R(F \times F(x,y,z)), \text{ for all } (x,y,z) \in X^3.$$

DEFINITION 4.2.88. [**57**] *An element $(x,y,z) \in X^3$ is called **lower-$R-$tripled fixed point** of F, if $F \times F(x,y,z) \in X_R(x,y,z)$.*

DEFINITION 4.2.89. *A sequence* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}} \subseteq X^3$ *is called* $R-$***monotone sequence*** *if* $(x_n, y_n, z_n) \in X_R(x_{n-1}, y_{n-1}, z_{n-1})$.

DEFINITION 4.2.90. [**60**] *Let* X *be a topological space and* $F : X^3 \to X$ *be a mixed R-monotone mapping. Then an element* $(x, y, z) \in X^3$ *is called a* ***tripled attractor basin element*** *of* F *with respect to* $(x^*, y^*, z^*) \in X^3$, *if* $F^n(x, y, z) \to x^*$, $F^n(y, x, y) \to y^*$ *and* $F^n(z, y, x) \to z^*$, *as* $n \to \infty$. *The set of these points* (x^*, y^*, z^*) *will be denoted by* $A_F(x^*, y^*, z^*)$, *and, when* $x^* = y^* = z^*$, *by* $A_F(x^*)$.

DEFINITION 4.2.91. [**57**] *Let* X *be a topological space and* $F : X^3 \to X$ *be a mixed R-monotone mapping. The mapping* F *is called* ***orbitally continuous*** *if* $(x, y, z), (a, b, c) \in X^3$ *and* $F^{n_k}(x, y, z) \to a, F^{n_k}(y, x, y) \to b, F^{n_k}(z, y, x) \to c$, *when* $k \to \infty$, *implies* $F^{n_k+1}(x, y, z) \to F(a, b, c)$, $F^{n_k+1}(y, x, y) \to F(b, a, b)$ *and* $F^{n_k+1}(z, y, x) \to F(c, b, a)$, *as* $k \to \infty$.

2.2. Uniqueness and existence theorems

The following result establishes the existence of a tripled fixed point of the mapping $F : X^3 \to X$.

THEOREM 4.2.92. [**57**] *Let* (X, d) *be a complete metric space and* R *be a binary reflexive relation on* X *such that* R *and* d *are compatible. If* $F : X^3 \to X$ *is a mapping such that*

(i) F *has a lower-R-tripled fixed point;*
(ii) F *has the mixed R-monotone property on* X;
(iii) F *is orbitally continuous;*
(iv) there exists $k \in [0, 1)$ *such that*

$$d(F(x, y, z), F(t, u, v)) \leq \frac{k}{3} \cdot [d(x, t) + d(y, u) + d(z, v)], \qquad \forall (x, y, z) \in X_R(t, u, v), k \in [0, 1). \tag{4.39}$$

Then:

(1) F *has a tripled fixed point, that is,* $\exists(\overline{x}, \overline{y}, \overline{z}), \in X^3$ *such that* $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{y}) = \overline{y}$, $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, *defined by* $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, y_n), z_{n+1} = F(z_n, y_n, x_n)$, *converge to* $\overline{x}$,$\overline{y}$ *and* $\overline{z}$, *respectively.*
(3) The error estimation that holds is: $\max_{n\in\mathbb{N}}\{d(x_n, \overline{x}), d(y_n, \overline{y}), d(z_n, \overline{z})\}$ $\leq \dfrac{k^n}{3(1-k)}[d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)]$.

Proof: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0, y_0, z_0) \in X\times X\times X$ be it, we have $F\times F(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$. Further, using the mixed R-monotone property of F, we have $F\times F(x_0, y_0, z_0) \in X_R(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0)) \in$$

$$\in X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0), F^{n-1}(z_0, y_0, x_0)). \tag{4.40}$$

We claim that, for $n \in \mathbb{N}$

$$d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) \leq$$

$$\frac{k^n}{3} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)]. \tag{4.41}$$

For $n = 1$, we get:

$$d(F^2(x_0, y_0, z_0), F(x_0, y_0, z_0)) = d(F(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)),$$

$$F(x_0, y_0, z_0)) \leq \frac{k}{3} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)].$$

Now, we assume that (4.41) holds. Using (iv) from the hypothesis we get:

$$d(F^{n+2}(x_0, y_0, z_0), F^{n+1}(x_0, y_0, z_0)) =$$

$$d(F(F^{n+1}(x_0, y_0, z_0), F^{n+1}(y_0, x_0, y_0), F^{n+1}(z_0, y_0, x_0)),$$

$$F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0))) \leq$$

$$\leq \frac{k}{3} \cdot [d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) +$$

$$d(F^{n+1}(y_0, x_0, y_0), F^n(y_0, x_0, y_0)) + d(F^{n+1}(z_0, y_0, x_0), F^n(z_0, y_0, x_0)] \leq$$

$$\leq \frac{k^{n+1}}{3} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), x_0)] \to 0, asn \to \infty.$$

This implies that $\{F^n(x_0, y_0, z_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Similarly, following the same steps, we can prove that $\{F^n(y_0, x_0, y_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0, y_0, x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since X is a complete metric space, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F^n(x_0, y_0, z_0) \to \overline{x}, F^n(y_0, x_0, y_0) \to \overline{y}, F^n(z_0, y_0, x_0) \to \overline{z}, \quad n \to \infty$.
Let $m > n$. Then:

$$d(F^m(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) \leq \sum_{j=n}^{m-1} d(F^{j+1}(x_0, y_0, z_0), F^j(x_0, y_0, z_0)) \leq$$

$$\leq (k^{n-1} + k^n + ... + k^{m-n}) \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)]$$

$$= \frac{k^n - k^m}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)] <$$

$$< \frac{k^n}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)].$$

But $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, x_n, y_n)$ and $z_{n+1} = F(z_n, y_n, x_n)$, $F^n(x_0, y_0, z_0) \to \overline{x}, F^n(y_0, x_0, y_0) \to \overline{y}, F^n(z_0, y_0, x_0) \to \overline{z}, \quad n \to \infty$ and F is orbitally continuous. Thus, applying maximum to the last relation, we get:

$$\max_{n\in\mathbb{N}}\{d(x_n, \overline{x}), d(y_n, \overline{y}), d(z_n, \overline{z})\} \le \frac{k^n}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) +$$

$$+ d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0)].$$

□

REMARK 4.2.93. *If in Theorem 4.2.92 we know that for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z),*
$(x_0, y_0, z_0) \in X_R(r, s, t)$, we can also prove the uniqueness of the tripled fixed point.

Thus, the following result comes as a completion of Theorem 4.2.92, adding to the conclusion the uniqueness of the tripled fixed point.

THEOREM 4.2.94. [**60**] *In addition to the hypothesis of Theorem 4.2.92, we suppose that, for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z),*
$(x_0, y_0, z_0) \in X_R(r, s, t)$. Then, F is a Picard operator.

Proof: From Theorem 4.2.92 it follows that there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{y}) = \overline{y}, F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
The next step is to show that $A_F(\overline{x}, \overline{y}, \overline{z}) = X \times X \times X$.
Let $(x, y, z) \in X^3$. Since F has the mixed $R-$monotone property on X, then there exists $(r, s, t) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(r, s, t)$. From $(x_0, y_0, z_0) \in X_R(r, s, t)$ and the fact that (X, d) is a complete metric space, it follows that for $n \in \mathbb{N}$

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0))$$

$$\in X_R(F^n(r, s, t), F^n(s, r, s), F^n(t, s, r)).$$

But F is orbitally continuous, so we have:

$$d(F^n(x_0, y_0, z_0), F^n(r, s, t)) \le \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)],$$

$$d(F^n(y_0, x_0, y_0), F^n(s, r, s)) \le \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)]$$

and

$$d(F^n(z_0, y_0, x_0), F^n(t, s, r)) \le \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)].$$

Now, using the fact that $(x_0, y_0, z_0) \in A_F(\overline{x}, \overline{y}, \overline{z})$, it follows that $(r, s, t) \in A_F(\overline{x}, \overline{y}, \overline{z})$.
Thus, $A_F(\overline{x}, \overline{y}, \overline{z}) = X^3$.

It is also true that $(\overline{y},\overline{x},\overline{y}),(\overline{z},\overline{y},\overline{x})\in A_F(\overline{x},\overline{y},\overline{z})$. Thus, we have $\overline{x}=\overline{y}=\overline{z}$. Therefore $A_F(\overline{x})=X$, so F is a Picard operator. □

In [**96**] and [**95**] Nieto and Rodríguez-López endow the metric space X with a regularity condition which assumes the existence of two monotonic sequences, one nonincreasing and the other nondecreasing, as presented in Theorems 2.1.18, 2.1.19, 2.1.20, 2.1.21 from Chapter 2. In the case of tripled fixed points of mixed$-R-$monotone, not necessarily orbitally continuous operators, the result we obtain is the following:

THEOREM 4.2.95. *Let (X,d) be a complete metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F:X^3\to X$ is a mapping having such that*

(i) F has a lower-R-tripled fixed point;
(ii) F has the mixed R-monotone property on X;
(iii) if an R-monotone sequence $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}\to(x,y,z)$, then $(x_n,y_n,z_n)\in X_R(x,y,z)$, for all $n\in\mathbb{N}$;
(iv) there exists $k\in[0,1)$ such that

$$d(F(x,y,z),F(t,u,v))\le\frac{k}{3}\cdot[d(x,t)+d(y,u)+d(z,v)],\quad\forall(x,y,z)\in X_R(t,u,v),k\in[0,1).\tag{4.42}$$

Then F has a tripled fixed point, that is, $\exists(\overline{x},\overline{y},\overline{z}),\in X^3$ such that $F(\overline{x},\overline{y},\overline{z})=\overline{x},F(\overline{y},\overline{x},\overline{y})=\overline{y}$, $F(\overline{z},\overline{y},\overline{x})=\overline{z}$.

Proof: Following the steps of the proof of Theorem 4.2.92, we only have to prove that $\exists(\overline{x},\overline{y},\overline{z})\in X^3$ such that $F(\overline{x},\overline{y},\overline{z})=\overline{x},F(\overline{y},\overline{x},\overline{y})=\overline{y}$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. Since $F^n(x_0,y_0,z_0)\to\overline{x}$, $F^n(y_0,x_0,y_0)\to\overline{y}$ and $F^n(z_0,y_0,x_0)\to\overline{z}$, using (iii) we get

$$d(F(\overline{x},\overline{y},\overline{z}),\overline{x})\le d(F(\overline{x},\overline{y},\overline{z}),F^{n+1}(x_0,y_0,z_0))+$$

$$d(F^{n+1}(x_0,y_0,z_0),\overline{x})=d(F(\overline{x},\overline{y},\overline{z}),F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0))+$$

$$d(F^{n+1}(x_0,y_0,z_0),\overline{x})\le\frac{k}{3}[d(\overline{x},F^n(x_0,y_0,z_0))+d(\overline{y},F^n(y_0,x_0,y_0))$$
$$+d(\overline{z},F^n(z_0,y_0,x_0))]+d(F^{n+1}(x_0,y_0,z_0),\overline{x})\to0,\ \text{as } n\to\infty.$$

This implies that $F(\overline{x},\overline{y},\overline{z})=\overline{x}$. Similarly, we can prove that $F(\overline{y},\overline{x},\overline{y})=\overline{y}$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. □

In the next result, we obtain the same conclusion as in Theorem 4.2.92, replacing the orbital continuity of F with an assumption on some R-monotone sequences in X^3:

THEOREM 4.2.96. *Let (X,d) be a metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F:X^3\to X$ is a mapping such that*

(i) F has a lower-R-tripled fixed point;

(ii) F has the mixed R-monotone property on X;

(iii) there exists $k \in [0,1)$ such that, $\forall (x,y,z) \in X_R(t,u,v)$:

$$d(F(x,y,z),F(t,u,v)) \leq \frac{k}{3} \cdot [d(x,t)+d(y,u)+d(z,v)] \tag{4.43}$$

(iv) if an $R-$monotone sequence $\{(x_n,y_n,z_n)\} \to (x,y,z)$, then $(x_n,y_n,z_n) \in X_R(x,y,z)$, for all $n \in \mathbb{N}$.

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x},\overline{y},\overline{z}), \in X^3$ such that $F(\overline{x},\overline{y},\overline{z}) = \overline{x}, F(\overline{y},\overline{x},\overline{y}) = \overline{y}$, $F(\overline{z},\overline{y},\overline{x}) = \overline{z}$.

(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n,y_n,z_n), y_{n+1} = F(y_n,x_n,y_n), z_{n+1} = F(z_n,y_n,x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

(3) The error estimation that holds is: $\max_{n\in\mathbb{N}}\{d(x_n,\overline{x}),d(y_n,\overline{y}),d(z_n,\overline{z})\}$ $\leq \frac{k^n}{3(1-k)}[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,y_0),y_0)+d(F(z_0,y_0,x_0),z_0)]$.

Proof: The proof of this theorem follows the steps of the proof of Theorem 4.2.92. The only thing left to show is the existence of the tripled fixed point (conditioned by the orbital continuity of F in Theorem 4.2.92.)
From the proof of Theorem 4.2.92, we have that $F^n(x_0,y_0,z_0) \to \overline{x}$, $F^n(y_0,x_0,y_0) \to \overline{y}$ and $F^n(z_0,y_0,x_0) \to \overline{z}$. Now, using (ii) from the hypothesis, we get

$$d(F(\overline{x},\overline{y},\overline{z}),\overline{x}) \leq d(F(\overline{x},\overline{y},\overline{z}),F^{n+1}(x_0,y_0,z_0)) + d(F^{n+1}(x_0,y_0,z_0),\overline{x})$$

$$= d(F(\overline{x},\overline{y},\overline{z}),F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0))) + d(F^{n+1}(x_0,y_0,z_0),\overline{x})$$

$$\leq \frac{k}{3}[d(\overline{x},F^n(x_0,y_0,z_0)) + d(\overline{y},F^n(y_0,x_0,y_0)) + d(\overline{z},F^n(z_0,y_0,x_0))]$$

$$+d(F^{n+1}(x_0,y_0,z_0),\overline{x}) \to 0, \text{ as } n \to \infty.$$

This implies that $F(\overline{x},\overline{y},\overline{z}) = \overline{x}$. Similarly, we can prove that $F(\overline{y},\overline{x},\overline{y}) = \overline{y}$ and $F(\overline{z},\overline{y},\overline{x}) = \overline{z}$. □

The following result comes as a completion of Theorem 4.2.92 and Theorem 4.2.96, respectively, adding to the conclusion of Theorem 4.2.96 the identity of the components of the tripled fixed point of the mapping F.

THEOREM 4.2.97. [**60**] *In addition to the hypothesis of Theorem 4.2.92 (resp. Theorem 4.2.96), let $(x_0,y_0,z_0) \in X^3$ such that for all $(x,y,z),(x_0,y_0,z_0) \in X^3$, there exists $(r,s,t) \in X^3$ such that (x,y,z),
$(x_0,y_0,z_0) \in X_R(r,s,t)$. Then $\overline{x} = \overline{y} = \overline{z}$.*

Proof: From the mixed $R-$monotone property of F, we have
$(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) \in$
$X_R(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$.
Next, it can be verified that, for all $n \in \mathbb{N}$,

$$(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0), F^{n-1}(z_0, y_0, x_0)) \in$$

$$X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0), F^{n-1}(z_0, y_0, x_0))$$

By using the contractivity of F we get

$$d(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0)),$$

$$F(F^{n-1}(y_0, x_0, y_0), F^{n-1}(x_0, y_0, z_0))) \leq k \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(y_0, z_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, y_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim\limits_{n\to\infty} F^n(x_0, y_0, z_0) = \lim\limits_{n\to\infty} F^n(y_0, x_0, y_0) = \overline{y}$.
On the other hand,

$$d(F^n(x_0, y_0, z_0), F^n(z_0, y_0, x_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(z_0, y_0, x_0)),$$

$$F(F^{n-1}(z_0, y_0, x_0), F^{n-1}(x_0, y_0, z_0))) \leq \frac{k}{3} \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(z_0, y_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, z_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim\limits_{n\to\infty} F^n(x_0, y_0, z_0) = \lim\limits_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{z}$.
In conclusion,

$$\overline{z} = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, y_0) = \overline{y},$$

thus we have the identity of the three components of the tripled fixed point. □

REMARK 4.2.98. *The conclusion of Theorem 4.2.97 is, in fact, equivalent to the existence of a fixed point of the mapping F, that is $F(\overline{x}, \overline{x}, \overline{x}) = \overline{x}$.*

If, in Theorem 4.2.92, we replace the contractive condition by a more general, we obtain the following results:

THEOREM 4.2.99. *Let (X, d) be a metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(1) F has a lower-R-tripled fixed point;
(2) F has the mixed R-monotone property on X;
(3) F is orbitally continuous;
(4) there exist $a, b, c \in [0, 1), a + b + c < 1$ such that

$$d(F(x,y,z),F(t,u,v)) \leq a \cdot d(x,t) + b \cdot d(y,u) + c \cdot d(z,v),$$
$$\forall (x,y,z) \in X_R(t,u,v), a+b+c<1. \tag{4.44}$$

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x},\overline{y},\overline{z}), \in X^3$ such that $F(\overline{x},\overline{y},\overline{z}) = \overline{x}, F(\overline{y},\overline{x},\overline{y}) = \overline{y},\ F(\overline{z},\overline{y},\overline{x}) = \overline{z}$.

(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n,y_n,z_n), y_{n+1} = F(y_n,x_n,y_n), z_{n+1} = F(z_n,y_n,x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

Proof: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0,y_0,z_0) \in X\times X\times X$ be it, we have $F\times F(x_0,y_0,z_0) \in X_R(x_0,y_0,z_0)$. Further, using the mixed R-monotone property of F, we have $F\times F(x_0,y_0,z_0) \in X_R(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0)) \in$$
$$\in X_R(F^{n-1}(x_0,y_0,z_0),F^{n-1}(y_0,x_0,y_0),F^{n-1}(z_0,y_0,x_0)). \tag{4.45}$$

We claim that, for $n \in \mathbb{N}$

$$d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0)) \leq$$
$$a^n \cdot d(F(x_0,y_0,z_0),y_0) + b^n \cdot d(F(y_0,x_0,y_0),y_0) + c^n \cdot d(F(z_0,y_0,x_0),z_0). \tag{4.46}$$

For $n=1$, we get:

$$d(F^2(x_0,y_0,z_0),F(x_0,y_0,z_0)) = d(F(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0)),$$
$$F(x_0,y_0,z_0)) \leq a \cdot [d(F(x_0,y_0,z_0),x_0) + b \cdot d(F(y_0,x_0,y_0),y_0) + c \cdot d(F(z_0,y_0,x_0),z_0)].$$

Now, we assume that (4.46) holds. Using (4), we get:

$$d(F^{n+2}(x_0,y_0,z_0),F^{n+1}(x_0,y_0,z_0)) =$$
$$d(F(F^{n+1}(x_0,y_0,z_0),F^{n+1}(y_0,x_0,y_0),F^{n+1}(z_0,y_0,x_0)),$$
$$F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0))) \leq$$
$$\leq a \cdot d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0)) +$$
$$b \cdot d(F^{n+1}(y_0,x_0,y_0),F^n(y_0,x_0,y_0)) + c \cdot d(F^{n+1}(z_0,y_0,x_0),F^n(z_0,y_0,x_0) \leq$$
$$\leq a^{n+1} \cdot d(F(x_0,y_0,z_0),x_0) + b^{n+1} \cdot d(F(y_0,x_0,y_0),y_0) + c^{n+1} \cdot d(F(z_0,y_0,x_0),z_0) \to 0, asn \to \infty.$$

This implies that the sequence $\{F^n(x_0,y_0,z_0\}_{n\in\mathbb{N}}$ is fundamental in X.
Similarly, following the same steps, it can be proved that $\{F^n(y_0,x_0,y_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0,y_0,x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since X is a complete metric space, there exist $\overline{x},\overline{y},\overline{z} \in X$ such that $F^n(x_0,y_0,z_0) \to \overline{x}, F^n(y_0,x_0,y_0) \to \overline{y}, F^n(z_0,y_0,x_0) \to \overline{z}, \quad n \to \infty$.

□

THEOREM 4.2.100. *Let (X,d) be a complete metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(i) F has a lower-R-tripled fixed point;
(ii) F has the mixed R-monotone property on X;
(iii) $\exists a,b,c \in [0,1), a+b+c<1$ such that, $\forall (x,y,z) \in X_R(t,u,v)$:

$$d(F(x,y,z),F(t,u,v)) \leq a \cdot d(x,t) + b \cdot d(y,u) + c \cdot d(z,v) \tag{4.47}$$

(iv) if an $R-$monotone sequence $\{(x_n,y_n,z_n)\} \to (x,y,z)$, then $(x_n,y_n,z_n) \in X_R(x,y,z)$, for all $n \in \mathbb{N}$.

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x},\overline{y},\overline{z}), \in X^3$ such that $F(\overline{x},\overline{y},\overline{z}) = \overline{x}, F(\overline{y},\overline{x},\overline{y}) = \overline{y},\ F(\overline{z},\overline{y},\overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n,y_n,z_n), y_{n+1} = F(y_n,x_n,y_n), z_{n+1} = F(z_n,y_n,x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

Proof: The proof of this theorem follows the steps of the proof of Theorem 4.2.99: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0,y_0,z_0) \in X \times X \times X$ be it, we have $F \times F(x_0,y_0,z_0) \in X_R(x_0,y_0,z_0)$. Further, using the mixed R-monotone property of F, we have $F \times F(x_0,y_0,z_0) \in X_R(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0)) \in$$

$$\in X_R(F^{n-1}(x_0,y_0,z_0),F^{n-1}(y_0,x_0,y_0),F^{n-1}(z_0,y_0,x_0)). \tag{4.48}$$

We suppose that, for $n \in \mathbb{N}$

$$d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0)) \leq \tag{4.49}$$

$$a^n \cdot d(F(x_0,y_0,z_0),y_0) + b^n \cdot d(F(y_0,x_0,y_0),y_0) + c^n \cdot d(F(z_0,y_0,x_0),z_0).$$

For $n=1$, we get:

$$d(F^2(x_0,y_0,z_0),F(x_0,y_0,z_0)) =$$

$$d(F(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0)),$$

$$F(x_0,y_0,z_0)) \leq a \cdot d(F(x_0,y_0,z_0),x_0)+$$

$$b \cdot d(F(y_0,x_0,z_0),y_0) + c \cdot d(F(z_0,y_0,x_0),z_0).$$

Now, we assume that (4.49) holds. We obtain:

$$d(F^{n+2}(x_0,y_0,z_0),F^{n+1}(x_0,y_0,z_0)) =$$
$$d(F(F^{n+1}(x_0,y_0,z_0),F^{n+1}(y_0,x_0,y_0),F^{n+1}(z_0,y_0,x_0)),$$
$$F(F^{n}(x_0,y_0,z_0),F^{n}(y_0,x_0,y_0),F^{n}(z_0,y_0,x_0))) \leq a\cdot d(F^{n+1}(x_0,y_0,z_0),$$
$$F^{n}(x_0,y_0,z_0)) + b\cdot d(F^{n+1}(y_0,x_0,y_0),F^{n}(y_0,x_0,y_0))+$$
$$c\cdot d(F^{n+1}(z_0,y_0,x_0),F^{n}(z_0,y_0,x_0)] \leq \frac{a^{n+1}}{3}\cdot d(F(x_0,y_0,z_0),x_0)+$$
$$\frac{b^{n+1}}{3}\cdot d(F(y_0,x_0,y_0),y_0) + \frac{c^{n+1}}{3}\cdot d(F(z_0,y_0,x_0),z_0) \to 0, \text{ as } n\to\infty.$$

This implies that $\{F^n(x_0,y_0,z_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Similarly, following the same steps, we can prove that $\{F^n(y_0,x_0,y_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0,y_0,x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since (X,d) is a complete metric space, there exist $\overline{x},\overline{y},\overline{z}\in X$ such that $F^n(x_0,y_0,z_0)\to \overline{x}, F^n(y_0,x_0,y_0)\to\overline{y}, F^n(z_0,y_0,x_0)\to\overline{z}, \quad n\to\infty$.
The only thing left to show is that $F(\overline{x},\overline{y},\overline{z})=\overline{x}, F(\overline{y},\overline{x},\overline{y})=\overline{y})$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. We have

$$\begin{aligned}
d(F(\overline{x},\overline{y},\overline{z}),\overline{x}) &\leq d(F(\overline{x},\overline{y},\overline{z}),F^{n+1}(x_0,y_0,z_0)) + d(F^{n+1}(x_0,y_0,z_0),\overline{x})\\
&= d(F(\overline{x},\overline{y},\overline{z}),F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0)))+\\
&+d(F^{n+1}(x_0,y_0,z_0),\overline{x}) \leq a\cdot d(\overline{x},F^n(x_0,y_0,z_0)) + b\cdot d(\overline{y},F^n(y_0,x_0,y_0))\\
&+c\cdot d(\overline{z},F^n(z_0,y_0,x_0)) + d(F^{n+1}(x_0,y_0,z_0),\overline{x}) \to 0 \quad (n\to\infty)
\end{aligned}$$

This implies that $F(\overline{x},\overline{y},\overline{z})=\overline{x}$. Similar to this case we can prove $F(\overline{y},\overline{x},\overline{y})=\overline{y})$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. □

REMARK 4.2.101. *If in Theorem 4.2.99, we know that $(x_0,y_0,z_0)\in X^3$ is such that for all $(x,y,z),(x_0,y_0,z_0)\in X^3$, there exists $(r,s,t)\in X^3$ such that (x,y,z), $(x_0,y_0,z_0)\in X_R(r,s,t)$, then we can also prove the uniqueness of the tripled fixed point.*

Thus, the following results comes as a completion of the ones presented above, adding to the conclusion the uniqueness of the tripled fixed point, respectively the identity of its' three components.

THEOREM 4.2.102. *In addition to the hypothesis of Theorem 4.2.99, we suppose that, for all $(x,y,z),(x_0,y_0,z_0)\in X^3$, there exists $(r,s,t)\in X^3$ such that $(x,y,z),(x_0,y_0,z_0)\in X_R(r,s,t)$,. Then, F is a Picard operator.*

Proof: From Theorem 4.2.99 it follows that there exist $\overline{x},\overline{y},\overline{z}\in X$ such that $F(\overline{x},\overline{y},\overline{z})=\overline{x}, F(\overline{y},\overline{x},\overline{y})=\overline{y}, F(\overline{z},\overline{y},\overline{x})=\overline{z}$.
The next step is to show that $A_F(\overline{x},\overline{y},\overline{z})=X\times X\times X$.

Let $(x, y, z) \in X^3$. Since F has the mixed $R-$monotone property on X, then there exists $(r, s, t) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(r, s, t)$. From $(x_0, y_0, z_0) \in X_R(r, s, t)$ and the fact that (X, d) is a complete metric space, it follows that for $n \in \mathbb{N}$

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0))$$
$$\in X_R(F^n(r, s, t), F^n(s, r, s), F^n(t, s, r)).$$

But F is orbitally continuous, so we have:

$$d(F^n(x_0, y_0, z_0), F^n(r, s, t)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t),$$

$$d(F^n(y_0, x_0, y_0), F^n(s, r, s)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t)$$

and

$$d(F^n(z_0, y_0, x_0), F^n(t, s, r)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t).$$

Now, using the fact that $(x_0, y_0, z_0) \in A_F(\overline{x}, \overline{y}, \overline{z})$, it follows that $(r, s, t) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, $A_F(\overline{x}, \overline{y}, \overline{z}) = X^3$.
It is also true that $(\overline{y}, \overline{x}, \overline{y}), (\overline{z}, \overline{y}, \overline{x}) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, we have $\overline{x} = \overline{y} = \overline{z}$. Therefore $A_F(\overline{x}) = X$, so F is a Picard operator. □

THEOREM 4.2.103. *In addition to the hypothesis of Theorem 4.2.99, let* $(x_0, y_0, z_0) \in X^3$ *such that for all* $(x, y, z), (x_0, y_0, z_0) \in X^3$*, there exists* $(r, s, t) \in X^3$ *such that* (x, y, z),
$(x_0, y_0, z_0) \in X_R(r, s, t)$. *Then* $\overline{x} = \overline{y} = \overline{z}$

Proof: From the mixed $R-$monotone property of F, we have
$(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) \in$
$X_R(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$.
Next, it can be verified that, for all $n \in \mathbb{N}$,

$$(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0), F^{n-1}(z_0, y_0, x_0)) \in$$
$$X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0), F^{n-1}(z_0, y_0, x_0))$$

By using the contractivity of F and letting $k := a + b + c < 1$, we get

$$d(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, y_0)),$$
$$F(F^{n-1}(y_0, x_0, y_0), F^{n-1}(x_0, y_0, z_0))) \leq k \cdot d(F^{n-1}(x_0, y_0, z_0),$$
$$F^{n-1}(y_0, x_0, y_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, y_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, y_0) = \overline{y}$.
On the other hand,

$$d(F^n(x_0, y_0, z_0), F^n(z_0, y_0, x_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(z_0, y_0, x_0)),$$
$$F(F^{n-1}(z_0, y_0, x_0), F^{n-1}(x_0, y_0, z_0))) \leq \frac{k}{3} \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(z_0, y_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, z_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{z}$.
In conclusion, $\overline{z} = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, y_0) = \overline{y}$, thus we have the identity of the three components of the tripled fixed point. $\square$

REMARK 4.2.104. *If, in Theorem 4.2.99, we take $R =\leq$ and assume that F is continuous, we obtain Theorem 7 in* [**28**].

REMARK 4.2.105. *If, in Theorem 4.2.92, we take $R =\leq$ and assume that F is continuous, we obtain Theorem 2.1.10 in* [**40**].

REMARK 4.2.106. *If, in Theorem 4.2.102, resp. 4.2.103, we take $R =\leq$ and assume that F is continuous, we obtain Theorem 9, resp. Theorem 10 in* [**28**].

REMARK 4.2.107. *If, in Theorems 4.2.94, resp. 4.2.97, we take $R =\leq$, we obtain Theorem 2.1.16, resp. Theorem 2.1.17 in* [**40**].

REMARK 4.2.108. *If, in Theorem 4.2.95, we take $R =\leq$ and let $k := j + k + l < 1$, we obtain Theorem 8 in* [**28**].

3. Tripled fixed points of monotone operators

3.1. Definitions

The following concepts are extensions of the notions presented in [**9**] in the case of coupled fixed points in the framework of tripled fixed points of monotone mappings.

NOTATION 3. *Let X be a nonempty set and let $F : X \times X \times X \to X$ be a mapping. Then*

(1) The cartesian product of F and itself is denoted by $F \times F$ and it is defined by

$$F \times F(x, y, z) = (F(x, y, z), F(y, x, z), F(z, y, x)).$$

(2) We will denote by $F^0(x, y, z) = x$ and $F^n(x, y, z) = F(F^{n-1}(x, y, z), F^{n-1}(y, x, y), F^{n-1}(z, y, x))$, for all $x, y, z \in X, n \in \mathbb{N}$.

REMARK 4.3.109. *Note that cartesian product of the two mappings is different from the one in Notation 2. In this case we have permutations of (x, y, z) in order to obtain the tripled fixed points of monotone operators defined by Borcut in Definition 4.1.82*

DEFINITION 4.3.110. *Let X be a nonempty set, R be a reflexive relation on X and $F : X^3 \to X$ be a monotone mapping. The mapping F has the $R-$**monotone property** on X if $F \times F(X_R(x, y, z)) \subseteq X_R(F \times F(x, y, z))$, for all $(x, y, z) \in X^3$.*

DEFINITION 4.3.111. *An element $(x, y, z) \in X^3$ is called **lower-$R-$tripled fixed point** of the $R-$monotone mapping F, if $F \times F(x, y, z) \in X_R(x, y, z)$, where $X_R(x, y, z) = \{(t, u, v) \in X \times X \times X : tRx \wedge yRu \wedge vRz\}$.*

DEFINITION 4.3.112. *Let X be a topological space and $F : X^3 \to X$ be an $R-$monotone mapping. An element $(x, y, z) \in X^3$ is called a **tripled attractor basin element** of F with respect to $(x^*, y^*, z^*) \in X^3$, if $F^n(x, y, z) \to x^*$, $F^n(y, x, z) \to y^*$ and $F^n(z, y, x) \to z^*$, as $n \to \infty$. The set of these points (x^*, y^*, z^*) will be denoted by $A_F(x^*, y^*, z^*)$, and, when $x^* = y^* = z^*$, by $A_F(x^*)$.*

DEFINITION 4.3.113. *Let X be a topological space and $F : X^3 \to X$ be an $R-$monotone mapping. The mapping F is called **orbitally continuous** if $(x, y, z), (a, b, c) \in X^3$ and $F^{n_k}(x, y, z) \to a, F^{n_k}(y, x, z) \to b, F^{n_k}(z, y, x) \to c$, when $k \to \infty$, implies $F^{n_k+1}(x, y, z) \to F(a, b, c)$, $F^{n_k+1}(y, x, z) \to F(b, a, b)$ and $F^{n_k+1}(z, y, x) \to F(c, b, a)$, as $k \to \infty$.*

REMARK 4.3.114. *Note that the definitions for tripled attractor basin element and orbital continuity of a mapping are different that in the case of mixed-R-monotone operators.*

3.2. Existence and uniqueness theorems

The following result establishes the existence of a tripled fixed point of the mapping $F : X^3 \to X$.

THEOREM 4.3.115. *Let (X, d) be a complete metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(i) F has a lower-R-tripled fixed point;
(ii) F has the R-monotone property on X;
(iii) F is orbitally continuous;
(iv) there exists $k \in [0, 1)$ such that

$$d(F(x, y, z), F(t, u, v)) \le \frac{k}{3} \cdot [d(x, t) + d(y, u) + d(z, v)], \tag{4.51}$$
$$\forall (x, y, z) \in X_R(t, u, v), k \in [0, 1).$$

Then:

(1) F has a tripled fixed point, that is, $\exists(\overline{x}, \overline{y}, \overline{z}), \in X^3$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}$, $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, z_n), z_{n+1} = F(z_n, y_n, x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

(3) The error estimation that holds is: $\max_{n\in\mathbb{N}}\{d(x_n,\overline{x}),d(y_n,\overline{y}),d(z_n,\overline{z})\}$
$$\leq \frac{k^n}{3(1-k)}[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,z_0),y_0)+d(F(z_0,y_0,x_0),z_0)].$$

Proof: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0,y_0,z_0)\in X\times X\times X$ be it, we have $F\times F(x_0,y_0,z_0)\in X_R(x_0,y_0,z_0)$. Further, using the R-monotone property of F, we have $F\times F(x_0,y_0,z_0)\in X_R(F(x_0,y_0,z_0),F(y_0,x_0,z_0),F(z_0,y_0,x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0,y_0,z_0),F^n(y_0,x_0,z_0),F^n(z_0,y_0,x_0))\in$$
$$\in X_R(F^{n-1}(x_0,y_0,z_0),F^{n-1}(y_0,x_0,z_0),F^{n-1}(z_0,y_0,x_0)). \tag{4.52}$$

We claim that, for $n\in\mathbb{N}$

$$d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0))\leq$$
$$\frac{k^n}{3}\cdot[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,z_0),y_0)+d(F(z_0,y_0,x_0),z_0)]. \tag{4.53}$$

For $n=1$, we get:

$$d(F^2(x_0,y_0,z_0),F(x_0,y_0,z_0))=d(F(F(x_0,y_0,z_0),F(y_0,x_0,z_0),F(z_0,y_0,x_0)),$$
$$F(x_0,y_0,z_0))\leq\frac{k}{3}\cdot[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,z_0),y_0)+d(F(z_0,y_0,x_0),z_0)].$$

Now, we assume that (4.53) holds. Using (iv), we get:

$$d(F^{n+2}(x_0,y_0,z_0),F^{n+1}(x_0,y_0,z_0))=$$
$$d(F(F^{n+1}(x_0,y_0,z_0),F^{n+1}(y_0,x_0,z_0),F^{n+1}(z_0,y_0,x_0)),$$
$$F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,z_0),F^n(z_0,y_0,x_0)))\leq$$
$$\leq\frac{k}{3}\cdot[d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0))+$$
$$d(F^{n+1}(y_0,x_0,z_0),F^n(y_0,x_0,z_0))+d(F^{n+1}(z_0,y_0,x_0),F^n(z_0,y_0,x_0)]\leq$$
$$\leq\frac{k^{n+1}}{3}\cdot[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,z_0),y_0)+d(F(z_0,y_0,x_0),x_0)]\to 0, asn\to\infty.$$

This implies that $\{F^n(x_0,y_0,z_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Similarly, following the same steps, we can prove that $\{F^n(y_0,x_0,z_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0,y_0,x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since X is a complete metric space, there exist $\overline{x},\overline{y},\overline{z}\in X$ such that $F^n(x_0,y_0,z_0)\to\overline{x}, F^n(y_0,x_0,z_0)\to\overline{y}, F^n(z_0,y_0,x_0)\to\overline{z},\quad n\to\infty$.
Let $m>n$. Then:

$$d(F^m(x_0,y_0,z_0),F^n(x_0,y_0,z_0))\leq\sum_{j=n}^{m-1}d(F^{j+1}(x_0,y_0,z_0),F^j(x_0,y_0,z_0))\leq$$
$$\leq(k^{n-1}+k^n+...+k^{m-n})\cdot[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,z_0),y_0)+d(F(z_0,y_0,x_0),z_0)]$$

$$= \frac{k^n - k^m}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)] <$$

$$< \frac{k^n}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)].$$

But $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, x_n, z_n)$ and $z_{n+1} = F(z_n, y_n, x_n)$, $F^n(x_0, y_0, z_0) \to \overline{x}$, $F^n(y_0, x_0, z_0) \to \overline{y}$, $F^n(z_0, y_0, x_0) \to \overline{z}$, $\quad n \to \infty$ and F is orbitally continuous. Thus, applying maximum to the last relation, we get:

$$\max_{n \in \mathbb{N}} \{d(x_n, \overline{x}), d(y_n, \overline{y}), d(z_n, \overline{z})\} \leq \frac{k^n}{3(1-k)} \cdot [d(F(x_0, y_0, z_0), x_0) +$$

$$+ d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)].$$

□

Remark 4.3.116. *If in Theorem 4.2.92 we know that $(x_0, y_0, z_0) \in X^3$ is such that for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z), $(x_0, y_0, z_0) \in X_R(r, s, t)$, then we can also prove tripled fixed point of F is unique.*

Thus, the following result comes as a completion of Theorem 4.2.92, adding to the conclusion the uniqueness of the tripled fixed point.

Theorem 4.3.117. *In addition to the hypothesis of Theorem 4.2.92, we suppose that, for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z), $(x_0, y_0, z_0) \in X_R(r, s, t)$. Then F is a Picard operator.*

Proof: From Theorem 4.2.92 it follows that there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}$, $F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}$, $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
The next step is to show that $A_F(\overline{x}, \overline{y}, \overline{z}) = X \times X \times X$.
Let $(x, y, z) \in X^3$. Since F has the $R-$monotone property on X, then there exists $(r, s, t) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(r, s, t)$. From $(x_0, y_0, z_0) \in X_R(r, s, t)$ and the fact that (X, d) is a complete metric space, it follows that for $n \in \mathbb{N}$

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0))$$

$$\in X_R(F^n(r, s, t), F^n(s, r, t), F^n(t, s, r)).$$

But F is orbitally continuous, so we have:

$$d(F^n(x_0, y_0, z_0), F^n(r, s, t)) \leq \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)],$$

$$d(F^n(y_0, x_0, y_0), F^n(s, r, t)) \leq \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)]$$

and

$$d(F^n(z_0, y_0, x_0), F^n(t, s, r)) \leq \frac{k^n}{3} \cdot [d(x_0, r) + d(y_0, s) + d(z_0, t)].$$

Now, using the fact that $(x_0, y_0, z_0) \in A_F(\overline{x}, \overline{y}, \overline{z})$, it follows that $(r, s, t) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, $A_F(\overline{x}, \overline{y}, \overline{z}) = X^3$.
It is also true that $(\overline{y}, \overline{x}, \overline{z}), (\overline{z}, \overline{y}, \overline{x}) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, we have $\overline{x} = \overline{y} = \overline{z}$. Therefore $A_F(\overline{x}) = X$, so F is a Picard operator. □

In the next result, we obtain the same conclusion as in Theorem 4.3.115, replacing the orbital continuity of F with an assumption on some R-monotone sequences in X^3:

THEOREM 4.3.118. *Let (X, d) be a complete metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(i) F has a lower-R-tripled fixed point;
(ii) F has the R-monotone property on X;
(iii) there exists $k \in [0, 1)$ such that, $\forall (x, y, z) \in X_R(t, u, v)$:

$$d(F(x, y, z), F(t, u, v)) \leq \frac{k}{3} \cdot [d(x, t) + d(y, u) + d(z, v)] \tag{4.54}$$

(iv) if an $R-$monotone sequence $\{(x_n, y_n, z_n)\} \to (x, y, z)$, then $(x_n, y_n, z_n) \in X_R(x, y, z)$, for all $n \in \mathbb{N}$.

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x}, \overline{y}, \overline{z}), \in X^3$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{z}) = \overline{y},\ F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, z_n), z_{n+1} = F(z_n, y_n, x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.
(3) The error estimation that holds is: $\max_{n\in\mathbb{N}}\{d(x_n, \overline{x}), d(y_n, \overline{y}), d(z_n, \overline{z})\}$ $\leq \dfrac{k^n}{3(1-k)}[d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)]$.

Proof: The proof of this theorem follows the steps of the proof of Theorem 4.3.115. The only thing left to show is the existence of the tripled fixed point (conditioned by the orbital continuity of F in Theorem 4.2.92.)
From the proof of Theorem 4.3.115, we have that $F^n(x_0, y_0, z_0) \to \overline{x}$, $F^n(y_0, x_0, z_0) \to \overline{y}$ and $F^n(z_0, y_0, x_0) \to \overline{z}$. Now, using (iii) from the hypothesis, we get

$$\begin{aligned} d(F(\overline{x}, \overline{y}, \overline{z}), \overline{x}) &\leq d(F(\overline{x}, \overline{y}, \overline{z}), F^{n+1}(x_0, y_0, z_0)) + d(F^{n+1}(x_0, y_0, z_0), \overline{x}) \\ &= d(F(\overline{x}, \overline{y}, \overline{z}), F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0))) + d(F^{n+1}(x_0, y_0, z_0), \overline{x}) \\ &\leq \frac{k}{3}[d(\overline{x}, F^n(x_0, y_0, z_0)) + d(\overline{y}, F^n(y_0, x_0, z_0)) + d(\overline{z}, F^n(z_0, y_0, x_0))] \\ &\quad + d(F^{n+1}(x_0, y_0, z_0), \overline{x}) \to 0, \text{ as } n \to \infty. \end{aligned}$$

This implies that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}$. Similarly, we can prove that $F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}$ and $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$. □

The following result comes as a completion of Theorem 4.3.115 and Theorem 4.3.118, respectively, adding to the conclusion of Theorem 4.3.118 the identity of the components of the tripled fixed point of the mapping F.

THEOREM 4.3.119. *In addition to the hypothesis of Theorem 4.3.115 (resp. Theorem 4.3.118), let $(x_0, y_0, z_0) \in X^3$ such that for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z),*
$(x_0, y_0, z_0) \in X_R(r, s, t)$. Then $\overline{x} = \overline{y} = \overline{z}$.

Proof: From the $R-$monotone property of F, we have
$(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0)) \in$
$X_R(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0))$.
Next, it can be verified that, for all $n \in \mathbb{N}$,

$$(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0)) \in$$

$$X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0))$$

By using the contractivity of F we get

$$d(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0)),$$

$$F(F^{n-1}(y_0, x_0, z_0), F^{n-1}(x_0, y_0, z_0))) \leq k \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(y_0, z_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, y_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, z_0) = \overline{y}$.
On the other hand,

$$d(F^n(x_0, y_0, z_0), F^n(z_0, y_0, x_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(z_0, y_0, x_0)),$$

$$F(F^{n-1}(z_0, y_0, x_0), F^{n-1}(x_0, y_0, z_0))) \leq \frac{k}{3} \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(z_0, y_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, z_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{z}$.
In conclusion,

$$\overline{z} = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, z_0) = \overline{y},$$

thus we have the identity of the three components of the tripled fixed point. □

REMARK 4.3.120. *The conclusion of Theorem 4.3.119 is, in fact, equivalent to the existence of a fixed point of the mapping F, that is $F(\overline{x}, \overline{x}, \overline{x}) = \overline{x}$.*

If, in Theorem 4.3.115, we replace the contractive condition by a more general, we obtain the following results:

THEOREM 4.3.121. *Let (X, d) be a complete metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(1) F has a lower-R-tripled fixed point;
(2) F has the R-monotone property on X;
(3) F is orbitally continuous;
(4) there exist $a, b, c \in [0, 1), a + b + c < 1$ such that

$$d(F(x, y, z), F(t, u, v)) \leq a \cdot d(x, t) + b \cdot d(y, u) + c \cdot d(z, v), \quad \forall (x, y, z) \in X_R(t, u, v), a + b + c < 1. \tag{4.55}$$

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x}, \overline{y}, \overline{z}), \in X^3$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}$, $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, z_n), z_{n+1} = F(z_n, y_n, x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

Proof: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0, y_0, z_0) \in X \times X \times X$ be it, we have $F \times F(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$. Further, using the R-monotone property of F, we have $F \times F(x_0, y_0, z_0) \in X_R(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0)) \in$$

$$\in X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0)). \tag{4.56}$$

We claim that, for $n \in \mathbb{N}$

$$d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) \leq$$

$$a^n \cdot d(F(x_0, y_0, z_0), y_0) + b^n \cdot d(F(y_0, x_0, z_0), y_0) + c^n \cdot d(F(z_0, y_0, x_0), z_0). \tag{4.57}$$

For $n = 1$, we get:

$$d(F^2(x_0, y_0, z_0), F(x_0, y_0, z_0)) = d(F(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0)),$$

$$F(x_0, y_0, z_0)) \leq a \cdot [d(F(x_0, y_0, z_0), x_0) + b \cdot d(F(y_0, x_0, z_0), y_0) + c \cdot d(F(z_0, y_0, x_0), z_0)].$$

Now, we assume that (4.57) holds. Using the forth assumption from the hypothesis, (iv), we get:

$$d(F^{n+2}(x_0, y_0, z_0), F^{n+1}(x_0, y_0, z_0)) =$$

$$d(F(F^{n+1}(x_0, y_0, z_0), F^{n+1}(y_0, x_0, z_0), F^{n+1}(z_0, y_0, x_0)),$$

$$F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0))) \leq$$

$$\leq a \cdot d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0))+$$

$$b \cdot d(F^{n+1}(y_0, x_0, z_0), F^n(y_0, x_0, z_0)) + c \cdot d(F^{n+1}(z_0, y_0, x_0), F^n(z_0, y_0, x_0) \leq$$

$$\leq a^{n+1} \cdot d(F(x_0, y_0, z_0), x_0) + b^{n+1} \cdot d(F(y_0, x_0, z_0), y_0) + c^{n+1} \cdot d(F(z_0, y_0, x_0), z_0) \to 0, \text{ as } n \to \infty.$$

This implies that the sequence $\{F^n(x_0, y_0, z_0\}_{n\in\mathbb{N}}$ is fundamental in X.
Similarly, following the same steps, we can show that $\{F^n(y_0, x_0, y_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0, y_0, x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since (X, d) is a complete metric space, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F^n(x_0, y_0, z_0) \to \overline{x}, F^n(y_0, x_0, z_0) \to \overline{y}, F^n(z_0, y_0, x_0) \to \overline{z}, \quad n \to \infty$.

□

THEOREM 4.3.122. *Let (X, d) be a metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \to X$ is a mapping such that*

(i) F has a lower-R-tripled fixed point;
(ii) F has the R-monotone property on X;
(iii) $\exists a, b, c \in [0, 1), a + b + c < 1$ such that, $\forall (x, y, z) \in X_R(t, u, v)$:

$$d(F(x, y, z), F(t, u, v)) \leq a \cdot d(x, t) + b \cdot d(y, u) + c \cdot d(z, v) \tag{4.58}$$

(iv) if an $R-$monotone sequence $\{(x_n, y_n, z_n)\} \to (x, y, z)$, then $(x_n, y_n, z_n) \in X_R(x, y, z)$, for all $n \in \mathbb{N}$.

Then:

(1) F has a tripled fixed point, that is, $\exists (\overline{x}, \overline{y}, \overline{z}), \in X^3$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}$, $F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
(2) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, z_n), z_{n+1} = F(z_n, y_n, x_n)$, converge to $\overline{x}$,$\overline{y}$ and $\overline{z}$, respectively.

Proof: The proof of this theorem follows the steps of the proof of Theorem 4.2.99: Since the mapping F admits a lower-R-tripled fixed point, let $(x_0, y_0, z_0) \in X \times X \times X$ be it, we have $F \times F(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$. Further, using the R-monotone property of F, we have $F \times F(x_0, y_0, z_0) \in X_R(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0)) \in$$

$$\in X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0)). \tag{4.59}$$

We suppose that, for $n \in \mathbb{N}$

$$d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) \leq \tag{4.60}$$

$$a^n \cdot d(F(x_0, y_0, z_0), y_0) + b^n \cdot d(F(y_0, x_0, z_0), y_0) + c^n \cdot d(F(z_0, y_0, x_0), z_0).$$

For $n = 1$, we get:

$$d(F^2(x_0, y_0, z_0), F(x_0, y_0, z_0)) =$$

$$d(F(F(x_0,y_0,z_0),F(y_0,x_0,z_0),F(z_0,y_0,x_0)),$$
$$F(x_0,y_0,z_0)) \leq a\cdot d(F(x_0,y_0,z_0),x_0)+$$
$$b\cdot d(F(y_0,x_0,z_0),y_0)+c\cdot d(F(z_0,y_0,x_0),z_0).$$

Now, we assume that (4.60) holds. Using (iii), we get:

$$d(F^{n+2}(x_0,y_0,z_0),F^{n+1}(x_0,y_0,z_0)) =$$
$$d(F(F^{n+1}(x_0,y_0,z_0),F^{n+1}(y_0,x_0,z_0),F^{n+1}(z_0,y_0,x_0)),$$
$$F(F^{n}(x_0,y_0,z_0),F^{n}(y_0,x_0,z_0),F^{n}(z_0,y_0,x_0))) \leq a\cdot d(F^{n+1}(x_0,y_0,z_0),$$
$$F^{n}(x_0,y_0,z_0))+b\cdot d(F^{n+1}(y_0,x_0,z_0),F^{n}(y_0,x_0,z_0))+$$
$$c\cdot d(F^{n+1}(z_0,y_0,x_0),F^{n}(z_0,y_0,x_0)] \leq \frac{a^{n+1}}{3}\cdot d(F(x_0,y_0,z_0),x_0)+$$
$$\frac{b^{n+1}}{3}\cdot d(F(y_0,x_0,z_0),y_0)+\frac{c^{n+1}}{3}\cdot d(F(z_0,y_0,x_0),z_0)\to 0 \text{ as } n\to\infty.$$

This implies that $\{F^n(x_0,y_0,z_0)\}_{n\in\mathbb{N}}$ is a fundamental sequence in X.
Similarly, following the same steps, we can prove that $\{F^n(y_0,x_0,z_0)\}_{n\in\mathbb{N}}$ and $\{F^n(z_0,y_0,x_0)\}_{n\in\mathbb{N}}$ are also Cauchy sequences in X.
Since X is a complete metric space, there exist $\overline{x},\overline{y},\overline{z}\in X$ such that $F^n(x_0,y_0,z_0)\to \overline{x}, F^n(y_0,x_0,z_0)\to\overline{y}, F^n(z_0,y_0,x_0)\to\overline{z}, \quad n\to\infty$.
The only thing left to show is that $F(\overline{x},\overline{y},\overline{z})=\overline{x}, F(\overline{y},\overline{x},\overline{z})=\overline{y})$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. We have

$$d(F(\overline{x},\overline{y},\overline{z}),\overline{x}) \leq d(F(\overline{x},\overline{y},\overline{z}),F^{n+1}(x_0,y_0,z_0))+d(F^{n+1}(x_0,y_0,z_0),\overline{x})$$
$$= d(F(\overline{x},\overline{y},\overline{z}),F(F^{n}(x_0,y_0,z_0),F^{n}(y_0,x_0,z_0),F^{n}(z_0,y_0,x_0)))+$$
$$+d(F^{n+1}(x_0,y_0,z_0),\overline{x}) \leq a\cdot d(\overline{x},F^{n}(x_0,y_0,z_0))+b\cdot d(\overline{y},F^{n}(y_0,x_0,z_0))$$
$$+c\cdot d(\overline{z},F^{n}(z_0,y_0,x_0))+d(F^{n+1}(x_0,y_0,z_0),\overline{x})\to 0 \quad (n\to\infty)$$

This implies that $F(\overline{x},\overline{y},\overline{z})=\overline{x}$. Similar to this case we can prove $F(\overline{y},\overline{x},\overline{z})=\overline{y})$ and $F(\overline{z},\overline{y},\overline{x})=\overline{z}$. □

REMARK 4.3.123. *If in Theorem 4.3.121, we know that* $(x_0,y_0,z_0)\in X^3$ *is such that for all* $(x,y,z),(x_0,y_0,z_0)\in X^3$*, there exists* $(r,s,t)\in X^3$ *such that* (x,y,z), $(x_0,y_0,z_0)\in X_R(r,s,t)$*, then we can also prove the uniqueness of the tripled fixed point.*

Thus, the following results comes as a completion of the ones presented above, adding to the conclusion the uniqueness of the tripled fixed point, respectively the identity of its' three components.

THEOREM 4.3.124. *In addition to the hypothesis of Theorem 4.3.121, we suppose that, for all* $(x,y,z),(x_0,y_0,z_0)\in X^3$*, there exists* $(r,s,t)\in X^3$ *such that* (x,y,z), $(x_0,y_0,z_0)\in X_R(r,s,t)$*. Then* F *is a Picard operator.*

Proof: From Theorem 4.2.99 it follows that there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{x}, \overline{z}) = \overline{y}, F(\overline{z}, \overline{y}, \overline{x}) = \overline{z}$.
The next step is to show that $A_F(\overline{x}, \overline{y}, \overline{z}) = X \times X \times X$.
Let $(x, y, z) \in X^3$. Since F has the $R-$monotone property on X, then there exists $(r, s, t) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(r, s, t)$. From $(x_0, y_0, z_0) \in X_R(r, s, t)$ and the fact that (X, d) is a complete metric space, it follows that for $n \in \mathbb{N}$

$$(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0))$$
$$\in X_R(F^n(r, s, t), F^n(s, r, t), F^n(t, s, r)).$$

But F is orbitally continuous, so we have:

$$d(F^n(x_0, y_0, z_0), F^n(r, s, t)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t),$$
$$d(F^n(y_0, x_0, y_0), F^n(s, r, t)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t)$$

and

$$d(F^n(z_0, y_0, x_0), F^n(t, s, r)) \leq a^n \cdot d(x_0, r) + b^n \cdot d(y_0, s) + c^n \cdot d(z_0, t).$$

Now, using the fact that $(x_0, y_0, z_0) \in A_F(\overline{x}, \overline{y}, \overline{z})$, it follows that $(r, s, t) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, $A_F(\overline{x}, \overline{y}, \overline{z}) = X^3$.
It is also true that $(\overline{y}, \overline{x}, \overline{z}), (\overline{z}, \overline{y}, \overline{x}) \in A_F(\overline{x}, \overline{y}, \overline{z})$. Thus, we have $\overline{x} = \overline{y} = \overline{z}$. Therefore $A_F(\overline{x}) = X$, so F is a Picard operator. □

THEOREM 4.3.125. *In addition to the hypothesis of Theorem 4.3.121, suppose that for all $(x, y, z), (x_0, y_0, z_0) \in X^3$, there exists $(r, s, t) \in X^3$ such that (x, y, z), $(x_0, y_0, z_0) \in X_R(r, s, t)$. Then $\overline{x} = \overline{y} = \overline{z}$*

Proof: From the $R-$monotone property of F, we have
$(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0)) \in$
$X_R(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0))$.
Next, it can be verified that, for all $n \in \mathbb{N}$,

$$(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0)) \in$$
$$X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0))$$

By using the contractivity of F and letting $k := a + b + c < 1$, we get

$$d(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0)),$$
$$F(F^{n-1}(y_0, x_0, z_0), F^{n-1}(x_0, y_0, z_0))) \leq k \cdot d(F^{n-1}(x_0, y_0, z_0),$$
$$F^{n-1}(y_0, x_0, z_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, y_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, z_0) = \overline{y}$.
On the other hand,

$$d(F^n(x_0, y_0, z_0), F^n(z_0, y_0, x_0)) = d(F(F^{n-1}(x_0, y_0, z_0), F^{n-1}(z_0, y_0, x_0)),$$

$$F(F^{n-1}(z_0, y_0, x_0), F^{n-1}(x_0, y_0, z_0))) \leq \frac{k}{3} \cdot d(F^{n-1}(x_0, y_0, z_0),$$

$$F^{n-1}(z_0, y_0, x_0)) \leq ... \leq \frac{k^n}{3} \cdot d(x_0, z_0) \to 0 \quad (n \to \infty)$$

This implies $\overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{z}$.
In conclusion, $\overline{z} = \lim_{n\to\infty} F^n(z_0, y_0, x_0) = \overline{x} = \lim_{n\to\infty} F^n(x_0, y_0, z_0) = \lim_{n\to\infty} F^n(y_0, x_0, z_0) = \overline{y}$, thus we have the identity of the three components of the tripled fixed point. □

Remark 4.3.126. *If, in Theorem 4.3.121, we take $R =\leq$, we obtain Theorem 2.2.26 in from* [**39**].

Remark 4.3.127. *If, in Theorem 4.3.115, we take $R =\leq$ and assume that F is continuous, we obtain Theorem 4.1.84 from* [**39**].

Remark 4.3.128. *If, in Theorem 4.3.124, resp. 4.3.125, we take $R =\leq$, we obtain Theorem 2.2.30, resp. Theorem 2.2.31 in* [**40**], *from* [**39**].

4. Examples and applications

4.1. Examples

Example 4.4.129. *Let $X = \mathbb{R}$, the metric $d(x, y) = |x - y|$, the relation R on X given by*

$$(x, y, z)R(t, u, v) \Leftrightarrow xRt \wedge yRu \wedge zRv,$$

where $xRt \Leftrightarrow x^2 + 2x = t^2 + 2t$ Let $F : X^3 \to X$ be defined by $F(x, y, z) = \dfrac{x + y + 4z}{7}$. So, $\forall (x, y, z) \in X^3$, we have :

$$X_R(x, y, z) = \{(x, y, z), (x, y + 2, z), (x + 2, y, z), (x + 2, y + 2, z), (x, y, z + 2),$$

$$(x + 2, y + 2, z + 2), (x + 2, y, z + 2), (x, y + 2, z + 2)\}.$$

$$F \times F(X_R(x, y, z)) = \{F \times F(x, y, z)\} \subseteq X_R(F \times F(x, y, z)).$$

Next, we will check the conditions of Theorem 4.3.124. The system

$$\begin{cases} F(x, y, z) = x \\ F(y, x, z) = y \\ F(z, y, x) = z \end{cases} \tag{4.62}$$

admits the unique solution $(0, 0, 0)$.

The contractive condition (4.55) *holds:*

$$d(F(x, y, z), F(t, u, v)) = \left|\frac{x + y + 4z}{7} - \frac{t + u + 4v}{7}\right| = \left|\frac{1}{7}(x - t) + \frac{1}{7}(y - u) + \frac{4}{7}(z - v)\right| \leq$$

$$\frac{1}{7}|x - t| + \frac{1}{7}|y - u| + \frac{4}{7}|z - v|,$$

which is condition (4.55) *for* $a = \frac{1}{7}$, $b = \frac{1}{7}$ *and* $c = \frac{4}{7}$. *Indeed,* $a + b + c = \frac{6}{7} < 1$. *It is clear that* F *has the* $R-$*monotone property. Furthermore,* F *is continuous and* $\exists(0,0,0) \in X^3$ *such that* $F(0,0,0) \in X_R(0,0,0)$. *Therefore, the hypotheses of the Theorem 4.3.121 are satisfied, whereas Theorem 2.2.26 from* [**40**] *(resp.* [**39**]*) cannot be applied because* R *is not antisymmetric, thus metric space is not partially ordered. The (unique) tripled fixed point of* F *is* $(0,0,0)$.

Example 4.4.130. *Let* $X = [0,\infty]$, *the metric* $d(x,y) = |x-y|$, *the relation* R *on* X *given by*

$$(x,y,z)R(t,u,v) \Leftrightarrow xRt \wedge yRu \wedge zRv,$$

where $xRt \Leftrightarrow x^2 + 4x - 9 = t^2 + 4t - 9$. *Let* $F : X^3 \to X$ *be defined by* $F(x,y,z) = \frac{x - 4y + 9z}{15}$. *So,* $\forall(x,y,z) \in X^3$, *we have :*

$$X_R(x,y,z) = \{(x,y,z), (x,-4-y,z), (-4-x,y,z), (-4-x,-4-y,z),$$

$$(x,y,-4-z), (-4-x,-4-y,-4-z), (-4-x,y,-4-y), (x,-4-y,-4-z)\}.$$

Next, we will check the conditions of Theorem 4.2.102. The system

$$\begin{cases} F(x,y,z) = x \\ F(y,x,y) = y \\ F(z,y,x) = z \end{cases} \tag{4.63}$$

admits the unique solution $(0,0,0)$. *The contractive condition* (4.44) *holds:*

$$d(F(x,y,z),F(t,u,v)) = \left|\frac{x-4y+9z}{15} - \frac{t-4u+9v}{15}\right| = \left|\frac{1}{15}(x-t) + \frac{4}{15}(u-y) + \frac{9}{15}(z-v)\right| \leq$$

$$\frac{1}{15}|x-t| + \frac{4}{15}|y-u| + \frac{9}{15}|z-v|,$$

which is condition (4.44) *for* $a = \frac{1}{15}$, $b = \frac{4}{15}$ *and* $c = \frac{9}{15}$. *Indeed,* $a + b + c = \frac{14}{5} < 1$. *It is clear that* F *has the mixed* $R-$*monotone property. Furthermore,* F *is orbitally continuous and* $\exists(0,0,0) \in X^3$ *such that* $F(0,0,0) \in X_R(0,0,0)$. *Therefore, the hypotheses of the Theorem 4.2.99 are satisfied. The (unique) tripled fixed point of* F *is* $(0,0,0)$.

4.2. Applications

Nonlinear matrix equations present great interest in the field among researchers. Many recent papers were dedicated to this topic (see [**32**], [**33**],[**90**]). In this section, we will study the following nonlinear matrix equation, which is an extension of the cases presented by Ran and Reurings in [**113**], and Asgari and Mousavi in [**10**] and [**9**], for the case of tripled fixed points:

$$X = C + \sum_{i=1}^{p} A_i^* F(X) A_i - \sum_{j=1}^{q} B_j^* G(X) B_j - \sum_{k=1}^{r} C_k^* H(X) C_k, \tag{4.64}$$

where $C \in \mathcal{P}(n)$ is a positive definite matrix (we denote $C > 0$), A_i, B_j, C_k are arbitrary $n \times n$ matrices and F, G, H are three continuous order preserving mappings from $\mathcal{H}(n)$ into $\mathcal{P}(n)$, such that $F(0) = G(0) = H(0) = 0$, where $\mathcal{M}(n)$ denotes the set of all $n \times n$ matrices, $\mathcal{H}(n)$ the set of all $n \times n$ Hermitian matrices and $\mathcal{P}(n)$ the set of all $n \times n$ positive definite matrices, $\mathcal{P}(n) \subset \mathcal{H}(n) \subset \mathcal{M}(n)$.

If, in $\mathcal{H}(n)$, we introduce a relation of order "$\geq$", we get a partially ordered set where every matrix has a lower and an upper bound. Furthermore, to take advantage of the results presented above (that is, Theorems 4.2.92-4.2.97), let $T : \mathcal{H}(n) \times \mathcal{H}(n) \times \mathcal{H}(n) \to \mathcal{H}(n)$ be a mapping having the mixed monotone property, $T(X, Y, Z) = C + \sum_{i=1}^{p} A_i^* F(X) A_i - \sum_{j=1}^{q} B_j^* G(Y) B_j - \sum_{k=1}^{r} C_k^* H(Z) C_k$ (see equation (4.64)).Thus,the fixed points of T are, in fact, the solutions of equation (4.64).
In the following results we will discuss the existence and uniqueness of a solution of equation (4.64).In order to prove our results, we need the following lemmas presented in [**113**]:

Lemma 4.4.131. [**113**] *Let A and B be two positive semidefinite matrices. Then* $0 \leq tr(AB) \leq \|A\| \cdot tr(B)$.

Lemma 4.4.132. [**113**] *Let $A \in \mathcal{H}(n)$ satisfy $A < I$ (that is, $I - A$ is a positive definite matrix) , where $\mathcal{H}(n)$ is the set of all $n \times n$ Hermitian matrices. Then $\|A\| < 1$.*

The next theorem assures the existence of a tripled fixed point of the mapping T.

Theorem 4.4.133. [**60**] *Let $C \in \mathcal{P}(n)$ and M a positive number such that:*
(i)$\forall (X, Y, Z) \in \mathcal{H}(n)_{\leq}(U, V, W)$, we have

$$|tr(F(U) - F(X))| \leq \frac{1}{M} |tr(U - X)|$$
$$|tr(G(Y) - G(V))| \leq \frac{1}{M} |tr(Y - V)|$$
$$|tr(H(W) - F(Z))| \leq \frac{1}{M} |tr(W - Z)|$$

(ii) $\sum_{i=1}^{p} A_i^* A_i < \frac{M}{2} I_n$, ; $\sum_{j=1}^{q} B_j^* B_j < \frac{M}{2} I_n$ and $\sum_{k=1}^{r} C_k^* C_k < \frac{M}{2} I_n$.
(iii) $\sum_{i=1}^{p} A_i^* F(2C) A_i < C$; $\sum_{j=1}^{q} B_j^* G(2Q) B_j < Q$ and $\sum_{k=1}^{r} C_k^* H(2Q) C_k < Q$. Then there exist $X^*, Y^*, Z^* \in \mathcal{H}(n)$ such that $T(X^*, Y^*, Z^*) = X^*$,
$T(Y^*, X^*, Y^*) = Y^*$ and $T(Z^*, Y^*, X^*) = Z^*$.

Proof: Let $(X,Y,Z) \in \mathcal{H}(n)_{\leq}(U,V,W)$. Then $F(X) \leq F(U), G(Y) \geq G(V)$ and $H(Z) \leq H(W)$. Thus, we have

$$\|T(U,V,W) - T(X,Y,Z)\|_1 = tr(T(U,V,W) - T(X,Y,Z)) =$$
$$\sum_{i=1}^{p} tr(A_i^*(F(U) - F(X))A_i) + \sum_{j=1}^{q} tr(B_j^*(G(Y) - G(V))B_j)$$
$$+ \sum_{k=1}^{r} tr(C_k^*(H(W) - H(Z))B_k) = \sum_{i=1}^{p} tr(A_i^* A_i(F(U) - F(X)))+$$
$$\sum_{j=1}^{q} tr(B_j B_j^*(G(Y) - F(V))) + \sum_{k=1}^{r} tr(C_k C_k^*(H(W) - H(Z))) =$$
$$tr((\sum_{i=1}^{p} A_i A_i^*)(F(U) - F(X))) + tr((\sum_{j=1}^{q} B_j B_j^*)(G(Y) - F(V)))+$$
$$tr((\sum_{k=1}^{r} C_k C_k^*(H(W) - H(Z))) \leq \left\|\sum_{i=1}^{p} A_i A_i^*\right\|$$
$$\|F(U) - F(X)\|_1 + \left\|\sum_{j=1}^{q} B_j B_j^*\right\| \|G(Y) - G(V)\|_1$$
$$+ \left\|\sum_{k=1}^{r} C_k C_k^*\right\| \|H(W) - H(Z)\|_1 \leq$$
$$\frac{\|\sum_{i=1}^{p} A_i A_i^*\|}{M} \|U - X\|_1 + \frac{\left\|\sum_{j=1}^{q} B_j B_j^*\right\|}{M} \|Y - V\|_1 +$$
$$\frac{\|\sum_{k=1}^{r} C_k C_k^*\|}{M} \|W - Z\|_1 \leq \frac{\lambda}{2}(\|U - X\|_1 + \|Y - V\|_1 + \|W - Z\|_1),$$
$$\lambda = 2 \cdot max \frac{\|\sum_{i=1}^{p} A_i A_i^*\|}{M}, \frac{\left\|\sum_{j=1}^{q} B_j B_j^*\right\|}{M}, \frac{\|\sum_{k=1}^{r} C_k C_k^*\|}{M}.$$

From the second assumption and Lemma 4.4.132, we have $\lambda < 1$, so the contractive condition of Theorem 4.2.92 is satisfied, $\forall (X,Y,Z) \in \mathcal{H}(n)_{\leq}(U,V,W)$. From the mixed monotone property of T and the last assumption we get the conclusion of the theorem, that is, there exist $X^*, Y^*, Z^* \in \mathcal{H}(n)$ such that $T(X^*, Y^*, Z^*) = X^*, T(Y^*, X^*, Y^*) = Y^*$ and $T(Z^*, Y^*, X^*) = Z^*$. □

THEOREM 4.4.134. [**60**] *Under the assumptions of Theorem 4.4.133, the equation* (4.64) *has a unique solution* $\overline{X} \in \mathcal{H}(n)$.

Proof: It is known that every $X, Y, Z \in \mathcal{H}(n)$ has an upper and a lower bound.Thus, for any $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \in \mathcal{H}(n) \times \mathcal{H}(n) \times \mathcal{H}(n)$ there exists $(U,V,W) \in \mathcal{H}(n) \times \mathcal{H}(n) \times \mathcal{H}(n)$ such that $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \in \mathcal{H}(n)_{\leq}(U,V,W)$. Now, from Theorem 4.2.94 we get that X^*, Y^*, Z^* from Theorem 4.4.133 are unique and $X^* = Y^* = Z^* = \overline{X}$. □

CHAPTER 5

Coupled coincidence point theorems in metric spaces endowed with a reflexive relation

1. Coincidence points of operators in partially ordered metric spaces-Preliminaries

In this paragraph we will present some basic concepts and fundamental results regarding coupled and tripled coincidence points. Coupled coincidence points were introduced by Ćirić and Lakshmikantham in [**53**], by generalizing the notion of coupled fixed point. The tripled coincidence points we present are obtained by Borcut in [**37**], [**38**] , [**41**].

Definition 5.1.135. [**53**] *Let $(X, \leq)$ a partially ordered space, the operator $F : X^3 \to X$ and the mapping $g : X \to X$. We say that F is mixed g-monotone if $F(x, y, z)$ is g-monotone increasing in x and it is g-monotone decreasing in y, that is, for any $x, y \in X$, we have*

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Similarly, we have the following definition in [**37**]:

Definition 5.1.136. [**37**] *Let $(X, \leq)$ a partially ordered space , the operator $F : X^3 \to X$ and the mapping $g : X \to X$. We say that F is mixed g-monotone if $F(x, y, z)$ is g-monotone increasing in x and z and it is g-monotone decreasing in y, that is, for any $x, y, z \in X$, we have*

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, g(z_2) \leq g(z_1) \Rightarrow F(x, y, z_2) \leq F(x, y, z_1).$$

Note that if g is the identity mapping, by Definitions 5.1.135 and 5.1.136, we obtain Definition 3.1.32 of mixed monotone mappings presented in [**36**], and, respectively, in [**28**].

Definition 5.1.137. [**53**] *An element* $(x,y) \in X^2$ *is called coupled coincidence point for the operator* $F : X^2 \to X$ *and the mapping* $g : X \to X$ *if*

$$F(x,y) = g(x),\ F(y,x) = g(y).$$

Definition 5.1.138. [**38**] *An element* $(x,y) \in X^2$ *is called tripled coincidence point for the mixed* g*-monotone operator* $F : X \times X \times x \to X$ *and* $g : X \to X$ *if*

$$F(x,y,z) = g(x), F(y,x,y) = g(y) \text{ and } F(z,y,x) = g(z).$$

Similarly , if g is the identity mapping, by Definitions 5.1.137 and 5.1.138, we obtain the classical definition of coupled fixed points from [**70**] and, respectively tripled fixed point for mixed monotone mappings in [**28**].

Definition 5.1.139. [**53**] *Let* X *be a nonempty set,* $F : X^2 \to X$ *and operator and* $g : X \to X$ *a mapping. We say that* F *and* g *commute if:*

$$g(F(x,y)) = F(g(x), g(y)), \forall x, y \in X.$$

Borcut provides similar definitions for commuting operators and mappings in [**37**] and [**38**]:

Definition 5.1.140. [**38**] *Let* X *be a nonempty set and let* $F : X^3 \to X$ *and* $g : X \to X$. *We say that* F *and* g *commute if* $g(F(x,y,z)) = F(g(x), g(y), g(z))$.

The main results in [**53**] and [**88**] are given by the next theorems:

Theorem 5.1.141. [**88**] *Let* $(X, \leq)$ *be a partially ordered metric space and* d *a metric on* X *such that* (X,d) *is a complete metric space. Let* $F : X^2 \to X$ *be an operator and* $g : X \to X$ *be a function, where* F *is mixed* g*-monotone. Suppose that there exists the constant* $k \in [0,1)$ *such that*

$$d\left(F\left(x,y\right), F\left(u,v\right)\right) \leq \frac{k}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] \tag{5.65}$$

for every $x, y, u, v \in X$ *with* $g\left(x\right) \leq g\left(u\right), g\left(y\right) \geq g\left(v\right)$.

Suppose that $F\left(X^2\right) \subseteq g\left(X\right)$, g *is continuous and it commutes with* F *and the following hold:*

(a) F *is continuous or*

(b) X *has the following properties:*

(i) *if there exists the increasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n,

(ii) *if there exists the decreasing sequence* $\{y_n\} \to y$, *then* $y_n \geq y$ *for all* n.

If there exist $x_0, y_0 \in X$ *such that*

$$g\left(x_0\right) \leq F\left(x_0, y_0\right) \text{ and } g\left(y_0\right) \geq F\left(y_0, x_0\right),$$

then there exist $x, y \in X$ *such that*

$$g\left(x\right) = F\left(x,y\right) \text{ and } g\left(y\right) = F\left(y,x\right).$$

THEOREM 5.1.142. [**53**] *Let $(X, \leq)$ a partially ordered space and let d be a metric on X such that the metric space (X, d) is complete. Suppose that there exists the mapping $\varphi : [0, \infty) \to [0, \infty)$, where $\varphi(t) < t$ and $\lim_{r \to t} \varphi(r) < t$ for any $t > 0$. Let $F : X^2 \to X$ be an operator and $g : X \to X$ a function where F is mixed g-monotone and*

$$d(F(x, y), F(u, v)) \leq \varphi \left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2} \right), \tag{5.66}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u), g(y) \geq g(v)$.

We suppose that $F(X^2) \subseteq g(X)$, g is continuous and it commutes with F and the following hold:

(1) F is continuous or

(2) X has the following properties:
- *if there exists an increasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n;*
- *if there exists a decreasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x).$$

THEOREM 5.1.143. [**53**] *In addition to the hypothesis of Theorem 5.1.142, suppose that for every $(x, y), (x^*, y^*) \in X^2$ there exists $(u, v) \in X \times X$, such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and to $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have a unique coincidence point, that is, there exists a unique point $(x, y) \in X^2$, such that*

$$x = g(x) = F(x, y) \text{ and } y = g(y) = F(y, x).$$

Berinde extends and generalizes these results in [**26**] and [**27**] by considering weaker symmetric contractive conditions:

THEOREM 5.1.144. [**26**] *Let $(X, \leq)$ a partially ordered space and let d be a metric on X such that (X, d) is a complete metric space. Let$g : X \to X$ and $F : X^2 \to X$ be a mixed g-monotone mapping for which there exists $\varphi : [0, \infty) \to [0, \infty)$ where $\varphi(t) < t$ and $\lim_{r \to t} \varphi(r) < t$ for any $t > 0$ and*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq 2\varphi \left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2} \right), \tag{5.67}$$

for all $x, y, u, v \in X$ with $g(x) \leq g(u), g(y) \geq g(v)$.

We suppose that $F(X^2) \subseteq g(X)$, g is continuous and it commutes with F and also suppose either:

(1) F is continuous or;

(2) X has the following properties:

- *if there exists an increasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n;*
- *if there exists a decreasing sequence $\{y_n\} \to y$, then $y_n \ge y$ for all n.*

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \le F(x_0, y_0) \text{ and } g(y_0) \ge F(y_0, x_0),$$

then there exist $\overline{x}, \overline{y} \in X$ such that
$g(\overline{x}) = F(\overline{x}, \overline{y})$ and $g(\overline{y}) = F(\overline{y}, \overline{x})$, that is, F and g have a coupled coincidence point.

THEOREM 5.1.145. [**26**] *Let $(X, \le)$ a partially ordered space and let d be a metric on X such that (X, d) is a complete metric space. Let $g : X \to X$ and $F : X^2 \to X$ be a mixed g-monotone mapping for which there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$ with $g(x) \le g(u), g(y) \ge g(v)$,*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \le k[d(g(x), g(u)) + d(g(y), g(v))]. \tag{5.68}$$

Suppose $F(X^2) \subseteq g(X)$, g is continuous and it commutes with F and also suppose either:

(1) F is continuous or;

(2) X has the following properties:

- *if there exists an increasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n;*
- *if there exists a decreasing sequence $\{y_n\} \to y$, then $y_n \ge y$ for all n.*

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \le F(x_0, y_0) \text{ and } g(y_0) \ge F(y_0, x_0),$$

then there exist $\overline{x}, \overline{y} \in X$ such that
$g(\overline{x}) = F(\overline{x}, \overline{y})$ and $g(\overline{y}) = F(\overline{y}, \overline{x})$, that is, F and g have a coupled coincidence point.

The following results are obtained by Borcut in the case of tripled coincidence points in [**37**], [**38**], [**40**]:

THEOREM 5.1.146. [**37**] *Let $(X, \le)$ a partially ordered space and let d be a metric on X such that (X, d) is a complete metric space. Let$g : X \to X$ and $F : X^3 \to X$ be a mixed g-monotone mapping.*
Suppose there exist $j, k, l \in [0, 1), j + k + l < 1$, such that

$$d(F(x,y,z), F(u,v,w)) \le j \cdot d(g(x), g(u)) + k \cdot d(g(y), g(v)) + l \cdot d(g(z), g(w)) \tag{5.69}$$

for all $x, y, z, u, v, w \in X$ with $g(x) \le g(u), g(y) \ge g(v)$ and $g(z) \le g(w)$.

We suppose that $F(X^2) \subseteq g(X)$, g is continuous and it commutes with F and also suppose either:

(1) F is continuous or

(2) X has the following properties:

- *if there exists an increasing sequence* $\{x_n\} \to x$*, then* $x_n \leq x$ *for all* n*;*
- *if there exists a decreasing sequence* $\{y_n\} \to y$*, then* $y_n \geq y$ *for all* n*.*

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$g(x_0) \leq F(x_0, y_0, z_0),\ g(y_0) \geq F(y_0, x_0, y_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $\overline{x}, \overline{y} \in X$ *such that*
$g(\overline{x}) = F(\overline{x}, \overline{y}, \overline{z})$ *and* $g(\overline{y}) = F(\overline{y}, \overline{x}, \overline{y})$ *and* $g(\overline{z}) = F(\overline{z}, \overline{y}, \overline{x})$*, that is,* F *and* g *have a coupled coincidence point.*

THEOREM 5.1.147. [**37**] *In addition to the hypothesis of Theorem 5.1.146, suppose that for every* $(x, y, z), (x^*, y^*, z^*) \in X^3$ *there exists* $(u, v, w) \in X^3$*, such that* $(F(u, v, w), F(v, u, w), F(w, v, u))$ *is comparable to* $(g(x), g(y), g(z))$ *and to* $(g(x^*), g(y^*), g(z^*))$*. Then* F *and* g *have a unique coincidence point, that is, there exists a unique point* $(x, y, z) \in X \times X$*, such that*

$$x = g(x) = F(x, y, z),\ y = g(y) = F(y, x, y) \text{ and } z = g(z) = F(z, y, x).$$

Similar results are obtained in the case of g-monotone operators by Borcut in [**38**]:

DEFINITION 5.1.148. [**38**] *Let* $(X, \leq)$ *a partially ordered space, the operator* $F : X^3 \to X$ *and the mapping* $g : X \to X$*. We say that* F *is* g*-monotone, if* $F(x, y, z)$ *is* g*-monotone increasing (decreasing) in* x, y, z *, that is for every* $x, y, z \in X$*, we have*

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \leq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, g(z_2) \leq g(z_1) \Rightarrow F(x, y, z_2) \geq F(x, y, z_1).$$

DEFINITION 5.1.149. [**38**] *An element* $(x, y, z) \in X^3$ *is a tripled coincidence point for the* g*-monotone operator* F *and the mapping* g *if*

$$F(x, y, z) = g(x),\ F(y, x, z) = g(y),\ F(z, y, x) = g(z).$$

REMARK 5.1.150. *Note that the concept of tripled coincidence point for* g*-monotone operators is different of that for mixed* g*-monotone operators.*

Borcut also provides a result regarding the existence of a tripled coincidence point for this kind of operators:

THEOREM 5.1.151. [**38**] *Let* $(X, \leq)$ *a partially ordered space and* d *a metric on* X*, such that* (X, d) *is a complete metric space. Let* $F : X^3 \to X$ *be an operator and* $g : X \to X$ *a mapping such that* F *is* g*-monotone. Suppose that there exist* $j, k, l \in [0, 1)$ *with* $j + k + l < 1$*, such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(g(x), g(u)) + kd(g(y), g(v))$$

$$+ld\left(g\left(z\right),g\left(w\right)\right),$$

for every $x,y,z,u,v,w\in X$ *cu* $g\left(x\right)\leq g\left(u\right),\ g\left(y\right)\geq g\left(v\right),\ g\left(z\right)\leq g\left(w\right)$.

Suppose that $F\left(X^{3}\right)\subseteq g\left(X\right)$, g *is continuous and it commutes with* F *and one of the following hold :*

- F *is continuous*
- X *has the following property:*

$$\text{if we have the increasing sequence } \{x_n\}\rightarrow x,\ \text{ cu } x_n\leq x \text{ for any } n, \tag{5.70}$$

If there exist $x_0,y_0,z_0\in X$, *such that*

$$g\left(x_0\right)\leq F\left(x_0,y_0,z_0\right),g\left(y_0\right)\leq F\left(y_0,x_0,z_0\right)\ \text{ and } g\left(z_0\right)\leq F\left(z_0,y_0,x_0\right)$$

then there exist $x,y,z\in X$, *such that*

$$g\left(x\right)=F\left(x,y,z\right),g\left(y\right)=F\left(y,x,z\right)\ \text{ and } g\left(z\right)=F\left(z,y,x\right).$$

The author also presents many variations of this result, based on this last theorem, by replacing the contractive condition by weaker ones, using one constant instead of three.(see [**38**]).

The following result establishes the uniqueness of the tripled coincidence point:

THEOREM 5.1.152. [**38**] *In addition to the hypothesis of Theorem 5.1.151, if, for every* $(x,y,z),(x^*,y^*,z^*)\in X^3$, *there exists* $(u,v,w)\in X^3$ *such that* $(F(u,v,w),F(v,u,w),F(w,v,u))$ *is comparable to* $(g(x),g(y),g(z))$ *and to* $(g(x^*),g(y^*),g(z^*))$, *then* F *and* g *have a unique tripled coincidence point.*

Another important result is provided in [**13**] by Aydi, Karapinar and Postolache, in the case of mixed$-g-$monotone operators. The improvement they brought to the results of Borcut is the symmetrization of the contractive condition, following the idea of Berinde in [**25**].

THEOREM 5.1.153. [**13**] *Let* $(X,\leq)$ *a partially ordered set and let d be a metric on* X *such that* (X,d) *is a complete metric space. Letg* $:X\rightarrow X$ *and* $F:X^3\rightarrow X$ *be a mixed g-monotone mapping Suppose there exist* $\varphi\in\Phi$, *such that*

$$\begin{aligned} &d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))\\ &\qquad\leq 3\cdot\varphi\left(\frac{d(g(x),g(u))+d(g(y),g(v))+d(g(z),g(w))}{3}\right)\end{aligned} \tag{5.71}$$

for all $x,y,z,u,v,w\in X$ *with* $g(x)\leq g(u),g(y)\geq g(v)$ *and* $g(z)\leq g(w)$.

We suppose that $F(X^3)\subset g(X)$, g *is continuous and it commutes with* F *and* F *is continuous.*

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$g(x_0) \leq F(x_0, y_0, z_0),\ g(y_0) \geq F(y_0, x_0, y_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $\overline{x}, \overline{y} \in X$ *such that*
$g(\overline{x}) = F(\overline{x}, \overline{y}, \overline{z})$ *and* $g(\overline{y}) = F(\overline{y}, \overline{x}, \overline{y})$ *and* $g(\overline{z}) = F(\overline{z}, \overline{y}, \overline{x})$*, that is* F *and* g *have a coupled coincidence point.*

2. Definitions

In this paragraph we will present the definitions of lower-R-coupled coincidence point, mixed $g - R-$monotony of a mapping and related concepts used for obtaining the results presented in the next section.

NOTATION 4. *Let* X *be a nonempty set and let* $f : X \times X \to X$ *and* $g : X \to X$ *be two mappings. Then*

(1) *The cartesian product of* f *and itself is denoted by* $f \times f$ *and it is defined by*

$$f \times f(x, y) = (f(x, y), f(y, x)).$$

(2) *We will denote by* $f^0(x, y) = x$ *and* $f^n(x, y) = f(f^{n-1}(x, y), f^{n-1}(y, x))$*, for all* $x, y \in X, n \in \mathbb{N}$.

(3) *The cartesian product of* f *and* g *is denoted by* $f \times g$ *and is defined by*

$$(f \times g)(x, y) = (g(f(x, y)), g(f(y, x)))$$

(4) *We will denote by* $g^0(x) = x$ *and* $g^n(x) = g(x^{n-1}(x))$*, for all* $x \in X, n \in \mathbb{N}$.

DEFINITION 5.2.154. [**58**] *Let* X *be a nonempty set and let* R *be a reflexive relation on* X*,* $f : X^2 \to X$*,* $g : X \to X$*. The mapping* f *has the* ***mixed*** $g - R-$***monotone property*** *on* X *if* $(f \times g)(X_R(x, y)) \subseteq X_R((f \times g)(x, y))$*, for all* $(x, y) \in X^2$.

DEFINITION 5.2.155. [**58**] *An element* $(x, y) \in X^2$ *is called* ***lower-***$R-$***coupled coincidence point*** *for* f *and* g*, if* $(f \times g)(x, y) \in X_R(x, y)$.

Next, starting from the orbital continuity presented in [**9**], we will define the orbital g-continuity of a mapping f.

DEFINITION 5.2.156. [**58**] *The mapping* f *is called* ***orbitally*** g***-continuous*** *if* $(x, y), (a, b) \in X^2$ *and* $f^{n_k}(x, y) \to a, f^{n_k}(y, x) \to b$*, when* $k \to \infty$*, implies* $f^{n_k+1}(x, y) \to g(a)$ *and* $f^{n_k+1}(y, x) \to g(b)$ *when* $k \to \infty$.

3. Existence and uniqueness theorems

THEOREM 5.3.157. [**58**] *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^2 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii) there exist $k, l \in [0, 1), k + l < 1$ such that

$$d(f(x, y), f(z, t)) \leq k \cdot d(g(x), g(z)) + l \cdot d(g(y), g(t)), \forall (x, y) \in X_R(z, t); \tag{5.72}$$

(iv) f and g have a lower-R-coupled coincidence point;
(v) $f(X^2) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a coupled coincidence point, that is, there exists $(x, y) \in X^2$ such that $f(x, y) = g(x)$ and $f(y, x) = g(y)$.

Proof: Since f and g have a lower-R-coupled coincidence point, let (x_0, y_0) be it. Thus, $(f \times g)(x_0, y_0) \in X_R(x_0, y_0)$.
From (i) we have that $(f \times g)(X_R(x_0, y_0)) \subseteq X_R((f \times g)(x_0, y_0))$.
Further, it can easily be checked that
$(g^n(f(x_0, y_0)), g^n(f(y_0, x_0))) \in X_R(g^{n-1}(f(x_0, y_0)), g^{n-1}(f(y_0, x_0)))$.
Since $f(X^2) \subseteq g(X)$, let $x_1, y_1 \in X$ such that $g(x_1) = f(x_0, y_0)$, $g(y_1) = f(y_0, x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$g(x_{n+1}) = f(x_n, y_n), g(y_{n+1}) = f(y_n, x_n) \tag{5.73}$$

Now, using (iii), we have that

$$\begin{gathered}
d(f(g^n(f(x_0, y_0)), g^n(f(y_0, x_0))), f(g^{n-1}(f(x_0, y_0)), g^{n-1}(f(y_0, x_0)))) \\
\leq k^n \cdot d(g(g^n(f(x_0, y_0))), g(g^{n-1}(f(x_0, y_0)))) + l^n \cdot d(g(g^n(f(y_0, x_0))), g(g^{n-1}(f(y_0, x_0)))) \\
\Leftrightarrow d(f(g^n(f(x_0, y_0)), g^n(f(y_0, x_0))), f(g^{n-1}(f(x_0, y_0)), g^{n-1}(f(y_0, x_0)))) \\
\leq k^n \cdot d(g^{n+1}(f(x_0, y_0)), g^n(f(x_0, y_0))) + l^n \cdot d(g^{n+1}(f(y_0, x_0))), g^n(f(y_0, x_0)) \\
\Leftrightarrow d(f(g^n(g(x_1)), g^n(g(y_1))), f(g^{n-1}(g(x_1)), g^{n-1}(g(y_1)))) \\
\leq k^n \cdot d(g^{n+1}(g(x_1)), g^n(g(x_1))) + l^n \cdot d(g^{n+1}(g(y_1)), g^n(g(y_1))) \\
\Leftrightarrow d(f(g^{n+1}(x_1), g^{n+1}(y_1)), f(g^n(x_1), g^n(y_1))) \\
\leq k^n \cdot d(g^{n+2}(x_1), g^{n+1}(x_1)) + l^n \cdot d(g^{n+2}(y_1), g^{n+1}(y_1))
\end{gathered}$$

For $n = 0$, we get

$$d(f(g(x_1), g(y_1)), f(x_1, y_1))) \leq d(g^2(x_1), g(x_1)) + d(g^2(y_1), g(y_1))$$

If $n \to \infty$, we get

$$d(f(g^{n+1}(x_1), g^{n+1}(y_1)), f(g^n(x_1), g^n(y_1))) \leq 0$$

But $d(x,y) \geq 0, \forall x, y \in \mathbb{R}$. We get $:f(g^{n+1}(x_1), g^{n+1}(y_1)) = f(g^n(x_1), g^n(y_1))$, so $g(x_1) = x_1 = f(x_0, y_0)$ This implies that $\{g^n(x_1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Similarly, we get that $\{g^n(y_1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Now, because (X,d) is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x, \lim_{n\to\infty} g(y_n) = y. \tag{5.74}$$

From the continuity of g, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \lim_{n\to\infty} g(g(y_n)) = g(y).$$

Because f and g commute, and from (5.73), we have

$$g(g(x_{n+1})) = g(f(x_n, y_n)) = f(g(x_n), g(y_n))$$

and

$$g(g(y_{n+1})) = g(f(y_n, x_n)) = f(g(y_n), g(x_n)).$$

From (5.74) and the orbital continuity of f we get

$$g(x) = f(x,y)$$

and

$$g(y) = f(y,x).$$

□

COROLLARY 5.3.158. [**58**] *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^2 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii) there exist $\alpha \in [0,1)$ such that

$$d(f(x,y), f(z,t)) \leq \frac{\alpha}{2}[d(g(x),g(z)) + d(g(y),g(t))], \forall (x,y) \in X_R(z,t); \tag{5.75}$$

(iv) f and g have a lower-R-coupled coincidence point;
(v) $f(X^2) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a coupled coincidence point, that is, there exists $(x,y) \in X^2$ such that $f(x,y) = g(x)$ and $f(y,x) = g(y)$.

Proof: From the proof of Theorem 5.3.157, for $k = l = \dfrac{\alpha}{2}, \alpha \in [0, 1)$, there exist $x, y \in X$ such that

$$g(x) = f(x, y)$$

and

$$g(y) = f(y, x).$$

□

THEOREM 5.3.159. [**58**] *In addition to the hypothesis of Theorem 5.3.157, suppose that for every $(x, y), (\overline{x}, \overline{y}) \in X^2$, there exists $(u, v) \in X^2$ such that $(g(x), g(y)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u))$. Then f and g have a unique coupled coincidence point.*

Proof: According to the proof of Theorem 5.3.157, there exist $\overline{x}, \overline{y} \in X$ such that $f(\overline{x}, \overline{y}) = g(\overline{x})$ and $f(\overline{y}, \overline{x}) = g(\overline{y})$. We have to show that, if (x, y) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y})), (g(x), g(y))) = 0.$$

Because both (x, y) and $(\overline{x}, \overline{y})$ are coupled coincidence points, we have

$$g(x) = f(x, y), g(y) = f(y, x)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}), g(\overline{y}) = f(\overline{y}, \overline{x}).$$

Now, let $u_0 = u$ and $v_0 = v$. Then, there exist $u_1, v_1 \in X^2$ such that $g(u_1) = f(u_0, v_0)$, $g(v_1) = f(v_0, u_0)$. Using the same procedure as in the proof of Theorem 5.3.157, we obtain the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(u_{n+1}) = f(u_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n).$$

Furthermore, let $x_0 = x, y_0 = y$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{\overline{x}_n\}_{n\in\mathbb{N}}$ and $\{\overline{y}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n) = f(x, y), g(y_n) = f(y, x)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}), g(\overline{y}_n) = f(\overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(u, v) \in X^2$ such that

$$(g(x), g(y)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u)).$$

From $(g(x_0), g(y_0)) \in X_R(f(u, v), f(v, u))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0)), f^n(g(y_0), g(x_0))) \in X_R(f^{n+1}(u, v), f^{n+1}(v, u))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0)), f^{n+1}(u, v)) \leq k^n \cdot d(g(x_0), f(u, v)) + l^n \cdot d(g(y_0), f(v, u))$$

and

$$d(f^n(g(y_0), g(x_0)), f^{n+1}(v, u)) \leq k^n \cdot d(g(x_0), f(u, v)) + l^n \cdot d(g(y_0), f(v, u)).$$

Summing up, we obtain that

$$d(f^n(g(x_0), g(y_0)), f^{n+1}(u, v)) + d(f^n(g(y_0), g(x_0)), f^{n+1}(v, u)) \leq$$
$$2k^n \cdot d(g(x_0), f(u, v)) + 2l^n \cdot d(g(y_0), f(v, u)).$$

But $x_0 = x$ and $y_0 = y$. We obtain

$$d(f^n(g(x), g(y)), f^{n+1}(u, v)) + d(f^n(g(y), g(x)), f^{n+1}(v, u)) \leq$$
$$2k^n \cdot d(g(x), f(u, v)) + 2l^n \cdot d(g(y), f(v, u)).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(y), f(v, u)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{y}), f(v, u)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x), g(\overline{x})) \leq d(g(x), f(u, v)) + d(f(u, v), g(\overline{x})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(y), g(\overline{y})) \leq d(g(y), f(v, u)) + d(f(v, u), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

Now, let's recall the definition of a mapping φ introduced in [**53**] by Ćirić and Lakshmikantham: Let $\varphi : [0, \infty) \to [0, \infty)$ satisfying :

i) $\varphi(t) < t, \forall t \in (0, \infty)$;

ii) $\lim_{r\to t_+} \varphi(r) < t, \forall t \in (0, \infty)$;

The set of all these mappings φ is denoted by Φ.

Replacing the contraction condition (5.72) with one that uses the mapping φ defined above, following the idea in [**53**], we obtain:

Theorem 5.3.160. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^2 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;

(ii) f is orbitally g-continuous;

(iii)

$$d(f(x,y),f(z,t)) \leq \varphi\left(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\right), \forall (x,y) \in X_R(z,t), \tag{5.76}$$

where $\varphi \in \Phi$;

(iv) f and g have a lower-R-coupled coincidence point;

(v) $f(X^2) \subseteq g(X)$;

(vi) g is continuous ;

(vii) f and g commute.

Then f and g have a coupled coincidence point, that is, there exists $(x,y) \in X^2$ such that $f(x,y) = g(x)$ and $f(y,x) = g(y)$.

Proof: Since f and g have a lower-R-coupled coincidence point, let (x_0,y_0) be it. Thus, $(f \times g)(x_0,y_0) \in X_R(x_0,y_0)$.
From (i) we have that $(f \times g)(X_R(x_0,y_0)) \subseteq X_R((f \times g)(x_0,y_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0,y_0)), g^n(f(y_0,x_0))) \in X_R(g^{n-1}(f(x_0,y_0)), g^{n-1}(f(y_0,x_0))). \tag{5.77}$$

Since $f(X^2) \subseteq g(X)$, let $x_1, y_1 \in X$ such that $g(x_1) = f(x_0,y_0)$, $g(y_1) = f(y_0,x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$g(x_{n+1}) = f(x_n,y_n), g(y_{n+1}) = f(y_n,x_n) \tag{5.78}$$

Let's consider the nonnegative sequence $\{z_n\}_{n\in\mathbb{N}^*}$ such that $z_n = d(g(x_{n+1}),g(x_n)) + d(g(y_{n+1}),g(y_n)), n \in \mathbb{N}^*$.
Now, using (5.76), (5.77) and letting $x := x_n$ and $y := y_n$, $z := x_{n-1}$ and $t := y_{n-1}$, we obtain

$$d(g(x_{n+1}),g(x_n)) = d(f(x_n,y_n),f(x_{n-1},y_{n-1})) \leq \varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))}{2}\right) =$$

$$\varphi\left(\frac{z_{n-1}}{2}\right)$$

and

$$d(g(y_{n+1}),g(y_n)) = d(f(y_n,x_n),f(y_{n-1},x_{n-1})) \leq \varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))}{2}\right) =$$

$$\varphi\left(\frac{z_{n-1}}{2}\right).$$

By summing up the last two relations, we get that

$$d(g(x_{n+1}),g(x_n)) + d(g(y_{n+1}),g(y_n)) = z_n \leq 2 \cdot \varphi\left(\frac{z_{n-1}}{2}\right).$$

Now, using the properties of φ, we have that

$$z_n \leq 2 \cdot \varphi\left(\frac{z_{n-1}}{2}\right) < 2 \cdot \frac{z_{n-1}}{2} = z_{n-1}. \tag{5.79}$$

Thus, the sequence $\{z_n\}_{n\in\mathbb{N}^*}$ is decreasing and nonnegative. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} z_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (5.79), let $n \to \infty$. Using (ii) (the second condition satisfied by φ), we have

$$\varepsilon_0 = \lim_{n\to\infty} z_n \leq 2 \cdot \lim_{n\to\infty} \varphi\left(\frac{z_{n-1}}{2}\right) = 2 \cdot \lim_{z_{n-1}\to\varepsilon_0+} \varphi\left(\frac{z_{n-1}}{2}\right) < \varepsilon_0,$$

which is a contradiction. Thus, $\lim_{n\to\infty} z_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0$.
Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$ and $\{g(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) \geq \delta, \tag{5.80}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{N}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (5.80). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) < \delta. \tag{5.81}$$

Now, using the triangle inequality and the last two inequalities ((5.80) and (5.81)), we have

$$\begin{aligned}\delta &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) \\ &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(x_{n_2(k)}), g(x_{n_1(k)})) \\ &+d(g(y_{n_2(k)}), g(y_{n_1(k)})) \leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + \delta.\end{aligned}$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty}[d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)}))] = \delta.$$

Now we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}s_k &= d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) \\ &\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) \\ &+d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) = z_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)}))\end{aligned}$$

$$\leq z_{n_1(k)} + z_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})). \tag{5.82}$$

But

$$d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1}))$$

$$= d(f(x_{n_1(k)}, y_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)})) + d(f(y_{n_1(k)}, x_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}))$$

$$\leq 2 \cdot \varphi\left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))}{2}\right)$$

$$\leq 2 \cdot \varphi\left(\frac{s_k}{2}\right).$$

Now, returning to (5.82), we have

$$s_k \leq z_{n_1(k)} + z_{n_2(k)} + 2 \cdot \varphi\left(\frac{s_k}{2}\right).$$

Let $k \to \infty$. Thus, using property ii of φ, we obtain

$$\delta \leq 2 \cdot \lim_{k\to\infty} \varphi\left(\frac{s_k}{2}\right) = 2 \cdot \lim_{s_k \to \delta_+} \varphi(\frac{k}{2}) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{g(x_n)\}_{n\in\mathbb{N}}$ and $\{g(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since X is complete, there exist $\overline{x}$ and $\overline{y}$ such that $g^n(x_n) \to \overline{x}$ and $g^n(y_n) \to \overline{y}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n) \to \overline{x}$ and $f^{n-1}(y_n, x_n) \to \overline{y}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n) \to g(\overline{x})$ and $f^n(y_n, x_n) \to g(\overline{y})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y})$ is a coupled coincidence point for f and g. □

THEOREM 5.3.161. *In addition to the hypothesis of Theorem 5.3.160, suppose that for every $(x^*, y^*), (\overline{x}, \overline{y}) \in X^2$, there exists $(u, v) \in X^2$ such that $(g(x^*), g(y^*)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u))$. Then f and g have a unique coupled coincidence point.*

Proof: From Theorem 5.3.160, there exist $\overline{x}, \overline{y} \in X$ such that $f(\overline{x}, \overline{y}) = g(\overline{x})$ and $f(\overline{y}, \overline{x}) = g(\overline{y})$. We have to show that, if (x^*, y^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y})), (g(x^*), g(y^*))) = 0.$$

Since (x^*, y^*) and $(\overline{x}, \overline{y})$ are both coupled coincidence points, it follows that

$$g(x^*) = f(x^*, y^*), g(y^*) = f(y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}), g(\overline{y}) = f(\overline{y}, \overline{x}).$$

Now, using the hypothesis of Theorem 5.3.160, from $f(X^2) \subseteq g(x)$, there exist $u_1, v_1 \in X^2$ such that $g(u_1) = f(u_0, v_0)$, $g(v_1) = f(v_0, u_0)$. Using the same procedure as in the proof of Theorem 5.3.157, we build the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(u_{n+1}) = f(u_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n).$$

Next, let $x_0 = x^*, y_0 = y^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{\overline{x}_n\}_{n\in\mathbb{N}}$ and $\{\overline{y}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*), g(y_n^*) = f(y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}), g(\overline{y}_n) = f(\overline{y}, \overline{x}).$$

From the hypothesis, we know that there exists $(u, v) \in X^2$ such that

$$(g(x^*), g(y^*)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u)).$$

From $(g(x_0), g(y_0)) \in X_R(f(u, v), f(v, u))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0)), f^n(g(y_0), g(x_0))) \in X_R(f^{n+1}(u, v), f^{n+1}(v, u))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0)), f^{n+1}(u, v)) \leq \varphi\left(\frac{d(g(x_0), f(u, v)) + d(g(y_0), f(v, u))}{2}\right)$$

and

$$d(f^n(g(y_0), g(x_0)), f^{n+1}(v, u)) \leq \varphi\left(\frac{d(g(x_0), f(u, v)) + d(g(y_0), f(v, u))}{2}\right).$$

Summing up, we obtain that

$$d(f^n(g(x_0), g(y_0)), f^{n+1}(u, v)) + d(f^n(g(y_0), g(x_0)), f^{n+1}(v, u)) \leq$$
$$2 \cdot \varphi\left(\frac{d(g(x_0), f(u, v)) + d(g(y_0), f(v, u))}{2}\right).$$

But $x_0 = x^*$ and $y_0 = y^*$. We obtain

$$d(f^n(g(x^*), g(y^*)), f^{n+1}(u, v)) + d(f^n(g(y^*), g(x^*)), f^{n+1}(v, u)) \leq$$
$$2 \cdot \varphi\left(\frac{d(g(x^*), f(u, v)) + d(g(y^*), f(v, u))}{2}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(y^*), f(v, u)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{y}), f(v, u)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(u, v)) + d(f(u, v), g(\overline{x})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(v, u)) + d(f(v, u), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

Now, by symmetrizing the contraction, using the idea in [**26**], we obtain the following result:

THEOREM 5.3.162. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^2 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g-R$-monotone;
(ii) f is orbitally g-continuous;
(iii)

(5.83)

$$d(f(x,y),f(z,t))+d(f(x,y),f(t,z)) \leq 2\cdot\varphi\left(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\right),\forall(x,y)\in X_R(z,t),$$

where $\varphi \in \Phi$;
(iv) f and g have lower-R-coupled coincidence point;
(v) $f(X^2) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a coupled coincidence point, that is, there exists $(x, y) \in X^2$ such that $f(x, y) = g(x)$ and $f(y, x) = g(y)$.

Proof: The proof of this theorem follows the steps of Theorem 5.3.160: Since f and g have a lower-R-coupled coincidence point, let (x_0, y_0) be it. Thus, $(f \times g)(x_0, y_0) \in X_R(x_0, y_0)$.
From (i) we have that $(f \times g)(X_R(x_0, y_0)) \subseteq X_R((f \times g)(x_0, y_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0,y_0)),g^n(f(y_0,x_0))) \in X_R(g^{n-1}(f(x_0,y_0)),g^{n-1}(f(y_0,x_0))). \tag{5.84}$$

Since $f(X^2) \subseteq g(X)$, let $x_1, y_1 \in X$ such that $g(x_1) = f(x_0, y_0)$, $g(y_1) = f(y_0, x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$g(x_{n+1}) = f(x_n,y_n), g(y_{n+1}) = f(y_n,x_n) \tag{5.85}$$

Let's consider the nonnegative sequence $\{z_n\}_{n\in\mathbb{N}^*}$ such that $z_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)), n \in \mathbb{N}^*$.
Now, using (5.76), (5.84) and letting $x := x_n$ and $y := y_n$, $z := x_{n-1}$ and $t := y_{n-1}$, we obtain

$$d(f(x_n,y_n),f(x_{n-1},y_{n-1}))+d(f(y_n,x_n),f(y_{n-1},x_{n-1})) = d(g(x_{n+1}),g(x_n))+d(g(y_{n+1}),g(y_n)) \leq$$
$$2\cdot\varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))}{2}\right) =$$

$$\varphi\left(\frac{z_n}{2}\right)$$

Thus, we have that

$$z_{n+1} \leq 2 \cdot \varphi\left(\frac{z_n}{2}\right) \tag{5.86}$$

Using the proof of Theorem 5.3.160 we have that $\{z_n\}_{n\in\mathbb{N}^*}$ is decreasing and nonnegative. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} z_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (5.86), let $n \to \infty$. Using (i), we have

$$\varepsilon_0 = \lim_{n\to\infty} z_{n+1} \leq 2 \cdot \lim_{n\to\infty} \varphi\left(\frac{z_n}{2}\right) = 2 \cdot \lim_{z_n\to\varepsilon_0+} \varphi\left(\frac{z_n}{2}\right) < \varepsilon_0.$$

So we have that $\varepsilon_0 < \varepsilon_o$, which is, clearly, a contradiction.
Thus, $\lim\limits_{n\to\infty} z_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0$ and

$$\lim_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0.$$

Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$ and $\{g(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) \geq \delta, \tag{5.87}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{N}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (5.87). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) < \delta. \tag{5.88}$$

Now, using the triangle inequality and the last two inequalities ((5.87) and (5.88)), we have

$$\delta \leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)}))$$
$$\leq d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)}))+$$
$$d(g(x_{n_2(k)}), g(x_{n_1(k)})) \leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + \delta.$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)}))] = \delta.$$

Now we will show that $\delta = 0$. Supposing the contrary, we have

$$s_k = d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)}))$$
$$\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_1(k)+1}))$$
$$+d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) = z_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)}))$$

$$\leq z_{n_1(k)} + z_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})). \tag{5.89}$$

But

$$d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1}))$$

$$= d(f(x_{n_1(k)}, y_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)})) + d(f(y_{n_1(k)}, x_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}))$$

$$\leq 2 \cdot \varphi\left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))}{2}\right)$$

$$\leq 2 \cdot \varphi\left(\frac{s_k}{2}\right).$$

Now, returning to (5.89), we have

$$s_k \leq z_{n_1(k)} + z_{n_2(k)} + 2 \cdot \varphi\left(\frac{s_k}{2}\right).$$

Let $k \to \infty$. Thus, using the property ii of φ, we obtain

$$\delta \leq 2 \cdot \lim_{k\to\infty} \varphi\left(\frac{s_k}{2}\right) = 2 \cdot \lim_{s_k\to\delta_+} \varphi\left(\frac{s_k}{2}\right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{g(x_n)\}_{n\in\mathbb{N}}$ and $\{g(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since X is complete, there exist $\overline{x}$ and $\overline{y}$ such that $g^n(x_n) \to \overline{x}$ and $g^n(y_n) \to \overline{y}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n) \to \overline{x}$ and $f^{n-1}(y_n, x_n) \to \overline{y}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n) \to g(\overline{x})$ and $f^n(y_n, x_n) \to g(\overline{y})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y})$ is a coupled coincidence point for f and g. ☐

THEOREM 5.3.163. *In addition to the hypothesis of Theorem 5.3.162, suppose that for every $(x^*, y^*), (\overline{x}, \overline{y}) \in X^2$, there exists $(u, v) \in X^2$ such that $(g(x^*), g(y^*)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u))$. Then f and g have a unique coupled coincidence point.*

Proof: According to Theorem 5.3.162, there exist $\overline{x}, \overline{y} \in X$ such that $f(\overline{x}, \overline{y}) = g(\overline{x})$ and $f(\overline{y}, \overline{x}) = g(\overline{y})$. We have to show that, if (x^*, y^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y})), (g(x^*), g(y^*))) = 0.$$

Because both (x^*, y^*) and $(\overline{x}, \overline{y})$ are coupled coincidence points, we have

$$g(x^*) = f(x^*, y^*), g(y^*) = f(y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}), g(\overline{y}) = f(\overline{y}, \overline{x}).$$

From $f(X^2) \subseteq g(X)$, there exist u_0, v_0 in X such that $g(u_1) = f(u_0, v_0)$, $g(v_1) = f(v_0, u_0)$. Following the procedure used in the proof of Theorem 5.3.157, we obtain the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(u_{n+1}) = f(u_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n).$$

Furthermore, let $x_0^* = x^*, y_0^* = y^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{\overline{x}_n\}_{n\in\mathbb{N}}$ and $\{\overline{y}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*), g(y_n^*) = f(y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}), g(\overline{y}_n) = f(\overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(u, v) \in X^2$ such that

$$(g(x^*), g(y^*)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u, v), f(v, u)).$$

From $(g(x_0), g(y_0)) \in X_R(f(u, v), f(v, u))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0)), f^n(g(y_0), g(x_0))) \in X_R(f^{n+1}(u, v), f^{n+1}(v, u))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0)), f^{n+1}(u, v)) + d(f^n(g(y_0), g(x_0)), f^{n+1}(v, u)) \leq \varphi\left(\frac{d(g(x_0), f(u, v)) + d(g(y_0), f(v, u))}{2}\right).$$

But $x_0^* = x^*$ and $y_0^* = y^*$. We obtain

$$d(f^n(g(x^*), g(y^*)), f^{n+1}(u, v)) + d(f^n(g(y^*), g(x^*)), f^{n+1}(v, u)) \leq \varphi\left(\frac{d(g(x^*), f(u, v)) + d(g(y^*), f(v, u))}{2}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(y^*), f(v, u)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(u, v)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{y}), f(v, u)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(u, v)) + d(f(u, v), g(\overline{x})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(v, u)) + d(f(v, u), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

which means that

$$g(x^*) = g(\overline{x})$$

and

$$g(y^*) = g(\overline{y}).$$

□

Next, we let $\varphi(t) = kt, k \in [0,1)$. It is easy to check that conditions (i) and (ii) still hold. We obtain the following result:

THEOREM 5.3.164. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^2 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g-R$-monotone;
(ii) f is orbitally $g-$continuous;
(iii) there exist $k \in [0,1)$ such that

(5.90)
$$d(f(x,y), f(z,t)) + d(f(y,x), f(t,z)) \leq k \cdot [d(g(x), g(z)) + d(g(y), g(t))], \forall (x,y) \in X_R(z,t);$$

(iv) f and g have a lower-R-coupled coincidence point;
(v) $f(X^2) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a coupled coincidence point, that is, there exists $(x,y) \in X^2$ such that $f(x,y) = g(x)$ and $f(y,x) = g(y)$.

Proof: In Theorem 5.3.162, let $\varphi(t) = kt, k \in [0,1)$. □

THEOREM 5.3.165. *In addition to the hypothesis of Theorem 5.3.164, suppose that for every $(x,y), (\overline{x},\overline{y}) \in X^2$, there exists $(u,v) \in X^2$ such that $(g(x), g(y)), (g(\overline{x}), g(\overline{y})) \in X_R(f(u,v), f(v,u))$. Then f and g have a unique coupled coincidence point.*

Proof: In Theorem 5.3.161, let $\varphi(t) = kt, k \in [0,1)$. □

REMARK 5.3.166. *If, in Theorem 5.3.158 we take $R = \leq$ and we assume that f is continuous, we obtain Theorem 5.1.141 from* [**53**].

REMARK 5.3.167. *If, in Theorem 5.3.159, we take $R = \leq$, we obtain Theorem 5.1.143 from* [**53**].

REMARK 5.3.168. *If, in Theorem 5.3.162, resp. 5.3.164, we take $R = \leq$ and we assume that f is continuous, we obtain Theorem 5.1.144, resp. 5.1.145 from* [**26**].

4. Examples and applications

4.1. Examples

EXAMPLE 5.4.169. *Let* $X = \mathbb{R}$, *the metric* $d(x,y) = |x-y|$, *the relation* R *on* X *given by*

$$xRy \Leftrightarrow \frac{x^2-2y}{3} = \frac{y^2-2x}{3}$$
$$\Leftrightarrow (x-y)(x+y+2) = 0.$$

Thus, we have $X_R(x,y) = (x,y)(x,-y-2),(-x-2,y),(-x-2,-y-2)$. *Let* $f : X^2 \to X$ *be defined by*

$$f(x,y) = \frac{2x-3y+1}{6}$$

and $g : X \to X$, *where*

$$g(x) = \frac{2x}{3}.$$

Hence, f *has the mixed* $g-R-$*monotone property. It can easily be checked that* f *and* g *satisfy all the other conditions of Theorem 5.3.157. The contraction also holds for* $k = \frac{1}{2}$ *and* $l = \frac{3}{4}$*:*

$$d(f(x,y), f(z,t)) = \left|\frac{2x-3y+1}{6} - \frac{2z-3t+1}{6}\right| =$$

$$\left|\frac{2(x-z)-3(y-t)}{6}\right| \leq \frac{3}{4}|g(x)-g(z)| + \frac{1}{2}|g(y)-g(t)|$$

$$= \frac{1}{2}d(g(x),g(z)) + \frac{3}{4}d(g(y),g(t)), \forall (x,y) \in X_R(z,t).$$

So, by Theorem 5.3.157, we obtain that f *and* g *have a coupled coincidence point,* $\left(\frac{1}{5}, \frac{1}{5}\right)$. *Note that, in this case, Theorem 5.1.141 from* [**88**] *cannot be applied because* R *is not antisymmetric, so it is not a relation of partial order.*

EXAMPLE 5.4.170. *Let* $X = \mathbb{R}_+$, *the metric* $d(x,y) = |x-y|$, *the relation* R *on* X *given by*

$$xRy \Leftrightarrow x^2 - x = y^2 - y.$$

Let $f : X^2 \to X$ *be defined by*

$$f(x,y) = \frac{3x-8y+1}{12}$$

and $g : X \to X$, *where*

$$g(x) = x - 1.$$

So, $\forall(x,y)\in X^2$*, we have :*

$$X_R(x,y)=\{(x,y),(x,1-y),(1-x,y),(1-x,1-y)\}.$$

$$f\times g(X_R(x,y))\subseteq X_R(f\times g(x,y))$$

So, f *has the mixed* $g-R-$*monotone property. It can easily be checked that* f *and* g *satisfy all the other conditions of Theorem 5.3.164.*

Condition (5.90) *is also satisfied by* f *and* g*, whereas* (5.75) *in Theorem 5.3.158 does not hold. Let's assume, to the contrary, that there exists* $\alpha\in[0,1)$*, such that* (5.75) *holds. This means*

$$d(f(x,y),f(z,t))=\left|\frac{3x-8y+1}{12}-\frac{3z-8t+1}{12}\right|=$$

$$\left|\frac{3(x-z)-8(y-t)}{12}\right|$$

$$\leq\frac{\alpha}{2}[|x-z|+|y-t|],\forall(x,y)\in X_R(z,t)$$

For $x=z$*, we have :*

$$\frac{2}{3}|y-t|\leq\frac{\alpha}{2}|y-t|\,,y\neq t$$

which would imply $\frac{2}{3}\leq\frac{\alpha}{2}\Leftrightarrow\frac{4}{3}\leq\alpha<1$*, a contradiction.*

For $x\neq z$*, we have Next, let's prove that, in this case,* (5.90) *holds:*

$$\left|\frac{3x+8y+1}{12}-\frac{3z+8t+1}{12}\right|\leq\frac{1}{4}|x-z|+\frac{2}{3}|y-t|\,,\forall(x,y)\in X_R(z,t)$$

and

$$\left|\frac{3y-8x+1}{12}-\frac{3t-8z+1}{12}\right|\leq\frac{1}{4}|y-t|+\frac{2}{3}|x-z|\,,\forall(x,y)\in X_R(z,t).$$

By summing up, we obtain:

$$d(f(x,y),f(z,t))+d(f(y,x),f(t,z))\leq\frac{11}{12}\cdot[d(g(x),g(z))+d(g(y),g(t))],\forall(x,y)\in X_R(z,t),$$

which is exactly (5.90)*, for* $k=\frac{11}{12}<1$*. So, by Theorem 5.3.164, we obtain that* f *and* g *have a coupled coincidence point,* $\left(\frac{13}{17},\frac{13}{17}\right)$.

EXAMPLE 5.4.171. *Let* $X=[-2,\infty)$*, the metric* $d(x,y)=|x-y|$*, the relation* R *on* X *given by*

$$xRy\Leftrightarrow x^2+x=y^2+y.$$

Let $f : X^2 \to X$ *be defined by*

$$f(x, y) = \frac{x - 2y + 2}{8}$$

and $g : X \to X$, *where*

$$g(x) = x + 1.$$

So, $\forall (x, y) \in X^2$, *we have :*

$$X_R(x, y) = \{(x, y), (x, -1 - y), (-1 - x, y), (-1 - x, -1 - y)\}.$$

$$f \times g(X_R(x, y)) \subseteq X_R(f \times g(x, y))$$

So, f *is mixed* $g - R-$*monotone. It can easily be checked that* f *and* g *satisfy all the other conditions of Theorem 5.3.160. Next, we will show that* f *and* g *satisfy condition* (5.76)*:*

$$d(f(x, y), f(z, t)) = \left| \frac{x - 2y + 2}{8} - \frac{z - 2t + 2}{8} \right|$$

$$= \left| \frac{(x - z) - 2(y - t)}{8} \right| \leq \frac{1}{8} d(g(x), g(z)) + \frac{1}{4} d(g(y), g(t))$$

$$< \frac{1}{4} d(g(x), g(z)) + \frac{1}{4} d(g(y), g(t)) = \varphi \left(\frac{d(g(x), g(z)) + d(g(y), g(t))}{2} \right), \forall (x, y) \in X_R(z, t),$$

which is exactly (5.76) *for* $\varphi(t) = \frac{t}{2}$. *So, by Theorem 5.3.164, we obtain that* f *and* g *have a coupled coincidence point,* $\left(-\frac{30}{53}, -\frac{30}{53} \right)$.

4.2. An application

Let us consider the following nonlinear matrix equation:

$$G(X) = Q + \sum_{i=1}^{p} A_i^* T(X) A_i - \sum_{j=1}^{q} B_j^* K(X) B_j \tag{5.91}$$

where $Q \in \mathcal{P}(n)$ is a positive definite matrix (we denote $Q > 0$), A_i, B_j are arbitrary $n \times n$ matrices and $G, K, T : \mathcal{H}(n) \to \mathcal{P}(n)$, are three continuous order preserving mappings such that $T(0) = G(0) = K(0) = 0$, where $\mathcal{M}(n)$ denotes the set of all $n \times n$ matrices, $\mathcal{H}(n)$ the set of all $n \times n$ Hermitian matrices and $\mathcal{P}(n)$ the set of all $n \times n$ positive definite matrices, $\mathcal{P}(n) \subset \mathcal{H}(n) \subset \mathcal{M}(n)$.

If, in $\mathcal{H}(n)$, we introduce a relation of order "$\leq$", we get a partially ordered set where every matrix has a lower and an upper bound. Furthermore, let $F : \mathcal{H}(n) \times \mathcal{H}(n) \to \mathcal{H}(n)$ be a mapping having the mixed monotone property, $F(X, Y) = Q + \sum_{i=1}^{p} A_i^* T(X) A_i - \sum_{j=1}^{q} B_j^* K(Y) B_j$ (see Equation (5.91)).Thus,the coupled coincidence points of F and G are, in fact, the solutions of Equation(5.91).

In the following results we will discuss the existence and uniqueness of a solution of Equation (5.91).

THEOREM 5.4.172. *Let $C \in \mathcal{P}(n)$ and M a positive number such that:*

(1) $\forall (X,Y) \in \mathcal{H}(n)_{\leq}(U,V)$, *we have*

$$|tr(T(U) - T(X))| \leq \frac{1}{M} |tr(U - X)|$$

and

$$|tr(K(Y) - K(V))| \leq \frac{1}{M} |tr(Y - V)|;$$

(2) $\sum_{i=1}^{p} A_i^* A_i < \frac{M}{2} I_n,$ *and* $\sum_{j=1}^{q} B_j^* B_j < \frac{M}{2} I_n$;

(3) $\sum_{i=1}^{p} A_i^* F(2C) A_i < C,$ *and* $\sum_{j=1}^{q} B_j^* G(2Q) B_j < Q.$

Then there exist $X^*, Y^* \in \mathcal{H}(n)$ such that $F(X^*, Y^*) = G(X^*)$, and $F(Y^*, X^*) = G(Y^*)$.

Proof: Let $(X,Y) \in \mathcal{H}(n)_{\leq}(U,V)$. Then $T(X) \leq T(U)$ and $K(Y) \geq K(V)$. Thus, we have

$$\|F(U,V) - F(X,Y)\|_1 = tr(F(U,V) - F(X,Y)) =$$

$$\sum_{i=1}^{p} tr(A_i^*(T(U) - T(X))A_i) + \sum_{j=1}^{q} tr(B_j^*(K(Y) - K(V))B_j) =$$

$$\sum_{i=1}^{p} tr(A_i^* A_i (T(U) - T(X))) + \sum_{j=1}^{q} tr(B_j B_j^*(K(Y) - K(V))) =$$

$$tr((\sum_{i=1}^{p} A_i A_i^*)(T(U) - T(X))) + tr((\sum_{j=1}^{q} B_j B_j^*)(K(Y) - K(V))) \leq$$

$$\left\| \sum_{i=1}^{p} A_i A_i^* \right\| \|T(U) - T(X)\|_1 +$$

$$\left\| \sum_{j=1}^{q} B_j B_j^* \right\| \|K(Y) - K(V)\|_1 \leq$$

$$\frac{\|\sum_{i=1}^{p} A_i A_i^*\|}{M} \|U - X\|_1 + \frac{\left\|\sum_{j=1}^{q} B_j B_j^*\right\|}{M} \|Y - V\|_1,$$

$$\lambda = 2 \cdot max \frac{\|\sum_{i=1}^{p} A_i A_i^*\|}{M}, \frac{\left\|\sum_{j=1}^{q} B_j B_j^*\right\|}{M}.$$

From the second assumption and Lemma 4.4.132, we have $\lambda < 1$, so the contractive condition of Theorem 5.3.158 is satisfied, $\forall (X,Y) \in \mathcal{H}(n)_{\leq}(U,V)$. From the mixed monotone property of F and the last assumption we get the conclusion of the theorem, that is, there exist $X^*, Y^* \in \mathcal{H}(n)$ such that $F(X^*, Y^*) = G(X^*)$, and $T(Y^*, X^*) = G(Y^*)$. $\square$

THEOREM 5.4.173. *Under the assumptions of Theorem 5.4.172, the Equation* (5.91) *has a unique solution* $\overline{X} \in \mathcal{H}(n)$.

Proof: It is known that every matrix $X \in \mathcal{H}(n)$ has an upper bound and a lower bound. In consequence, for any matrix pairs $(X_1, Y_1), (X_2, Y_2) \in \mathcal{H}(n) \times \mathcal{H}(n)$ there exists a pair $(U, V) \in \mathcal{H}(n) \times \mathcal{H}(n)$ such that $(X_1, Y_1), (X_2, Y_2) \in \mathcal{H}(n)_{\leq}(U, V)$. The conclusion follows using Theorem 5.3.159. Thus, we get that X^*, Y^* from Theorem 5.4.172 are unique and $X^* = Y^* = \overline{X}$. $\square$

REMARK 5.4.174. *If, in Theorem 5.4.172, resp. 5.4.173 we let* $G(X) = X$, *we obtain Theorem 3.3, resp. 3.5 in* [**9**].

CHAPTER 6

Tripled coincidence point theorems in metric spaces endowed with a reflexive relation

1. Tripled coincidence points of mixed $g-R-$monotone operators

The following concepts are extensions of the notions presented in [**37**] and [**38**] for the case of tripled coincidence points in metric spaces endowed with a reflexive relation.

1.1. Definitions

In this section we extend and generalize notions related to coupled and tripled coincidence points in partially ordered metric spaces for the case of tripled coincidence points in metric spaces endowed with a reflexive relation.

Notation 5. *Let X be a nonempty set and let $f : X \times X \times X \to X$ and $g : X \to X$ be two mappings. Then*

(1) We will denote by $f^0(x, y, z) = x$ and

$$f^n(x, y, z) = f(f^{n-1}(x, y, z), f^{n-1}(y, x, y), f^{n-1}(z, y, x)),$$

for all $x, y, z \in X, n \in \mathbb{N}$.

(2) We will denote by $g^0(x) = x$ and $g^n(x) = g(x^{n-1}(x))$, for all $x \in X, n \in \mathbb{N}$.

(3) The cartesian product of f and itself is denoted by $f \times f$ and it is defined by

$$f \times f(x, y, z) = (f(x, y, z), f(y, x, y), f(z, y, x)).$$

(4) The cartesian product of f and g is denoted by $f \times g$ and is defined by

$$(f \times g)(x, y, z) = (g(f(x, y, z)), g(f(y, x, y)), g(f(z, y, x)))$$

Definition 6.1.175. *Let X be a nonempty set and let R be a reflexive relation on X, $f : X^3 \to X$, $g : X \to X$. The mapping f has the **mixed-**$g-R-$**monotone property** on X if $(f \times g)(X_R(x, y, z)) \subseteq X_R((f \times g)(x, y, z))$, for all $(x, y, z) \in X^3$.*

Definition 6.1.176. *An element $(x, y, z) \in X^3$ is called **lower-**$R-$**tripled coincidence point** for f and g, if $(f \times g)(x, y, z) \in X_R(x, y, z)$.*

Next, starting from the orbital continuity presented in [**9**], we will define the orbital g-continuity of a mapping f.

DEFINITION 6.1.177. *Let X be a topological space and $f : X^3 \to X$ be a mixed $g-R-$monotone mapping, $g : X \to X$. We say that f is **orbitally g-continuous** if $(x,y,z),(a,b,c) \in X^3$ and $f^{n_k}(x,y,z) \to a, f^{n_k}(y,x,y) \to b$, $f^{n_k}(z,y,x) \to c$, when $k \to \infty$, implies $f^{n_k+1}(x,y,z) \to g(a)$, $f^{n_k+1}(y,x,y) \to g(b)$ and $f^{n_k+1}(z,y,x) \to g(c)$, when $k \to \infty$.*

1.2. Existence and uniqueness theorems

THEOREM 6.1.178. [**61**] *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g-R$-monotone;
(ii) f is orbitally g-continuous;
(iii) there exist $k,l,m \in [0,1), k+l+m<1$ such that

$$d(f(x,y,z),f(t,u,v)) \leq k \cdot d(g(x),g(t)) + l \cdot d(g(y),g(u)) + m \cdot d(g(z),g(v)), \tag{6.92}$$

$$\forall (x,y,z) \in X_R(t,u,v);$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x,y,z) \in X^3$ such that $f(x,y,z) = g(x)$, $f(y,x,y) = g(y)$ and $f(z,y,x) = g(z)$.

Proof: Since f and g have a lower-R-tripled coincidence point, let (x_0,y_0,z_0) be it. Thus, $(f \times g)(x_0,y_0,z_0) \in X_R(x_0,y_0,z_0)$.
From (i) we have that $(f \times g)(X_R(x_0,y_0,z_0)) \subseteq X_R((f \times g)(x_0,y_0,z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(z_0,y_0,x_0))$$
$$\in X_R(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0)) g^{n-1}(f(z_0,y_0,x_0))).$$

Since $f(X^3) \subseteq g(X)$, let $x_1,y_1,z_1 \in X$ such that $g(x_1) = f(x_0,y_0,z_0)$, $g(y_1) = f(y_0,x_0,y_0)$, $g(z_1) = f(z_0,y_0,x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1}) = f(x_n,y_n,z_n), g(y_{n+1}) = f(y_n,x_n,y_n), g(z_{n+1}) = f(z_n,y_n,x_n). \tag{6.93}$$

Now, using (iii), we have that

$$d(f(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(z_0,y_0,x_0)),$$
$$f(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0)), g^{n-1}(f(z_0,y_0,x_0))))$$

$$\leq k^n \cdot d(g(g^n(f(x_0,y_0,z_0))), g(g^{n-1}(f(x_0,y_0,z_0))))+$$
$$l^n \cdot d(g(g^n(f(y_0,x_0,y_0))), g(g^{n-1}(f(y_0,x_0,y_0))))$$
$$+m^n \cdot d(g(g^n(f(z_0,y_0,x_0))), g(g^{n-1}(f(z_0,y_0,x_0))))$$
$$\Leftrightarrow d(f(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(f(z_0,y_0,x_0))),$$
$$f(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0)), g^n(f(z_0,y_0,x_0))))$$
$$\leq k^n \cdot d(g^{n+1}(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)))+$$
$$l^n \cdot d(g^{n+1}(f(y_0,x_0,y_0)), g^n(f(y_0,x_0,y_0)), g^n(f(y_0,x_0,y_0)))$$
$$+m^n \cdot d(g^{n+1}(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)))$$
$$\Leftrightarrow d(f(g^n(g(x_1)), g^n(g(y_1)), g^n(g(z_1))), f(g^{n-1}(g(x_1)), g^{n-1}(g(y_1)), g^{n-1}(g(z_1))))$$
$$\leq k^n \cdot d(g^{n+1}(g(x_1)), g^n(g(x_1)))+l^n \cdot d(g^{n+1}(g(y_1)), g^n(g(y_1)))+m^n d(g^{n+1}(g(z_1)), g^n(g(z_1)))$$
$$\Leftrightarrow d(f(g^{n+1}(x_1), g^{n+1}(y_1), g^{n+1}(z_1)), f(g^n(x_1), g^n(y_1)g^n(z_1)))$$
$$\leq k^n \cdot d(g^{n+2}(x_1), g^{n+1}(x_1)) + l^n \cdot d(g^{n+2}(y_1), g^{n+1}(y_1)) + m^n \cdot d(g^{n+2}(z_1), g^{n+1}(z_1))$$

This implies that $\{g^n(x_1)\}_{n\in\mathbb{N}}$ is a fundamental sequence in X.
Now, because (X,d) is a complete metric space, there exist $x,y,z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x,\ \lim_{n\to\infty} g(y_n) = y,\ \lim_{n\to\infty} g(z_n) = z. \tag{6.94}$$

From the continuity of g, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(x),\ \lim_{n\to\infty} g(g(y_n)) = g(y),\ \lim_{n\to\infty} g(g(z_n)) = g(z).$$

Because f and g commute, and from (6.93), we have

$$g(g(x_{n+1})) = g(f(x_n,y_n,z_n)) = f(g(x_n), g(y_n), g(z_n)),$$

$$g(g(y_{n+1})) = g(f(y_n,x_n,y_n)) = f(g(y_n), g(x_n), g(y_n)$$

and

$$g(g(z_{n+1})) = g(f(z_n,y_n,x_n)) = f(g(z_n), g(y_n), g(x_n)$$

From (6.94) and the orbital continuity of f we get

$$g(x) = f(x,y,z),$$

$$g(y) = f(y,x,y)$$

and

$$g(z) = f(z,y,x).$$

□

Corollary 6.1.179. [**61**] *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;
(ii) f is orbitally g-continuous;

(iii) there exist $\alpha \in [0,1)$ *such that*

(6.95)

$$d(f(x,y,z), f(t,u,v)) \le \frac{\alpha}{3}\cdot [d(g(x),g(u)) + d(g(y),g(v)) + d(g(z),g(t))], \forall (x,y,z) \in X_R(t,u,v);$$

(iv) f *and* g *have a lower-R-tripled coincidence point;*

(v) $f(X^3) \subseteq g(X)$*;*

(vi) g *is continuous ;*

(vii) f *and* g *commute.*

Then f *and* g *have a tripled coincidence point, i.e., there exists* $(x,y,z) \in X^3$ *such that* $f(x,y,z) = g(x)$*,* $f(y,x,y) = g(y)$ *and* $f(z,y,x) = g(z)$*.*

Proof: From the proof of Theorem 6.1.178, for $k = l = m = \dfrac{\alpha}{3}, \alpha \in [0,1)$, there exist $x, y, z \in X$ such that

$$g(x) = f(x,y,z),$$

$$g(y) = f(y,x,y)$$

and

$$g(z) = f(z,y,x).$$

□

THEOREM 6.1.180. *In addition to the hypothesis of Theorem 6.1.178, suppose that for every* $(x,y,z),(\overline{x},\overline{y},\overline{z}) \in X^3$*, there exists* $(t,u,v) \in X^3$ *such that* $(g(x),g(y),g(z))$*,* $(g(\overline{x}),g(\overline{y}),g(\overline{z})) \in X_R(f(t,u,v),f(u,t,u),f(v,u,t))$*. Then* f *and* g *have a unique tripled coincidence point.*

Proof: According to the proof of Theorem 6.1.178, there exist $\overline{x},\overline{y},\overline{z} \in X$ such that $f(\overline{x},\overline{y},\overline{z}) = g(\overline{x})$, $f(\overline{y},\overline{x},\overline{y}) = g(\overline{y})$ and $f(\overline{z},\overline{y},\overline{x}) = g(\overline{z})$. We have to show that, if (x,y,z) is another coincidence point for f and g,

$$d((g(\overline{x}),g(\overline{y}),g(\overline{z}),(g(x),g(y),g(z))) = 0.$$

Because both (x,y,z) and $(\overline{x},\overline{y},\overline{z})$ are tripled coincidence points, we have

$$g(x) = f(x,y,z), g(y) = f(y,x,y), g(z) = f(z,y,x)$$

and

$$g(\overline{x}) = f(\overline{x},\overline{y},\overline{z}), g(\overline{y}) = f(\overline{y},\overline{x},\overline{y}),$$

$$g(\overline{z}) = f(\overline{z},\overline{y},\overline{x}).$$

Now, let $u_0 = u$, $v_0 = v$ and $t_0 = t$. Then, there exist $t_1, u_1, v_1 \in X$ such that $g(t_1) = f(t_0,u_0,v_0)$, $g(u_1) = f(u_0,t_0,u_0)$ and $g(v_1 f(v_0,u_0,t_0)$. Using the same procedure as in the proof of Theorem 6.1.178, we obtain the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ where

$$g(t_{n+1} = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, u_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Furthermore, let $x_0 = x, y_0 = y, z_0 = z$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{z}$. Thus, we obtain the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$ and $\{\overline{x}_n\}_{n\in\mathbb{N}}, \{\overline{y}_n\}_{n\in\mathbb{N}}, \{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n) = f(x, y, z), g(y_n) = f(y, x, y), g(z_n) = f(z, y, x)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(t, u, v) \in X^3$ such that

$$(g(x), g(y), g(z)), (g(\overline{x}), g(\overline{y}), g(\overline{z}) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(y_0)), f^n(g(z_0), g(y_0), g(x_0)))$$

$$\in X_R(f^{n+1}(t, u, v), f^{n+1}(u, t, u), f^{n+1}(v, u, t))$$

Also, by using the contractivity condition, we have

$$d((f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, u))$$
$$+ m^n \cdot d(g(z_0), f(v, u, t)),$$
$$d((f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u, t, u)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, u))$$
$$+ m^n \cdot d(g(z_0), f(v, u, t))$$

and

$$d((f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, u))$$
$$+ m^n \cdot d(g(z_0), f(v, u, t)).$$

Summing up, we obtain that

$$d((f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) + d((f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u, t, u)) +$$
$$d((f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq 3k^n \cdot d(g(x_0), f(t, u, v))$$
$$+ 3l^n \cdot d(g(y_0), f(u, t, u)) + 3m^n \cdot d(g(z_0), f(v, u, t)).$$

But $x_0 = x$, $y_0 = y$ and $z_0 = z$. We obtain

$$d((f^n(g(x), g(y), g(z)), f^{n+1}(t, u, v)) + d((f^n(g(y), g(x), g(y)), f^{n+1}(u, t, u)) +$$
$$d((f^n(g(z), g(y), g(x)), f^{n+1}(v, u, t)) \leq 3k^n \cdot d(g(x), f(t, u, v))$$
$$+ 3l^n \cdot d(g(y), f(u, t, u)) + 3m^n \cdot d(g(z), f(v, u, t)).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(y), f(u,t,u)) = 0$$

$$\text{and } \lim_{n\to\infty} d(g(z), f(v,u,t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(\overline{y}), f(u,t,u)) = 0$$

$$\text{and } \lim_{n\to\infty} d(g(\overline{z}), f(v,u,t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x), g(\overline{x})) \le d(g(x), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y), g(\overline{y})) \le d(g(y), f(u,v,t)) + d(f(u,t,u), g(\overline{y})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(z), g(\overline{z})) \le d(g(z), f(v,t,u)) + d(f(v,u,t), g(\overline{z})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

The next result is obtained by replacing the contraction (6.92) with one that uses the mapping φ defined in Chapter 5, following the idea in [**53**]. Thus, we obtain :

THEOREM 6.1.181. [**61**] *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g-R$-monotone;
(ii) f is orbitally g-continuous;
(iii)

(6.96)

$$d(f(x,y,z), f(t,u,v)) \le \varphi\left(\frac{d(g(x),g(t)) + d(g(y),g(u)) + d(g(z),g(v))}{3}\right), \forall (x,y,z) \in X_R(t,u,v),$$

where $\varphi \in \Phi$;
(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x,y,z) \in X^3$ such that $f(x,y,z) = g(x)$, $f(y,x,y) = g(y)$ and $f(z,y,x) = g(z)$.

Proof: From the hypothesis, we know that f and g have a lower-R-triple coincidence point; let (x_0, y_0, z_0) be it. Thus, using the definition of the lower-R-tripled coincidence point, it follows that $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.
From (i) we know that $(f \times g)(X_R(x_0, y_0, z_0)) \subseteq X_R((f \times g)(x_0, y_0, z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0, y_0, z_0)), g^n(f(y_0, x_0, y_0)), g^n(f(z_0, y_0, x_0)))$$
$$\in X_R(g^{n-1}(f(x_0, y_0, z_0)), g^{n-1}(f(y_0, x_0, y_0)), g^{n-1}(f(z_0, y_0, x_0))).$$

Since $f(X^3) \subseteq g(X)$, let $x_1, y_1, z_1 \in X$ such that $g(x_1) = f(x_0, y_0, z_0)$, $g(y_1) = f(y_0, x_0, y_0)$, $g(z_0) = f(z_0, y_0, x_0)$ and so on. Thus, we obtain the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1}) = f(x_n, y_n, z_n), g(y_{n+1}) = f(y_n, x_n, y_n) \text{ and } g(z_{n+1} = f(z_n, y_n, x_n). \tag{6.97}$$

Let's consider the nonnegative sequence $\{\eta_n\}_{n\in\mathbb{N}^*}$ such that $\eta_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}, g(z_n)), n \in \mathbb{N}^*$.
Now, using (6.96), (1.2) and letting $x := x_n$, $y := y_n$ and $z := z_n$, $t := x_{n-1}$, $u := y_{n-1}$ and $v := z_{n-1}$, we obtain

$$d(g(x_{n+1}), g(x_n)) = d(f(x_n, y_n, z_n), f(x_{n-1}, y_{n-1}, z_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right),$$
$$d(g(y_{n+1}), g(y_n)) = d(f(y_n, x_n, y_n), f(y_{n-1}, x_{n-1}, y_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

and

$$d(g(z_{n+1}), g(z_n)) = d(f(z_n, y_n, x_n), f(z_{n-1}, y_{n-1}, x_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

By summing up the last three relations, we obtain that

$$d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}), g(z_n)) = \eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

Now, using the properties of φ, we have that

$$\eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right) < 3 \cdot \frac{\eta_{n-1}}{3} = \eta_{n-1}. \tag{6.98}$$

Thus, $\{\eta_n\}_{n\in\mathbb{N}^*}$ is a decreasing and nonnegative sequence. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} \eta_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (6.98), let $n \to \infty$. Using (ii) (the second condition satisfied by φ) from Chapter 5, we have

$$\varepsilon_0 = \lim_{n\to\infty} \eta_n \leq 3 \cdot \lim_{n\to\infty} \varphi\left(\frac{\eta_{n-1}}{3}\right) = 3 \cdot \lim_{\eta_{n-1}\to\varepsilon_0+} \varphi\left(\frac{\eta_{n-1}}{3}\right) < \varepsilon_0,$$

which is a contradiction. Thus, $\lim\limits_{n\to\infty} \eta_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0, \lim\limits_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(z_{n+1}), g(z_n)) = 0$.
Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \geq \delta, \tag{6.99}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{N}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (6.99). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(z_{n_2(k)}), g(z_{n_1(k)-1}) < \delta. \tag{6.100}$$

Now, using the triangle inequality and the last two inequalities ((6.99) and (6.100)), we have

$$\begin{aligned}\delta &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(x_{n_1(k)}), g(x_{n_2(k)}))\\ &\quad + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))\\ &\quad d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))\\ &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) + \delta.\end{aligned}$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))] = \delta.$$

Next, we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}s_k &= d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ &\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)}))\\ &\quad + d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)}))\\ &\quad + d(g(z_{n_1(k)}), g(z_{n_1(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\\ &= \eta_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\end{aligned}$$

(6.101)

$$\leq \eta_{n_1(k)} + \eta_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})).$$

But

$$d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1}))d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1}))$$

$$= d(f(x_{n_1(k)}, y_{n_1(k)}, z_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)}, z_{n_2(k)}))$$
$$+ d(f(y_{n_1(k)}, x_{n_1(k)}, y_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}, y_{n_2(k)}))$$
$$+ d(f(z_{n_1(k)}, y_{n_1(k)}, x_{n_1(k)}), f(z_{n_2(k)}, y_{n_2(k)}, x_{n_2(k)}))$$
$$\leq 2 \cdot \varphi \left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}) + d(g(z_{n_1(k)}), g(z_{n_2(k)})}{3} \right) +$$
$$\leq 2 \cdot \varphi \left(\frac{s_k}{3} \right).$$

Now, returning to (6.101), we have

$$s_k \leq \eta_{n_1(k)} + \eta_{n_2(k)} + 2 \cdot \varphi \left(\frac{s_k}{3} \right).$$

Let $k \to \infty$. We obtain

$$\delta \leq 3 \cdot \lim_{k \to \infty} \varphi \left(\frac{s_k}{3} \right) = 3 \cdot \lim_{s_k \to \delta_+} \varphi \left(\frac{s_k}{3} \right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{g(x_n)\}_{n \in \mathbb{N}}$, $\{g(y_n)\}_{n \in \mathbb{N}}$ and $\{g(z_n)\}_{n \in \mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since X is complete, there exist $\overline{x}$,$\overline{y}$ and $\overline{z}$ such that $g^n(x_n) \to \overline{x}$, $g^n(y_n) \to \overline{y}$ and $g^n(z_n) \to \overline{y}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n, z_n) \to \overline{x}$, $f^{n-1}(y_n, x_n, y_n) \to \overline{y}$ and $f^{n-1}(z_n, y_n, x_n) \to \overline{z}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n, z_n) \to g(\overline{x})$, $f^n(y_n, x_n, y_n) \to g(\overline{y})$ and $f^n(z_n, y_n, x_n) \to g(\overline{z})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y}, \overline{z})$ is a tripled coincidence point for f and g. □

THEOREM 6.1.182. [**61**] *In addition to the hypothesis of Theorem 6.1.181, suppose that for every $(x^*, y^*, z^*), (\overline{x}, \overline{y}, \overline{z}) \in X^3$, there exists $(t, u, v) \in X^3$ such that $(g(x^*), g(y^*), g(z^*)), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$. Then f and g have a unique tripled coincidence point.*

Proof: From Theorem 6.1.181, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{y}) = g(\overline{y})$ and $f(, \overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x^*, y^*, z^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z}), (g(x^*), g(y^*), g(z^*))) = 0.$$

Since (x^*, y^*, z^*) and $(\overline{x}, \overline{y}, \overline{z})$ are both tripled coincidence points, it follows that

$$g(x^*) = f(x^*, y^*, z^*), g(y^*) = f(y^*, x^*, y^*), g(z^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

Now, using the hypothesis of Theorem 6.1.181, from $f(X^3) \subseteq g(x)$, there exist $t_1, u_1, v_1 \in X^3$ such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, u_0)$, $g(v_1) = f(v_0, u_0, t_0)$. Using the same procedure as in the proof of Theorem 6.1.178, we build the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(t_{n+1}) = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, u_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Next, let $x_0 = x^*, y_0 = y^*, z_0 = z^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{z_n^*\}_{n\in\mathbb{N}}$ $\{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*, z^*), g(y_n^*) = f(y^*, x^*, y^*), g(z_n^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we know that there exists $(t, u, v) \in X^3$ such that

$$(g(x^*), g(y^*), g(z^*), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(y_0)), f^n(g(z_0), g(y_0), g(x_0)))$$
$$\in X_R(f^{n+1}(t, u, v), f^{n+1}(u, t, u), f^{n+1}(v, u, t))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t, u, v)) + d(g(y_0), f(u, t, u)) + d(g(z_0), f(v, u, t))}{3}\right),$$
$$d(f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u, t, u)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t, u, v)) + d(g(y_0), f(u, t, u)) + d(g(z_0), f(v, u, t))}{3}\right)$$

and

$$d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t, u, v)) + d(g(y_0), f(u, t, u)) + d(g(z_0), f(v, u, t))}{3}\right).$$

By summing up, we obtain that

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) + d(f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u, t, u))$$
$$+d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t))$$
$$\leq 3 \cdot \varphi\left(\frac{d(g(x_0), f(t, u, v)) + d(g(y_0), f(u, t, u)) + d(g(z_0), f(v, u, t))}{3}\right).$$

But $x_0 = x^*$, $y_0 = y^*$ and $z_0 = z^*$. We obtain

$$d(f^n(g(x^*), g(y^*), g(z^*)), f^{n+1}(t, u, v)) + d(f^n(g(y^*), g(x^*), g(y^*)), f^{n+1}(u, t, u))$$
$$+d(f^n(g(z^*), g(y^*), g(x^*)), f^{n+1}(v, u, t)) \leq$$

$$3 \cdot \varphi\left(\frac{d(g(x^*), f(t,u,v)) + d(g(y^*), f(u,t,u)) + d(g(z^*), f(v,u,t))}{3}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(y^*), f(u,t,u)) = 0 \text{ and } \lim_{n\to\infty} d(g(z^*), f(v,u,t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(\overline{z}), f(u,t,u)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{z}), f(v,u,t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(u,t,u)) + d(f(u,t,u), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

and

$$d(g(z^*), g(\overline{z})) \leq d(g(z^*), f(v,u,t)) + d(f(v,u,t), g(\overline{z})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

Now, by symmetrizing the contraction condition, using the idea in [**26**], we obtain the following result:

THEOREM 6.1.183. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii)

$$d(f(x,y,z), f(t,u,v)) + d(f(y,x,y), f(u,t,u)) + d(f(z,y,x), f(v,u,t) \leq \tag{6.102}$$
$$3 \cdot \varphi\left(\frac{d(g(x), g(t)) + d(g(y), g(y)) + d(g(z), g(v))}{3}\right), \forall (x,y,z) \in X_R(t,u,v);$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x,y,z) \in X^3$ such that $f(x,y,z) = g(x)$, $f(y,x,y) = g(y)$ and $f(z,y,x) = g(z)$.

Proof: The prove this result, we will follow the steps from the proof of Theorem 6.1.181: Since f and g have lower-R-tripled coincidence point, let (x_0, y_0, z_0) be it. Thus, $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.

From (i) we have that $(f\times g)(X_R(x_0,y_0,z_0))\subseteq X_R((f\times g)(x_0,y_0,z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0,y_0,z_0)),g^n(f(y_0,x_0,y_0)),g^n(f(z_0,y_0,x_0)))\in \\ X_R(g^{n-1}(f(x_0,y_0,z_0)),g^{n-1}(f(y_0,x_0,y_0)),g^{n-1}(f(z_0,y_0,x_0))). \tag{6.103}$$

Since $f(X^3)\subseteq g(X)$, let $x_1,y_1,z_1\in X$ such that $g(x_1)=f(x_0,y_0,z_0)$, $g(y_1)=f(y_0,x_0,y_0)$, $g(z_1)=f(z_0,y_0,x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1})=f(x_n,y_n,z_n),g(y_{n+1})=f(y_n,x_n,y_n),g(z_{n+1})=f(z_n,y_n,x_n) \tag{6.104}$$

Let's consider the nonnegative sequence $\{\eta_n\}_{n\in\mathbb{N}^*}$ such that $\eta_n=d(g(x_{n+1}),g(x_n))+d(g(y_{n+1}),g(y_n))+d(g(z_{n+1},g(z_n)),n\in\mathbb{N}^*$.
Now, using (6.102), (6.103) and letting $x:=x_n$, $y:=y_n$ and $z:=z_n$, $t:=x_{n-1}$, $u:=y_{n-1}$ and $v:=z_{n-1}$, we obtain

$$d(f(x_n,y_n,z_n),f(x_{n-1},y_{n-1},z_{n-1}))+d(f(y_n,x_n,y_n),f(y_{n-1},z_{n-1},x_{n-1}))$$
$$+d(f(z_n,y_n,x_n),f(z_{n-1},x_{n-1},y_{n-1}))=d(g(x_{n+1}),g(x_n))$$
$$+d(g(y_{n+1}),g(y_n))+d(g(z_{n+1}),g(z_n))\leq$$
$$3\varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))d(g(z_n),g(z_{n-1}))}{3}\right)=\varphi\left(\frac{\eta_n}{3}\right)$$

Thus, we have that

$$\eta_{n+1}\leq 3\cdot\varphi\left(\frac{\eta_n}{3}\right) \tag{6.105}$$

Using the proof of Theorem 6.1.181 we have that $\{\eta_n\}_{n\in\mathbb{N}^*}$ is decreasing and nonnegative. Therefore, there exists $\varepsilon_0\geq 0$ such that

$$\lim_{n\to\infty}\eta_n=\varepsilon_0.$$

Now, we will prove that $\varepsilon_0=0$. In (6.105), let $n\to\infty$. Using (ii) (the second condition satisfied by φ) from Chapter 5, we have

$$\varepsilon_0=\lim_{n\to\infty}\eta_{n+1}\leq 3\cdot\lim_{n\to\infty}\varphi\left(\frac{\eta_n}{3}\right)=3\cdot\lim_{\eta_n\to\varepsilon_0+}\varphi\left(\frac{\eta_n}{3}\right)<\varepsilon_0.$$

So we have that $\varepsilon_0<\varepsilon_o$, which is, clearly, a contradiction.
Thus, $\lim\limits_{n\to\infty}\eta_n=0$ and, consequently, $\lim\limits_{n\to\infty}d(g(x_{n+1}),g(x_n))=0$, $\lim\limits_{n\to\infty}d(g(y_{n+1}),g(y_n))=0$ and $\lim\limits_{n\to\infty}d(g(z_{n+1}),g(z_n))=0$.
Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta>0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k:=d(g(x_{n_2(k)}),g(x_{n_1(k)}))+d(g(y_{n_2(k)}),g(y_{n_1(k)}))+d(g(z_{n_2(k)}),g(z_{n_1(k)}))\geq\delta, \tag{6.106}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{N}^*$. Next, let's chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (6.106). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) < \delta. \tag{6.107}$$

Now, using the triangle inequality and the last two inequalities ((6.106) and (6.107)), we have

$$\begin{aligned}\delta \leq\ & d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ \leq\ & d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(x_{n_(k)}), g(x_{n_2(k)}))\\ & + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))\\ & d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))\\ \leq\ & d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) + \delta.\end{aligned}$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))] = \delta.$$

Now we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}s_k =\ & d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ \leq\ & d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)}))\\ & + d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)}))\\ & + d(g(z_{n_1(k)}), g(z_{n_1(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\\ =\ & \eta_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\end{aligned}$$

(6.108)

$$\leq \eta_{n_1(k)} + \eta_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})).$$

But

$$\begin{aligned}& d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1}))\\ &= d(f(x_{n_1(k)}, y_{n_1(k)}, z_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)}, z_{n_1(k)}))\\ &\quad + d(f(y_{n_1(k)}, x_{n_1(k)}, y_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}, y_{n_2(k)}))\\ &\quad + d(f(z_{n_1(k)}, y_{n_1(k)}, x_{n_1(k)}), f(z_{n_2(k)}, y_{n_2(k)}), x_{n_2(k)}))\\ &\leq 3 \cdot \varphi\left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))}{3}\right)\\ &\leq 3 \cdot \varphi\left(\frac{s_k}{3}\right).\end{aligned}$$

Now, returning to (6.108), we have

$$s_k \leq \eta_{n_1(k)} + \eta_{n_2(k)} + 3 \cdot \varphi\left(\frac{s_k}{3}\right).$$

Let $k \to \infty$. We obtain

$$\delta \leq 3 \cdot \lim_{k\to\infty} \varphi\left(\frac{s_k}{3}\right) = 3 \cdot \lim_{s_k\to\delta_+} \varphi\left(\frac{s_k}{3}\right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{g(x_n)\}_{n\in\mathbb{N}}$,$\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since (X, d) is complete, there exist $\overline{x}$ and $\overline{y}$ such that $g^n(x_n) \to \overline{x}$, $g^n(y_n) \to \overline{y}$ and $g^n(z_n) \to \overline{z}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n, z_n) \to \overline{x}$, $f^{n-1}(y_n, x_n, y_n) \to \overline{y}$ and $f^{n-1}(z_n, y_n, x_n) \to \overline{z}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n, z_n) \to g(\overline{x})$, $f^n(y_n, x_n, y_n) \to g(\overline{y})$ and $f^n(z_n, y_n, x_n) \to g(\overline{z})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y}, \overline{z})$ is a tripled coincidence point for f and g.

□

THEOREM 6.1.184. *In addition to the hypothesis of Theorem 6.1.183, suppose that for every* $(x^*, y^*, z^*), (\overline{x}, \overline{y}, \overline{z}) \in X^3$, *there exists* $(t, u, v) \in X^3$ *such that* $(g(x^*), g(y^*), g(z^*))$, $(g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$. *Then* f *and* g *have a unique tripled coincidence point.*

Proof: According to Theorem 6.1.183, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{y}) = g(\overline{y})$ and $f(\overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x^*, y^*, z^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z}), (g(x^*), g(y^*), g(z^*))) = 0.$$

Because both (x^*, y^*, z^*) and $(\overline{x}, \overline{y}, \overline{z})$ are tripled coincidence points, we have

$$g(x^*) = f(x^*, y^*, z^*), g(y^*) = f(y^*, x^*, y^*), g(z^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

From $f(X^3) \subseteq g(X)$, there exist t_0, u_0, v_0 in X such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, u_0)$, $g(v_1) = f(v_0, u_0, t_0)$. Following the procedure used in the proof of Theorem 6.1.178, we obtain the sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(t_{n+1}) = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, u_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Furthermore, let $x_0^* = x^*, y_0^* = y^*, z_0^* = z^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{z}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{z_n^*\}_{n\in\mathbb{N}}, \{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*, z^*), g(y_n^*) = f(y^*, x^*, y^*), g(z_n^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(t,u,v)\in X^3$ such that

$$(g(x^*),g(y^*),g(z^*),(g(\overline{x}),g(\overline{y}),g(\overline{z}))\in X_R(f(t,u,v),f(u,t,u),f(v,u,t)).$$

From $(g(x_0),g(y_0),g(z_0))\in X_R(f(t,u,v),f(u,t,u),f(v,u,t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0),g(y_0),g(z_0)),f^n(g(y_0),g(z_0),g(x_0)),f^n(g(z_0),g(x_0),g(y_0)))$$
$$\in X_R(f^{n+1}(t,u,v),f^{n+1}(u,t,u),f^{n+1}(v,u,t))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0),g(y_0),g(z_0)),f^{n+1}(t,u,v))+d(f^n(g(y_0),g(z_0),g(x_0)),f^{n+1}(u,t,u))$$
$$+d(f^n(g(z_0),g(x_0),g(y_0)),f^{n+1}(v,u,t))\le$$
$$\varphi\left(\frac{d(g(x_0),f(t,u,v))+d(g(y_0),f(u,t,u))+d(g(z_0),f(v,u,t))}{3}\right).$$

But $x_0^*=x^*$, $y_0^*=y^*$ and $z_0^*=z^*$. We obtain

$$d(f^n(g(x^*),g(y^*),g(z^*)),f^{n+1}(t,u,v))+d(f^n(g(y^*),g(x^*),g(y^*)),f^{n+1}(u,t,u))$$
$$+d(f^n(g(z^*),g(y^*),g(x^*)),f^{n+1}(v,u,t))\le$$
$$\varphi\left(\frac{d(g(x^*),f(t,u,v))+d(g(y^*),f(u,t,u))+d(g(z^*),f(v,u,t))}{3}\right).$$

Letting $n\to\infty$ we obtain that

$$\lim_{n\to\infty}d(g(x^*),f(t,u,v))=0,\ \lim_{n\to\infty}d(g(y^*),f(u,t,u))=0 \text{ and } \lim_{n\to\infty}d(g(z^*),f(v,u,t))=0.$$

Similarly, we obtain that

$$\lim_{n\to\infty}d(g(\overline{x}),f(t,u,v))=0,\ \lim_{n\to\infty}d(g(\overline{y}),f(u,t,u))=0 \text{ and } \lim_{n\to\infty}d(g(\overline{z}),f(v,u,t))=0.$$

Now, using the triangle inequality, we have

$$d(g(x^*),g(\overline{x}))\le d(g(x^*),f(t,u,v))+d(f(t,u,v),g(\overline{x}))\to 0,\ \text{when } n\to\infty,$$

$$d(g(y^*),g(\overline{y}))\le d(g(y^*),f(u,t,u))+d(f(u,t,u),g(\overline{y}))\to 0,\ \text{when } n\to\infty$$

and

$$d(g(z^*),g(\overline{z}))\le d(g(z^*),f(v,u,t))+d(f(v,u,t),g(\overline{z}))\to 0,\ \text{when } n\to\infty$$

which means that

$$g(x^*)=g(\overline{x}),$$
$$g(y^*)=g(\overline{y})$$

and

$$g(z^*)=g(\overline{z}).$$

□

Next, we let $\varphi(t) = kt, k \in [0, 1)$. It is easy to check that conditions (i) and (ii) still hold. We obtain the following result:

THEOREM 6.1.185. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is mixed $g - R$-monotone;
(ii) f is orbitally $g-$continuous;
(iii) there exist $k \in [0, 1), k < 1$ such that

$$d(f(x,y,z), f(t,u,v)) + d(f(y,x,y), f(u,t,u)) + d(f(z,y,x), f(v,u,t)) \tag{6.109}$$
$$\leq k \cdot [d(g(x), g(t)) + d(g(y), g(u)) + d(g(z), g(v))], \forall (x,y,z) \in X_R(t,u,v);$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x, y, z) \in X^3$ such that $f(x, y, z) = g(x)$, $f(y, x, y) = g(y)$ and $f(z, y, x) = g(z)$.

Proof: In Theorem 6.1.183, let $\varphi(t) = kt, k \in [0, 1)$. □

THEOREM 6.1.186. *In addition to the hypothesis of Theorem 6.1.185, suppose that for every $(x, y, z), (\overline{x}, \overline{y}, \overline{z}) \in X^3$, there exists $(t, u, v) \in X^3$ such that $(g(x), g(y), g(z))$, $(g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$. Then f and g have a unique tripled coincidence point.*

Proof: In Theorem 6.1.182, let $\varphi(t) = kt, k \in [0, 1)$. □

REMARK 6.1.187. *If, in Theorem 6.1.178, resp 6.1.180 we let $R =\leq$, assume that either f is continuous or the regularity condition 2 holds, we obtain Theorems 5.1.146, resp. 5.1.147 from* [**37**].

REMARK 6.1.188. *If, in Theorem 6.1.183, we let $R =\leq$ and we assume that f is continuous, we obtain Theorem 5.1.153 from* [**13**].

2. Tripled coincidence points of $R-$monotone operators

The following concepts are extensions of the notions presented in [**37**] and [**38**] for the case of tripled coincidence points in metric spaces endowed with a reflexive relation.

2.1. Definitions

In this section we extend and generalize notions related to coupled and tripled coincidence points in partially ordered metric spaces for the case of tripled coincidence points in metric spaces endowed with a reflexive relation.

NOTATION 6. *Let X be a nonempty set and let $f : X \times X \times X \to X$ and $g : X \to X$ be two mappings. Then*

(1) We will denote by $f^0(x, y, z) = x$ and

$$f^n(x, y, z) = f(f^{n-1}(x, y, z), f^{n-1}(y, x, z), f^{n-1}(z, y, x)),$$

for all $x, y, z \in X, n \in \mathbb{N}$.

(2) We will denote by $g^0(x) = x$ and $g^n(x) = g(x^{n-1}(x))$, for all $x \in X, n \in \mathbb{N}$.

(3) The cartesian product of f and itself is denoted by $f \times f$ and it is defined by

$$f \times f(x, y, z) = (f(x, y, z), f(y, x, z), f(z, y, x)).$$

(4) The cartesian product of f and g is denoted by $f \times g$ and is defined by

$$(f \times g)(x, y, z) = (g(f(x, y, z)), g(f(y, x, z)), g(f(z, y, x)))$$

REMARK 6.2.189. *Note that the cartesian product of two mappings is different to the one in Notation 5. In this case we have permutations of (x, y, z) in order to obtain the tripled coincidence points of monotone operators defined by Borcut in Definition 5.1.149.*

DEFINITION 6.2.190. *Let X be a nonempty set and let R be a reflexive relation on X, $f : X^3 \to X$, $g : X \to X$. The mapping f has the $g - R-$**monotone property** on X if $(f \times g)(X_R(x, y, z)) \subseteq X_R((f \times g)(x, y, z))$, for all $(x, y, z) \in X^3$.*

Next, starting from the orbital continuity presented in [**9**], we will define the orbital g-continuity of an $R-$monotone mapping f.

DEFINITION 6.2.191. *Let X be a topological space and $f : X^3 \to X$ be a $g - R-$monotone mapping, $g : X \to X$. We say that f is **orbitally g-continuous** if $(x, y, z), (a, b, c) \in X^3$ and $f^{n_k}(x, y, z) \to a, f^{n_k}(y, x, z) \to b$, $f^{n_k}(z, y, x) \to c$, when $k \to \infty$, implies $f^{n_k+1}(x, y, z) \to g(a)$, $f^{n_k+1}(y, x, z) \to g(b)$ and $f^{n_k+1}(z, y, x) \to g(c)$, when $k \to \infty$.*

REMARK 6.2.192. *Note that the orbital $g-$continuity of the $g - R$-monotone mapping f is different than in the case of mixed $g - R-$monotone operators.*

2.2. Existence and uniqueness theorems

THEOREM 6.2.193. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii) there exist $k, l, m \in [0, 1), k + l + m < 1$ such that

$$d(f(x, y, z), f(t, u, v)) \leq k \cdot d(g(x), g(t)) + l \cdot d(g(y), g(u)) + m \cdot d(g(z), g(v)), \tag{6.110}$$

$$\forall (x, y, z) \in X_R(t, u, v);$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence points, i.e., there exists $(x, y, z) \in X^3$ such that $f(x, y, z) = g(x)$, $f(y, x, z) = g(y)$ and $f(z, y, x) = g(z)$.

Proof: Since f and g have a lower-R-tripled coincidence point, let (x_0, y_0, z_0) be it. Thus, $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.
From (i) we have that $(f \times g)(X_R(x_0, y_0, z_0)) \subseteq X_R((f \times g)(x_0, y_0, z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0, y_0, z_0)), g^n(f(y_0, x_0, z_0)), g^n(z_0, y_0, x_0))$$
$$\in X_R(g^{n-1}(f(x_0, y_0, z_0)), g^{n-1}(f(y_0, x_0, z_0))g^{n-1}(f(z_0, y_0, x_0))).$$

Since $f(X^3) \subseteq g(X)$, let $x_1, y_1, z_1 \in X$ such that $g(x_1) = f(x_0, y_0, z_0)$, $g(y_1) = f(y_0, x_0, z_0)$, $g(z_1) = f(z_0, y_0, x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1}) = f(x_n, y_n, z_n), g(y_{n+1}) = f(y_n, x_n, z_n), g(z_{n+1}) = f(z_n, y_n, x_n). \tag{6.111}$$

Now, using (iii), we have that

$$\begin{gathered}
d(f(g^n(f(x_0, y_0, z_0)), g^n(f(y_0, x_0, y_0)), g^n(z_0, y_0, x_0)),\\
f(g^{n-1}(f(x_0, y_0, z_0)), g^{n-1}(f(y_0, x_0, z_0)), g^{n-1}(f(z_0, y_0, x_0))))\\
\leq k^n \cdot d(g(g^n(f(x_0, y_0, z_0))), g(g^{n-1}(f(x_0, y_0, z_0))))+\\
l^n \cdot d(g(g^n(f(y_0, x_0, y_0))), g(g^{n-1}(f(y_0, x_0, z_0))))\\
+m^n \cdot d(g(g^n(f(z_0, y_0, x_0))), g(g^{n-1}(f(z_0, y_0, x_0))))\\
\Leftrightarrow d(f(g^n(f(x_0, y_0, z_0)), g^n(f(y_0, x_0, z_0)), g^n(f(z_0, y_0, x_0))),\\
f(g^{n-1}(f(x_0, y_0, z_0)), g^{n-1}(f(y_0, x_0, z_0)), g^n(f(z_0, y_0, x_0))))
\end{gathered}$$

$$\leq k^n \cdot d(g^{n+1}(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)))+$$
$$l^n \cdot d(g^{n+1}(f(y_0,x_0,z_0)), g^n(f(y_0,x_0,z_0)), g^n(f(y_0,x_0,z_0)))$$
$$+m^n \cdot d(g^{n+1}(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)))$$
$$\Leftrightarrow d(f(g^n(g(x_1)), g^n(g(y_1)), g^n(g(z_1))), f(g^{n-1}(g(x_1)), g^{n-1}(g(y_1)), g^{n-1}(g(z_1))))$$
$$\leq k^n \cdot d(g^{n+1}(g(x_1)), g^n(g(x_1)))+l^n \cdot d(g^{n+1}(g(y_1)), g^n(g(y_1)))+m^n d(g^{n+1}(g(z_1)), g^n(g(z_1)))$$
$$\Leftrightarrow d(f(g^{n+1}(x_1), g^{n+1}(y_1), g^{n+1}(z_1)), f(g^n(x_1), g^n(y_1) g^n(z_1)))$$
$$\leq k^n \cdot d(g^{n+2}(x_1), g^{n+1}(x_1)) + l^n \cdot d(g^{n+2}(y_1), g^{n+1}(y_1)) + m^n \cdot d(g^{n+2}(z_1), g^{n+1}(z_1))$$

This implies that $\{g^n(x_1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Now, because (X,d) is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x,\ \lim_{n\to\infty} g(y_n) = y,\ \lim_{n\to\infty} g(z_n) = z. \tag{6.112}$$

From the continuity of g, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(x),\ \lim_{n\to\infty} g(g(y_n)) = g(y),\ \lim_{n\to\infty} g(g(z_n)) = g(z).$$

Because f and g commute, and from (6.93), we have

$$g(g(x_{n+1})) = g(f(x_n, y_n, z_n)) = f(g(x_n), g(y_n), g(z_n)),$$

$$g(g(y_{n+1})) = g(f(y_n, x_n, z_n)) = f(g(y_n), g(x_n), g(y_n))$$

and

$$g(g(z_{n+1})) = g(f(z_n, y_n, x_n)) = f(g(z_n), g(y_n), g(x_n))$$

From (6.94) and the orbital continuity of f we get

$$g(x) = f(x, y, z),$$

$$g(y) = f(y, x, z)$$

and

$$g(z) = f(z, y, x).$$

□

Corollary 6.2.194. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is $g-R$-monotone;
(ii) f is orbitally g-continuous;
(iii) there exist $\alpha \in [0,1)$ such that

(6.113)

$$d(f(x,y,z), f(t,u,v)) \leq \frac{\alpha}{3} \cdot [d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(t))], \forall (x,y,z) \in X_R(t,u,v);$$

(iv) f and g have a lower-R-tripled coincidence point;

(v) $f(X^3) \subseteq g(X)$*;*

(vi) g *is continuous ;*

(vii) f *and* g *commute.*

Then f *and* g *have a tripled coincidence point, i.e., there exists* $(x, y, z) \in X^3$ *such that* $f(x, y, z) = g(x)$*,* $f(y, x, z) = g(y)$ *and* $f(z, y, x) = g(z)$*.*

Proof: From the proof of Theorem 6.2.193, for $k = l = m = \dfrac{\alpha}{3}, \alpha \in [0, 1)$, there exist $x, y, z \in X$ such that

$$g(x) = f(x, y, z),$$

$$g(y) = f(y, x, z)$$

and

$$g(z) = f(z, y, x).$$

□

THEOREM 6.2.195. *In addition to the hypothesis of Theorem 6.2.193, suppose that for every* $(x, y, z), (\overline{x}, \overline{y}, \overline{z}) \in X^3$*, there exists* $(t, u, v) \in X^3$ *such that* $(g(x), g(y), g(z))$*,* $(g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t))$*. Then* f *and* g *have a unique tripled coincidence point.*

Proof: According to the proof of Theorem 6.2.193, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{z}) = g(\overline{y})$ and $f(\overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x, y, z) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z}), (g(x), g(y), g(z))) = 0.$$

Because both (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ are tripled coincidence points, we have

$$g(x) = f(x, y, z), g(y) = f(y, x, z), g(z) = f(z, y, x)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{z}),$$

$$g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

Now, let $u_0 = u$, $v_0 = v$ and $t_0 = t$. Then, there exist $t_1, u_1, v_1 \in X$ such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, v_0)$ and $g(v_1 f(v_0, u_0, t_0)$. Using the same procedure as in the proof of Theorem 6.1.178, we obtain the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ where

$$g(t_{n+1} = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Furthermore, let $x_0 = x, y_0 = y, z_0 = z$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{z}$. Thus, we obtain the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$ and $\{\overline{x}_n\}_{n\in\mathbb{N}}, \{\overline{y}_n\}_{n\in\mathbb{N}}, \{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n) = f(x, y, z), g(y_n) = f(y, x, z), g(z_n) = f(z, y, x)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{z}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(t, u, v) \in X^3$ such that

$$(g(x), g(y), g(z)), (g(\overline{x}), g(\overline{y}), g(\overline{z}) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(z_0)), f^n(g(z_0), g(y_0), g(x_0)))$$
$$\in X_R(f^{n+1}(t, u, v), f^{n+1}(u, t, u), f^{n+1}(v, u, t))$$

Also, by using the contractivity condition, we have

$$d((f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, v))$$
$$+m^n \cdot d(g(z_0), f(v, u, t)),$$
$$d((f^n(g(y_0), g(x_0), g(z_0)), f^{n+1}(u, t, u)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, v))$$
$$+m^n \cdot d(g(z_0), f(v, u, t))$$

and

$$d((f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq k^n \cdot d(g(x_0), f(t, u, v)) + l^n \cdot d(g(y_0), f(u, t, v))$$
$$+m^n \cdot d(g(z_0), f(v, u, t)).$$

Summing up, we obtain that

$$d((f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) + d((f^n(g(y_0), g(x_0), g(z_0)), f^{n+1}(u, t, v))+$$
$$d((f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq 3k^n \cdot d(g(x_0), f(t, u, v))$$
$$+3l^n \cdot d(g(y_0), f(u, t, v)) + 3m^n \cdot d(g(z_0), f(v, u, t)).$$

But $x_0 = x$, $y_0 = y$ and $z_0 = z$. We obtain

$$d((f^n(g(x), g(y), g(z)), f^{n+1}(t, u, v)) + d((f^n(g(y), g(x), g(z)), f^{n+1}(u, t, v))+$$
$$d((f^n(g(z), g(y), g(x)), f^{n+1}(v, u, t)) \leq 3k^n \cdot d(g(x), f(t, u, v))$$
$$+3l^n \cdot d(g(y), f(u, t, v)) + 3m^n \cdot d(g(z), f(v, u, t)).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x), f(t, u, v)) = 0, \lim_{n\to\infty} d(g(y), f(u, t, v)) = 0$$
$$\text{and } \lim_{n\to\infty} d(g(z), f(v, u, t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t, u, v)) = 0, \lim_{n\to\infty} d(g(\overline{y}), f(u, t, v)) = 0$$
$$\text{and } \lim_{n\to\infty} d(g(\overline{z}), f(v, u, t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x), g(\overline{x})) \leq d(g(x), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y), g(\overline{y})) \leq d(g(y), f(u,v,t)) + d(f(u,t,v), g(\overline{y})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(z), g(\overline{z})) \leq d(g(z), f(v,t,u)) + d(f(v,u,t), g(\overline{z})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

The next result is obtained by replacing the contraction (6.110) with one that uses the mapping φ defined in Chapter 5, following the idea in [**53**]. Consequently, we have:

THEOREM 6.2.196. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii)

(6.114)

$$d(f(x,y,z), f(t,u,v)) \leq \varphi\left(\frac{d(g(x),g(t)) + d(g(y),g(u)) + d(g(z),g(v))}{3}\right), \forall (x,y,z) \in X_R(t,u,v),$$

where $\varphi \in \Phi$;
(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x,y,z) \in X^3$ such that $f(x,y,z) = g(x)$, $f(y,x,z) = g(y)$ and $f(z,y,x) = g(z)$.

Proof: From the hypothesis, we know that f and g have lower-R-triple coincidence point; let (x_0, y_0, z_0) be it. Thus, using the definition of the lower-R-tripled coincidence point, it follows that $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.
From (i) we know that $(f \times g)(X_R(x_0, y_0, z_0)) \subseteq X_R((f \times g)(x_0, y_0, z_0))$.
Further, it can easily be checked that

$$\begin{aligned} &(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,z_0)), g^n(f(z_0,y_0,x_0))) \\ &\in X_R(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,z_0)), g^{n-1}(f(z_0,y_0,x_0))). \end{aligned}$$

Since $f(X^3) \subseteq g(X)$, let $x_1, y_1, z_1 \in X$ such that $g(x_1) = f(x_0,y_0,z_0)$, $g(y_1) = f(y_0,x_0,z_0)$, $g(z_0) = f(z_0,y_0,x_0)$ and so on. Thus, we obtain the sequences $\{x_n\}$,$\{y_n\}$

and $\{z_n\}$ such that

(6.115) $\quad g(x_{n+1}) = f(x_n, y_n, z_n), g(y_{n+1}) = f(y_n, x_n, z_n)$ and $g(z_{n+1} = f(z_n, y_n, x_n)$.

Let's consider the nonnegative sequence $\{\eta_n\}_{n\in\mathbb{N}^*}$ such that $\eta_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}, g(z_n)), n \in \mathbb{N}^*$.

Now, using (6.114), (2.2) and letting $x := x_n$, $y := y_n$ and $z := z_n$, $t := x_{n-1}$, $u := y_{n-1}$ and $v := z_{n-1}$, we obtain

$$d(g(x_{n+1}), g(x_n)) = d(f(x_n, y_n, z_n), f(x_{n-1}, y_{n-1}, z_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right),$$
$$d(g(y_{n+1}), g(y_n)) = d(f(y_n, x_n, z_n), f(y_{n-1}, x_{n-1}, z_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

and

$$d(g(z_{n+1}), g(z_n)) = d(f(z_n, y_n, x_n), f(z_{n-1}, y_{n-1}, x_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

By summing up the last three relations, we obtain that

$$d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}), g(z_n)) = \eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

Now, using the properties of φ, we have that

$$\eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right) < 3 \cdot \frac{\eta_{n-1}}{3} = \eta_{n-1}. \tag{6.116}$$

Thus, $\{\eta_n\}_{n\in\mathbb{N}^*}$ is a decreasing and nonnegative sequence. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} \eta_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (6.116), let $n \to \infty$. Using (ii) (the second condition satisfied by φ) from Chapter 5, we have

$$\varepsilon_0 = \lim_{n\to\infty} \eta_n \leq 3 \cdot \lim_{n\to\infty} \varphi\left(\frac{\eta_{n-1}}{3}\right) = 3 \cdot \lim_{\eta_{n-1}\to\varepsilon_0+} \varphi\left(\frac{\eta_{n-1}}{3}\right) < \varepsilon_0,$$

which is a contradiction. Thus, $\lim\limits_{n\to\infty} \eta_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0, \lim\limits_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(z_{n+1}), g(z_n)) = 0$.

Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \geq \delta, \tag{6.117}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{Z}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (6.99). Then, we have

(6.118) $d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(z_{n_2(k)}), g(z_{n_1(k)-1}) < \delta.$

Now, using the triangle inequality and the last two inequalities ((6.117) and (6.118)), we have

$$\begin{aligned}\delta &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(x_{n_1(k)}), g(x_{n_2(k)}))\\ &\quad + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))\\ &\quad d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))\\ &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) + \delta.\end{aligned}$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))] = \delta.$$

Next, we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}s_k &= d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))\\ &\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)}))\\ &\quad + d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)}))\\ &\quad + + d(g(z_{n_1(k)}), g(z_{n_1(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\\ &= \eta_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))\end{aligned}$$

(6.119)

$$\leq \eta_{n_1(k)} + \eta_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})).$$

But

$$\begin{aligned}&d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1}))d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1}))\\ &= d(f(x_{n_1(k)}, y_{n_1(k)}, z_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)}, z_{n_2(k)}))\\ &\quad + d(f(y_{n_1(k)}, x_{n_1(k)}, z_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}, z_{n_2(k)}))\\ &\quad + d(f(z_{n_1(k)}, y_{n_1(k)}, x_{n_1(k)}), f(z_{n_2(k)}, y_{n_2(k)}, x_{n_2(k)}))\\ &\leq 2 \cdot \varphi\left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}) + d(g(z_{n_1(k)}), g(z_{n_2(k)})}{3}\right) +\\ &\leq 2 \cdot \varphi\left(\frac{s_k}{3}\right).\end{aligned}$$

Now, returning to (6.119), we have

$$s_k \leq \eta_{n_1(k)} + \eta_{n_2(k)} + 2 \cdot \varphi\left(\frac{s_k}{3}\right).$$

Let $k \to \infty$. We obtain

$$\delta \leq 3 \cdot \lim_{k\to\infty} \varphi\left(\frac{s_k}{3}\right) = 3 \cdot \lim_{s_k\to\delta_+} \varphi\left(\frac{s_k}{3}\right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.
Consequently, $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Thus, there exist $\overline{x}$,$\overline{y}$ and $\overline{z}$ such that $g^n(x_n) \to \overline{x}$, $g^n(y_n) \to \overline{y}$ and $g^n(z_n) \to \overline{y}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n, z_n) \to \overline{x}$, $f^{n-1}(y_n, x_n, z_n) \to \overline{y}$ and $f^{n-1}(z_n, y_n, x_n) \to \overline{z}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n, z_n) \to g(\overline{x})$, $f^n(y_n, x_n, z_n) \to g(\overline{y})$ and $f^n(z_n, y_n, x_n) \to g(\overline{z})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y}, \overline{z})$ is a tripled coincidence point of f and g. □

THEOREM 6.2.197. *In addition to the hypothesis of Theorem 6.2.196, suppose that for every* $(x^*, y^*, z^*), (\overline{x}, \overline{y}, \overline{z}) \in X^3$, *there exists* $(t, u, v) \in X^3$ *such that* $(g(x^*), g(y^*), g(z^*))$, $(g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t))$. *Then f and g have a unique tripled coincidence point.*

Proof: From Theorem 6.2.196, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{z}) = g(\overline{y})$ and $f(, \overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x^*, y^*, z^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z}), (g(x^*), g(y^*), g(z^*))) = 0.$$

Since (x^*, y^*, z^*) and $(\overline{x}, \overline{y}, \overline{z})$ are both tripled coincidence points, it follows that

$$g(x^*) = f(x^*, y^*, z^*), g(y^*) = f(y^*, x^*, z^*), g(z^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{z}), g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

Now, using the hypothesis of Theorem 6.2.196, from $f(X^3) \subseteq g(x)$, there exist $t_1, u_1, v_1 \in X^3$ such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, v_0)$, $g(v_1) = f(v_0, u_0, t_0)$. Using the same procedure as in the proof of Theorem 6.2.193, we build the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(t_{n+1}) = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Next, let $x_0 = x^*, y_0 = y^*, z_0 = z^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{z_n^*\}_{n\in\mathbb{N}}$ $\{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*, z^*), g(y_n^*) = f(y^*, x^*, z^*), g(z_n^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{z}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we know that there exists $(t, u, v) \in X^3$ such that

$$(g(x^*), g(y^*), g(z^*), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t,u,v), f(u,t,v), f(v,u,t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(z_0)), f^n(g(z_0), g(y_0), g(x_0)))$$

$$\in X_R(f^{n+1}(t,u,v), f^{n+1}(u,t,v), f^{n+1}(v,u,t))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t,u,v)) \leq$$

$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,v)) + d(g(z_0), f(v,u,t))}{3}\right),$$

$$d(f^n(g(y_0), g(x_0), g(z_0)), f^{n+1}(u,t,u)) \leq$$

$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,v)) + d(g(z_0), f(v,u,t))}{3}\right)$$

and

$$d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v,u,t)) \leq$$

$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,v)) + d(g(z_0), f(v,u,t))}{3}\right).$$

By summing up, we obtain that

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t,u,v)) + d(f^n(g(y_0), g(x_0), g(z_0)), f^{n+1}(u,t,v))$$

$$+d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v,u,t))$$

$$\leq 3\cdot\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,v)) + d(g(z_0), f(v,u,t))}{3}\right).$$

But $x_0 = x^*$, $y_0 = y^*$ and $z_0 = z^*$. We obtain

$$d(f^n(g(x^*), g(y^*), g(z^*)), f^{n+1}(t,u,v)) + d(f^n(g(y^*), g(x^*), g(z^*)), f^{n+1}(u,t,v))$$

$$+d(f^n(g(z^*), g(y^*), g(x^*)), f^{n+1}(v,u,t)) \leq$$

$$3\cdot\varphi\left(\frac{d(g(x^*), f(t,u,v)) + d(g(y^*), f(u,t,v)) + d(g(z^*), f(v,u,t))}{3}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(y^*), f(u,t,v)) = 0 \text{ and } \lim_{n\to\infty} d(g(z^*), f(v,u,t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(\overline{z}), f(u,t,v)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{z}), f(v,u,t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(u,t,v)) + d(f(u,t,v), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

and

$$d(g(z^*), g(\overline{z})) \leq d(g(z^*), f(v, u, t)) + d(f(v, u, t), g(\overline{z})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. $\square$

Now, by symmetrizing the contraction, we obtain the following result:

THEOREM 6.2.198. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is $g - R$-monotone;
(ii) f is orbitally g-continuous;
(iii)

$$d(f(x, y, z), f(t, u, v)) + d(f(y, x, z), f(u, t, v)) + d(f(z, y, x), f(v, u, t) \leq \\ 3 \cdot \varphi \left(\frac{d(g(x), g(t)) + d(g(y), g(y)) + d(g(z), g(v))}{3} \right), \forall (x, y, z) \in X_R(t, u, v); \tag{6.120}$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3) \subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x, y, z) \in X^3$ such that $f(x, y, z) = g(x)$, $f(y, x, z) = g(y)$ and $f(z, y, x) = g(z)$.

Proof: The prove this result, we will follow the steps from the proof of Theorem 6.2.196: Since f and g have lower-R-tripled coincidence point, let (x_0, y_0, z_0) be it. Thus, $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.
From (i) we have that $(f \times g)(X_R(x_0, y_0, z_0)) \subseteq X_R((f \times g)(x_0, y_0, z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0, y_0, z_0)), g^n(f(y_0, x_0, z_0)), g^n(f(z_0, y_0, x_0))) \in \\ X_R(g^{n-1}(f(x_0, y_0, z_0)), g^{n-1}(f(y_0, x_0, z_0)), g^{n-1}(f(z_0, y_0, x_0))). \tag{6.121}$$

Since $f(X^3) \subseteq g(X)$, let $x_1, y_1, z_1 \in X$ such that $g(x_1) = f(x_0, y_0, z_0)$, $g(y_1) = f(y_0, x_0, z_0)$, $g(z_1) = f(z_0, y_0, x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1}) = f(x_n, y_n, z_n), g(y_{n+1}) = f(y_n, x_n, z_n), g(z_{n+1}) = f(z_n, y_n, x_n) \tag{6.122}$$

Let's consider the nonnegative sequence $\{\eta_n\}_{n \in \mathbb{N}^*}$ such that $\eta_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}, g(z_n)), n \in \mathbb{N}^*$.

Now, using (6.120), (6.121) and letting $x := x_n$, $y := y_n$ and $z := z_n$, $t := x_{n-1}$, $u := y_{n-1}$ and $v := z_{n-1}$, we obtain

$$d(f(x_n, y_n, z_n), f(x_{n-1}, y_{n-1}, z_{n-1})) + d(f(y_n, x_n, z_n), f(y_{n-1}, z_{n-1}, x_{n-1}))$$
$$+d(f(z_n, y_n, x_n), f(z_{n-1}, x_{n-1}, y_{n-1})) = d(g(x_{n+1}), g(x_n))$$
$$+d(g(y_{n+1}), g(y_n)) + d(g(z_{n+1}), g(z_n)) \leq$$
$$3\varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))d(g(z_n), g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_n}{3}\right)$$

Thus, we have that

$$\eta_{n+1} \leq 3 \cdot \varphi\left(\frac{\eta_n}{3}\right) \tag{6.123}$$

Using the proof of Theorem 6.2.196 we have that $\{\eta_n\}_{n\in\mathbb{N}^*}$ is decreasing and nonnegative. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} \eta_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (6.105), let $n \to \infty$. Using (ii) (the second condition satisfied by φ) from Chapter 5, we have

$$\varepsilon_0 = \lim_{n\to\infty} \eta_{n+1} \leq 3 \cdot \lim_{n\to\infty} \varphi\left(\frac{\eta_n}{3}\right) = 3 \cdot \lim_{\eta_n\to\varepsilon_{0+}} \varphi\left(\frac{\eta_n}{3}\right) < \varepsilon_0.$$

So we have that $\varepsilon_0 < \varepsilon_o$, which is, clearly, a contradiction.
Thus, $\lim\limits_{n\to\infty} \eta_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0$, $\lim\limits_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(z_{n+1}), g(z_n)) = 0$.
Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \geq \delta, \tag{6.124}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{Z}^*$. Next, let's chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (6.106). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) < \delta. \tag{6.125}$$

Now, using the triangle inequality and the last two inequalities ((6.124) and (6.125)), we have

$$\delta \leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))$$
$$\leq d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(x_{n_1(k)}), g(x_{n_2(k)}))$$
$$+d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}))$$
$$d(g(z_{n_2(k)}), g(z_{n_1(k)-1})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))$$

$$\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + +d(g(z_{n_2(k)}), g(z_{n_1(k)})) + \delta.$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))] = \delta.$$

Now we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}
s_k = d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \\
\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) \\
+d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) \\
+d(g(z_{n_1(k)}), g(z_{n_1(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))
\end{aligned}$$

$$= \eta_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)}))$$

(6.126)

$$\leq \eta_{n_1(k)} + \eta_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})).$$

But

$$\begin{aligned}
d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})) \\
= d(f(x_{n_1(k)}, y_{n_1(k)}, z_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)}, z_{n_1(k)})) \\
+d(f(y_{n_1(k)}, x_{n_1(k)}, z_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}, z_{n_2(k)})) \\
+d(f(z_{n_1(k)}, y_{n_1(k)}, x_{n_1(k)}), f(z_{n_2(k)}, y_{n_2(k)}), x_{n_2(k)})) \\
\leq 3 \cdot \varphi \left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)})) + d(g(z_{n_1(k)}), g(z_{n_2(k)}))}{3} \right) \\
\leq 3 \cdot \varphi \left(\frac{s_k}{3} \right).
\end{aligned}$$

Now, returning to (6.108), we have

$$s_k \leq \eta_{n_1(k)} + \eta_{n_2(k)} + 3 \cdot \varphi \left(\frac{s_k}{3} \right).$$

Let $k \to \infty$. We obtain

$$\delta \leq 3 \cdot \lim_{k\to\infty} \varphi \left(\frac{s_k}{3} \right) = 3 \cdot \lim_{s_k \to \delta_+} \varphi \left(\frac{s_k}{3} \right) < \delta$$

Thus, we have that $\delta < \delta$ which is clearly a contradiction.

Consequently, $\{g(x_n)\}_{n\in\mathbb{N}}$,$\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since (X, d) is complete, there exist $\overline{x}$ and $\overline{y}$ such that $g^n(x_n) \to \overline{x}$, $g^n(y_n) \to \overline{y}$ and $g^n(z_n) \to \overline{z}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n, z_n) \to \overline{x}$, $f^{n-1}(y_n, x_n, z_n) \to \overline{y}$ and $f^{n-1}(z_n, y_n, x_n) \to \overline{z}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n, z_n) \to g(\overline{x})$, $f^n(y_n, x_n, z_n) \to g(\overline{y})$ and $f^n(z_n, y_n, x_n) \to g(\overline{z})$, as $n \to \infty$, that is, $(\overline{x}, \overline{y}, \overline{z})$ is a tripled coincidence point of f and g.

$\square$

THEOREM 6.2.199. *In addition to the hypothesis of Theorem 6.2.198, suppose that for every* $(x^*, y^*, z^*), (\overline{x}, \overline{y}, \overline{z}) \in X^3$, *there exists* $(t, u, v) \in X^3$ *such that* $(g(x^*), g(y^*), g(z^*))$, $(g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t))$. *Then* f *and* g *have a unique tripled coincidence point.*

Proof: According to Theorem 6.2.198, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{y}) = g(\overline{y})$ and $f(\overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x^*, y^*, z^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z})), (g(x^*), g(y^*), g(z^*))) = 0.$$

Because both (x^*, y^*, z^*) and $(\overline{x}, \overline{y}, \overline{z})$ are tripled coincidence points, we have

$$g(x^*) = f(x^*, y^*, z^*), g(y^*) = f(y^*, x^*, z^*), g(z^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{z}), g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

From $f(X^3) \subseteq g(X)$, there exist t_0, u_0, v_0 in X such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, v_0)$, $g(v_1) = f(v_0, u_0, t_0)$. Following the procedure used in the proof of Theorem 6.2.193, we obtain the sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, where

$$g(t_{n+1}) = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, v_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Furthermore, let $x_0^* = x^*, y_0^* = y^*, z_0^* = z^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{z}$. Thus, we obtain the sequences $\{x_n^*\}_{n\in\mathbb{N}}, \{y_n^*\}_{n\in\mathbb{N}}, \{z_n^*\}_{n\in\mathbb{N}}, \{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}}$ and $\{\overline{z}_n\}_{n\in\mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*, z^*), g(y_n^*) = f(y^*, x^*, z^*), g(z_n^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{z}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we have that there exists $(t, u, v) \in X^3$ such that

$$(g(x^*), g(y^*), g(z^*), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t, u, v), f(u, t, v), f(v, u, t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(z_0)), f^n(g(z_0), g(y_0), g(x_0)))$$

$$\in X_R(f^{n+1}(t, u, v), f^{n+1}(u, t, v), f^{n+1}(v, u, t))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t, u, v)) + d(f^n(g(y_0), g(x_0), g(z_0)), f^{n+1}(u, t, v))$$

$$+ + d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v, u, t)) \leq$$

$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,v)) + d(g(z_0), f(v,u,t))}{3}\right).$$

But $x_0^* = x^*$, $y_0^* = y^*$ and $z_0^* = z^*$. We obtain

$$d(f^n(g(x^*), g(y^*), g(z^*)), f^{n+1}(t,u,v)) + d(f^n(g(y^*), g(x^*), g(z^*)), f^{n+1}(u,t,v))$$

$$+d(f^n(g(z^*), g(y^*), g(x^*)), f^{n+1}(v,u,t)) \leq$$

$$\varphi\left(\frac{d(g(x^*), f(t,u,v)) + d(g(y^*), f(u,t,v)) + d(g(z^*), f(v,u,t))}{3}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(y^*), f(u,t,v)) = 0 \text{ and } \lim_{n\to\infty} d(g(z^*), f(v,u,t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(\overline{y}), f(u,t,v)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{z}), f(v,u,t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(u,t,u)) + d(f(u,t,v), g(\overline{y})) \to 0, \text{ when } n \to \infty$$

and

$$d(g(z^*), g(\overline{z})) \leq d(g(z^*), f(v,u,t)) + d(f(v,u,t), g(\overline{z})) \to 0, \text{ when } n \to \infty$$

which means that

$$g(x^*) = g(\overline{x}),$$

$$g(y^*) = g(\overline{y})$$

and

$$g(z^*) = g(\overline{z}).$$

□

Next, we let $\varphi(t) = kt, k \in [0,1)$. It is easy to check that conditions (i) and (ii) still hold. We obtain the following result:

THEOREM 6.2.200. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(i) f is $g-R$-monotone;
(ii) f is orbitally $g-$continuous;

(iii) there exist $k \in [0,1)$ such that

$$d(f(x,y,z),f(t,u,v))+d(f(y,x,z),f(u,t,v))+d(f(z,y,x),f(v,u,t)) \leq k\cdot[d(g(x),g(t))+d(g(y),g(u))+d(g(z),g(v))], \forall(x,y,z)\in X_R(t,u,v); \tag{6.127}$$

(iv) f and g have a lower-R-tripled coincidence point;
(v) $f(X^3)\subseteq g(X)$;
(vi) g is continuous ;
(vii) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x,y,z)\in X^3$ such that $f(x,y,z)=g(x)$, $f(y,x,z)=g(y)$ and $f(z,y,x)=g(z)$.

Proof: In Theorem 6.2.198, let $\varphi(t)=kt, k\in[0,1)$. □

THEOREM 6.2.201. *In addition to the hypothesis of Theorem 6.2.200, suppose that for every $(x,y,z),(\overline{x},\overline{y},\overline{z})\in X^3$, there exists $(t,u,v)\in X^3$ such that $(g(x),g(y),g(z))$, $(g(\overline{x}),g(\overline{y}),g(\overline{z}))\in X_R(f(t,u,v),f(u,t,v),f(v,u,t))$. Then f and g have a unique tripled coincidence point.*

Proof: In Theorem 6.2.197, let $\varphi(t)=kt, k\in[0,1)$. □

REMARK 6.2.202. *If, in Theorem 6.2.193, resp 6.2.195 we let $R=\leq$, suppose that either the mapping f is only continuous or the condition 5.70 holds, we obtain Theorems 5.1.151, resp. 5.1.152 from* [**37**].

3. Examples and applications

3.1. Examples

Next, we will present some examples for the results presented above:

EXAMPLE 6.3.203. *Let $X=\mathbb{R}$, $d=|x-y|$,t he relation R on X given by*

$$(x,y,z)R(t,u,v)\Leftrightarrow xRt\wedge yRu\wedge zRv,$$

where $xRt\Leftrightarrow x^2+2x=t^2+2t$. Let $f:X^3\to X$ be defined by

$$f(x,y,z)=\frac{x+y+2z-1}{6}$$

and $g:X\to X$, where

$$g(x)=2x-1.$$

So, $\forall(x,y,z)\in X^3$, we have :

$$X_R(x,y,z)=\{(x,y,z),(x,y+2,z),(x+2,y,z),(x+2,y+2,z),(x,y,z+2),$$
$$(x+2,y+2,z+2),(x+2,y,z+2),(x,y+2,z+2)\}.$$

$$f \times g(X_R(x,y,z)) \subseteq X_R(f \times g(x,y,z))$$

So, f has the $g-R-$monotone property. It can easily be checked that f and g satisfy all the other conditions of Theorem 6.2.193. The contraction also holds for $k = l = \frac{1}{6}$ and $m = \frac{1}{3}$:

$$d(f(x,y,z), f(t,u,v)) = \left| \frac{x+y+2z-1}{6} - \frac{t+u+2v-1}{6} \right| =$$

$$\left| \frac{1}{6}(x-t) + \frac{1}{6}(y-u) + \frac{1}{3}(z-v) \right| < 2 \left| \frac{1}{6}(x-t) + \frac{1}{6}(y-u) + \frac{1}{3}(z-v) \right|$$

$$\leq \frac{1}{3}|x-t| + \frac{1}{3}|y-u| + \frac{2}{3}|z-v|$$

$$= \frac{1}{6}|g(x)-g(t)| + \frac{1}{6}|g(y)-g(u)| + \frac{1}{3}|g(z)-g(v)|$$

$$= \frac{1}{6}d(g(x),g(t)) + \frac{1}{6}d(g(y),g(u)) + \frac{1}{3}d(g(z),g(v)),$$

$$\text{where, clearly, } \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3} < 1.$$

So, by Theorem 6.2.193, we obtain that f and g have a (unique) tripled coincidence point, $\left(\frac{5}{8}, \frac{5}{8}, \frac{5}{8}\right)$, obtained by solving the following system of equations:

$$\begin{cases} f(x,y,z) = g(x) \\ f(y,x,y) = g(y) \\ f(z,y,x) = g(z). \end{cases} \tag{6.128}$$

It can easily be checked that for solving the system above, we use Cramer's rule (so its' solution is unique and so is the tripled coincidence point of the two mappings).

EXAMPLE 6.3.204. *Let $X = \mathbb{R}$, $d = |x-y|$, the relation R on X given by*

$$(x,y,z)R(t,u,v) \Leftrightarrow xRt \wedge yRu \wedge zRv,$$

where $xRt \Leftrightarrow x^2 + x = t^2 + t$. Let $f : X^3 \to X$ be defined by

$$f(x,y,z) = \frac{x-y+3z-2}{6}$$

and $g : X \to X$, where

$$g(x) = x - 1.$$

So, $\forall (x,y,z) \in X^3$, we have :

$X_R(x,y,z) = \{(x,y,z), (x,-y-1,z), (-x-1,y,z), (-x-1,-x-1,z), (x,y,-z-1),$

$$(-x-1,-y-1,-z-1),(-x-1,y,-z-1),(x,-y-1,-z-1)\}.$$

$$f\times g(X_R(x,y,z))\subseteq X_R(f\times g(x,y,z))$$

So, f has the mixed $g-R-$monotone property. It can easily be checked that f and g satisfy all the other conditions of Theorem 6.1.181. The contraction also holds for $\varphi(t)=\dfrac{kt}{3}, k\in[0,1)$. So, by Theorem 6.1.181, we obtain that f and g have a (unique) tripled coincidence point, $\left(\dfrac{4}{3},\dfrac{4}{3},\dfrac{4}{3}\right)$, obtained by solving the following system of equations:

$$\begin{cases} f(x,y,z)=g(x)\\ f(y,x,y)=g(y)\\ f(z,y,x)=g(z).\end{cases} \tag{6.129}$$

The determinant of the matrix associated to the system above is $\Delta=-144\neq 0$, so the solution is, indeed, unique.

Example 6.3.205. *Let $X=\mathbb{R}$, $d=|x-y|$, the relation R on X given by*

$$(x,y,z)R(t,u,v)\Leftrightarrow xRt\wedge yRu\wedge zRv,$$

where $xRt\Leftrightarrow x^2+2x=t^2+2t$. Let $f:X^3\to X$ be defined by

$$f(x,y,z)=\frac{2x+2y-3z+1}{18}$$

and $g:X\to X$, where

$$g(x)=\frac{2x}{3}.$$

So, $\forall(x,y,z)\in X^3$, we have :

$$X_R(x,y,z)=\{(x,y,z),(x,y+2,z),(x+2,y,z),(x+2,y+2,z),(x,y,z+2),$$

$$(x+2,y+2,z+2),(x+2,y,z+2),(x,y+2,z+2)\}.$$

$$f\times g(X_R(x,y,z))\subseteq X_R(f\times g(x,y,z))$$

So, f has the mixed $g-R-$monotone property. It can easily be checked that f and g satisfy all the other conditions of Theorem 6.1.185.
Condition (6.109) *is also satisfied by f and g, whereas* (6.95) *in Theorem 6.1.179 does not hold. Let's assume, to the contrary, that there exists $\alpha\in[0,1)$, such that* (6.95) *holds. This means*

$$d(f(x,y,z),f(t,u,v))=\left|\frac{2x+2y-3z+1}{18}-\frac{2t+2u-3v+1}{18}\right|=$$

$$\leq\frac{\alpha}{3}[\frac{2}{3}|x-t|+\frac{2}{3}|y-u|+\frac{2}{3}|z-v|]$$

For $x = t$, we have :

$$\left|\frac{2(y-u)-3(z-v)}{18}\right| \le \frac{2\alpha}{9}[|y-u|+|z-v|]$$

Next, for $y = u$, we obtain

$$\left|\frac{-3(z-v)}{18}\right| \le \frac{2\alpha}{9}|z-v|$$

$$\Leftrightarrow \left|\frac{(v-z)}{6}\right| \le \frac{2\alpha}{9}|z-v|$$

a contradiction,$\forall\alpha \in [0,1)$. Hence, f and g do not satisfy (6.95).
Next, we will prove that (6.109) *holds.*

$$\left|\frac{2x+2y-3z+1}{18} - \frac{2t+2u-3v+1}{18}\right| \le \frac{1}{6}|g(x)-g(t)|+\frac{1}{6}|g(y)-g(u)|+\frac{1}{4}|g(z)-g(v)|,$$

$$\left|\frac{2y+2z-3x+1}{18} - \frac{2u+2v-3t+1}{18}\right| \le \frac{1}{6}|g(y)-g(u)|+\frac{1}{6}|g(z)-g(v)|+\frac{1}{4}|g(x)-g(t)|$$

and

$$\left|\frac{2z+2x-3y+1}{18} - \frac{2v+2t-3u+1}{18}\right| \le \frac{1}{6}|g(z)-g(v)|+\frac{1}{6}|g(x)-g(t)|+\frac{1}{4}|g(y)-g(u)|.$$

By summing up these three inequalities, we obtain exactly (6.109), *i.e.,*

$$d(f(x,y,z),f(t,u,v)) + d(f(y,x,y),f(u,t,u)) + d(f(z,y,x),f(v,u,t))$$

$$\le k[d(g(x),g(t)) + d(g(y),g(u)) + d(g(z),g(v))],$$

where $k = \dfrac{7}{12} < 1$. So, by Theorem 6.1.185, we obtain that f and g have a (unique) tripled coincidence point, $\left(\dfrac{1}{11},\dfrac{1}{11},\dfrac{1}{11}\right)$, obtained by solving the following system of equations

$$\begin{cases} f(x,y,z) = g(x) \\ f(y,x,y) = g(y) \\ f(z,y,x) = g(z). \end{cases} \tag{6.130}$$

3.2. An application

Motivated by the work of Eshi, Das and Debnath in [**63**], let us consider the following system of integral equations:

$$
\begin{cases}
x(t) = \int_0^T f(t, x(s), y(s), z(s))ds, & t \in [0, T] \\
y(t) = \int_0^T f(t, y(s), x(s), y(s))ds, & t \in [0, T] \\
z(t) = \int_0^T f(t, z(s), y(s), x(s))ds, & t \in [0, T]
\end{cases}
\tag{6.131}
$$

where $T \in \mathbb{R}_+$ and $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$.

Let $X = C([0, T], \mathbb{R})$ and define a metric d on X such that

$$d(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)|\,, \forall x, y \in X$$

Further, we define the relation R using partial order:

$$xRy \Leftrightarrow x(t) \le y(t), t \in [0, T].$$

It is easy to check that (X, d) is a complete metric space and that R is a reflexive relation.

Theorem 6.3.206. *Considering the system of integral equations* (6.131)*, let us suppose that the following hold:*

i). *f is continuous;*

ii). *for all $t \in [0, T]$ and $x, y, z, u, v, w \in \mathbb{R}$ we have $x \le u, y \ge v, z \le w$ and $f(t, x, y, z) \le f(t, u, v, w)$;*

iii). *for each $t \in [0, T]$ and $x, y, z, u, v, w \in \mathbb{R}$, with $x \le u, y \ge v, z \le w$, there exists $k \in [0, 1)$ such that*

$$|f(t, x, y, z) - f(t, u, v, w)| \le \frac{k}{3T} (|x - u| + |y - v| + |z - w|)\,;$$

iv). *there exists $(x_0, y_0, z_0) \in X^3$ such that, $\forall t \in [0, T]$ we have*

$$x_0(t) \le \int_0^T f(t, x_0(s), y_0(s), z_0(s))ds,$$

$$y_0(t) \ge \int_0^T f(t, y_0(s), x_0(s), y_0(s))ds$$

and

$$z_0(t) \le \int_0^T f(t, z_0(s), y_0(s), x_0(s))ds.$$

Then (6.131) *has at least one solution.*

Proof: Let $f : X^3 \to X$ and $g : X \to X$,

$$f(x, y, z) = \int_0^T f(t, x(s), y(s), z(s))ds, \quad t \in [0, T]$$

and

$$g(x(t)) = x(t).$$

Using these notations, (6.131) becomes:

$$\begin{cases} g(x) = f(x,y,z) \\ g(y) = f(y,x,y) \\ g(z) = f(z,x,y). \end{cases} \tag{6.132}$$

This means that the solution of (6.131) is, in fact, the tripled coincidence point for f and g, provided the fact that the hypothesis of Theorem 6.1.179 is checked.

Suppose that $x, y, z, u, v, w \in X$ such that $g(x) \leq g(u), g(y) \geq g(v), g(z) \leq g(w)$. Then, we have

$$\begin{aligned} f(x,y,z)(t) &= \int_0^T f(t, x(s), y(s), z(s))ds \\ &= \int_0^T f(t, g(x(s)), g(y(s)), g(z(s)))ds \\ &\leq \int_0^T f(t, g(u(s)), g(v(s)), g(w(s)))ds \\ &= f(u,v,w)(t), \text{ for each } t \in [0,T]. \end{aligned}$$

This means that $f(x,y,z)(t) \leq f(u,v,w)(t), \forall t \in [0,T]$. Similarly, we can prove that $f(y,x,y)(t) \geq f(v,u,v)(t)$ and $f(z,x,y)(t) \leq f(w,v,u)(t), \forall t \in [0,T]$. Thus, f is mixed $g-R-$monotone. Next, we will show that the contractive condition in Theorem 6.1.179 holds:

$$\begin{aligned} &|f(x,y,z)(t) - f(u,v,w)(t)| = \\ &\left| \int_0^T f(t, x(s), y(s), z(s))ds - \int_0^T f(t, u(s), v(s), w(s))ds \right| \\ &\leq \int_0^T |f(t, x(s), y(s), z(s)) - f(t, u(s), v(s), w(s))| \, ds \\ &\leq \frac{k}{3T} \int_0^T (|x(s) - u(s)| + |y(s) - v(s)| + |z(s) - w(s)|) \, ds \\ &\leq \frac{k}{3T} \int_0^T \left(\sup_{n \in [0,T]} |x(n) - u(n)| + \sup_{n \in [0,T]} |y(n) - v(n)| + \sup_{n \in [0,T]} |z(n) - w(n)| \right) ds \\ &\leq \frac{k}{3T} \left(\sup_{n \in [0,T]} |x(n) - u(n)| + \sup_{n \in [0,T]} |y(n) - v(n)| + \sup_{n \in [0,T]} |z(n) - w(n)| \right), \end{aligned}$$

which means that

$$d(f(x,y,z), f(u,v,w)) \leq \frac{k}{3} \cdot [d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))],$$
$$\forall (x,y,z) \in X_R(u,v,w), \text{ where } R = \leq .$$

Next, from condition (iv), we have

$$x_0(t) \leq \int_0^T f(t, x_0(s), y_0(s), z_0(s))ds,$$

$$y_0(t) \geq \int_0^T f(t, y_0(s), x_0(s), y_0(s))ds$$

and

$$z_0(t) \leq \int_0^T f(t, z_0(s), y_0(s), x(s))ds,$$
$$\forall t \in [0, T].$$

But $x_0(t) = g(x_0(t)), y_0(t) = g(y_0(t))$ and $z_0(t) = g(z_0(t))$. Then, $g(x_0) \leq f(x_0, y_0, z_0), g(y_0) \geq f(y_0, x_0, y_0)$ and $g(z_0) \leq f(z_0, y_0, x_0)$, which means that f and g admit an R-tripled coincidence point.
It can also be easily checked that f and g commute and $f(X^3) \subseteq g(X)$.
Also, condition (ii) in Theorem 6.1.179 follows from condition (i) in the hypothesis. Thus, all the conditions in the hypothesis of Theorem 6.1.179 are fulfilled.
Hence, there exists a point $(x^*, y^*, z^*) \in X^3$ such that

$$g(x^*) = f(x^*, y^*, z^*),$$

$$g(y^*) = f(y^*, z^*, x^*$$

and

$$g(z^*) = f(z^*, y^*, x^*).$$

Now, using the definition of g, we have

$$x^* = g(x^*) = f(x^*, y^*, z^*),$$
$$y^* = g(y^*) = f(y^*, z^*, x^*$$

and

$$z^* = g(z^*) = f(z^*, y^*, x^*),$$

which means that (x^*, y^*, z^*) is a solution for (6.131). □

Remark 6.3.207. *If, in Theorem 6.3.206, we take $R = \leq$ we obtain an extension in the case of tripled coincidence points for Theorem 3.1 in* [**63**].

CHAPTER 7

Conclusions

As presented in the beginning of this thesis, the fixed point theory is one of the most productive and dynamic sub-domains of nonlinear analysis, meeting a great development in the last decades. Undoubtedly, the cornerstone of this vast theory is the Banach-Cacciappoli-Picard principle, a result intensively extended and generalized by researchers in various contexts, with several applications in integral equations, nonlinear matrix equations, differential equations , systems of functional equations, etc.

An important step in the development of this theory was made by Ran and Reurings in [**113**], when extending the famous contraction principle in partially ordered (complete) metric spaces, followed by Nieto and Rodríguez-López [**96**], [**95**], [**94**] who refined the results in [**113**] by removing the continuity of the mapping in the case of nonincreasing and nondecreasing, respectively, mappings. Not much later, Guo and Lakshmikantham, in [**70**], then Bhaskar and Lakshmikantham, in [**36**], define the coupled fixed points for mixed-monotone mappings, in a similar context, as in partially ordered metric spaces. Related to these concepts, coupled common and coincidence points of two mappings were shortly after introduced and discussed by Ćirić and Lakshmikantham in [**53**] and Jungck and Rhoades in [**75**]. It is important to mention that coupled fixed points were first studied by Opoitsev in [**97**], [**99**], [**98**].

Berinde and Borcut, in [**28**], laid the foundation of a new chapter in the metrical theory of fixed points, when introducing the tripled fixed points of a mapping. These were studied for mixed-monotone mappings in [**28**], [**40**], [**42**] and for monotone mappings in [**39**], [**41**] in partially ordered metric spaces. Tripled coincidence points were also introduced by Borcut in [**37**], [**38**].

Despite the three major directions that the development of fixed point theory followed, presented in Chapter 1, (theorems in metric spaces, topological spaces, ordered sets), a great part of the results in the field reveal the same tendency: they were obtained in metric spaces endowed with a relation of (partial) order. It was interesting to discover the articles and books where this relation was replaced by an amorphous binary one, a reflexive one, or a transitive one as presented in [**19**], [**10**], [**126**] and others. This was one of the ideas that lead to the results presented in this thesis.

Another fructuous way to obtain more general results and extend famous ones is by weakening the contraction condition. One of the pioneers of this idea is Berinde, obtaining important results for coupled fixed points and coupled coincidence points in [**27**], [**20**], [**26**],[**23**] and others. An example of contraction used for coupled fixed points is

$$d(f(x,y),f(t,u))+d(f(y,x),f(u,t))\le k[d(x,t)+d(y,u)],\text{ where } k\in[0,1)$$

and for coupled coincidence points,

$$d(f(x,y),f(t,u)+d(f(y,x),f(u,t))\le k[d(g(x),g(t))+d(g(y),g(u))],\text{ where } k\in[0,1).$$

Following these directions, we introduced R-tripled fixed points, R-coupled and tripled coincidence points and extended related notions(R-monotone sequence, mixed R-monotony of a mapping, orbital continuity of a mapping,in X^3, mixed $g-R-$continuity of a mapping and others, see Definitions 4.2.90-4.2.89, 5.2.154-5.2.156, 6.1.175-6.1.177).

As seen in Chapters 3-6, we also provided existence and uniqueness theorems for them, using different types of contractions, as listed below :

$$d(F(x,y),F(z,t))+d(F(y,x),F(t,z))\le k[d(x,z)+d(y,t)],\forall(x,y)\in X_R(z,t);$$

$$d(f_1(x,y),f_1(z,t))+d(f_2(x,y),f_2(z,t))\le k\cdot[d(x,z)+d(y,t)],\forall(x,y)\in X_R(z,t);$$

$$d(F(x,y,z),F(t,u,v))\le\frac{k}{3}\cdot[d(x,t)+d(y,u)+d(z,v)],\forall(x,y,z)\in X_R(t,u,v),k\in[0,1);$$

$$d(F(x,y,z),F(t,u,v))\le a\cdot d(x,t)+b\cdot d(y,u)+c\cdot d(z,v),\forall(x,y,z)\in X_R(t,u,v),a+b+c<1;$$

$$d(f(x,y),f(z,t))+d(f(y,x),f(t,z))\le k\cdot[d(g(x),g(z))+d(g(y),g(t))],\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y),f(z,t))+d(f(x,y),f(t,z))\le 2\cdot\varphi\left(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\right),\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y),f(z,t))\le\varphi\left(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\right),\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y),f(z,t))\le\frac{\alpha}{2}[d(g(x),g(z))+d(g(y),g(t))],\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y),f(z,t))\le k\cdot d(g(x),g(z))+l\cdot d(g(y),g(t)),\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y,z),f(t,u,v))\le k\cdot d(g(x),g(t))+l\cdot d(g(y),g(u))+m\cdot d(g(z),g(v)),\forall(x,y,z)\in X_R(t,u,v);$$

$$d(f(x,y,z),f(t,u,v))\le\frac{\alpha}{3}\cdot[d(g(x),g(u))+d(g(y),g(v))+d(g(z),g(t))],\forall(x,y,z)\in X_R(t,u,v);$$

$$d(f(x,y,z),f(t,u,v))\le\varphi\left(\frac{d(g(x),g(t))+d(g(y),g(u))+d(g(z),g(v))}{3}\right),\forall(x,y,z)\in X_R(t,u,v);$$

$$d(f(x,y,z),f(t,u,v))+d(f(y,z,x),f(u,v,t))+d(f(z,y,x),f(v,t,u))\leq$$

$$3\cdot\varphi\left(\frac{d(g(x),g(t))+d(g(y),g(y))+d(g(z),g(v))}{3}\right),\forall(x,y)\in X_R(z,t);$$

$$d(f(x,y,z),f(t,u,v))+d(f(y,z,x),f(u,v,t))+d(f(z,x,y),f(v,t,u))$$

$$\leq k\cdot[d(g(x),g(t))+d(g(y),g(u))+d(g(z),g(v))],\forall(x,y,z)\in X_R(t,u,v),$$

where φ is the one recalled in Chapter 5.

We consider that our results present great importance from the perspective of their applicability in solving different types of problems. To sustain this idea, we presented applications in first-order periodic boundary systems in Chapter 3, nonlinear matrix equations in Chapters 4, 5, integral equations in Chapter 6 and provided illustrative examples for our results (see Examples 3.4.68, 3.4.69, 4.4.129, 4.4.130, 5.4.169, 5.4.170, 6.3.203, 6.3.204, 6.3.205).

Also, our research can be extended in the following directions:

(1) extending the results for quadrupled fixed points, as introduced in [**83**],[**84**],[**80**] or for higher dimensional points, as presented in [**35**], [**127**];
(2) using other types of contractions, for example:
   - Chatterjea, in [**48**]:

$$d(Tx,Ty)\leq b\cdot[d(x,Ty)+d(y,Tx)],\ \text{where } b\in\left[0,\frac{1}{2}\right),x,y\in X;$$

   - Zamfirescu, in [**150**]:
   One of the following holds:

$$d(Tx,Ty)\leq a\cdot d(x,y);$$

$$d(Tx,Ty)\leq b\cdot[d(x,Tx)+d(y,Ty)];$$

$$d(Tx,Ty)\leq c\cdot[d(x,Ty)+d(y,Tx)],$$

   where $0\leq a<1$, $b<\frac{1}{2}$, $c<\frac{1}{2}$, for every $x,y\in X$.
   - Rus, in [**120**]:

$$d(Tx,Ty)\leq a\cdot d(x,y)+b\cdot[d(x,Tx)+d(y,Ty)],\forall x,y\in X,a+2b<1;$$

   - Agarwal, El-Gebeily and O'Regan in [**1**] and [**2**]:

$$d(fx,fy)\leq\varphi\left(max\left\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\right\}\right),\forall x,y\in X,x\geq y;$$

- Hardy and Rogers, in [**72**]:

$$d(Tx,Ty) \leq a \cdot d(x,Tx) + b \cdot d(y,Ty) + c \cdot d(x,Ty) + e \cdot d(y,Tx)$$
$$+ f \cdot d(x,y), \forall x,y \in X, a+b+c+e+f < 1;$$

- Babu et. al. in [**16**]:

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot min\left\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\},$$
$$0 < \delta < 1, L \geq 0, x,y \in X$$

(3) extending our results in the case of a metric space endowed with a transitive relation, or an amorphous relation, following the ideas in [**126**], [**19**] and redefining all the related concepts for that particular case (i.e. coupled fixed point, mixed-monotony);

(4) following the approach in [**108**] and [**145**] regarding coupled fixed points, extending tripled fixed points using three operators, instead of one. An example for this kind of approach could be

$$f_1(x,y,z) = x, f_2(x,y,z) = y, f_3(x,y,z) = z;$$

(5) following the same idea, coupled coincidence points and tripled coincidence points could also be generalized and redefined:

$$f_1(x,y) = g(x), f_2(x,y) = g(y), \text{ in the case of coupled coincidence points}$$

and

$$f_1(x,y,z) = g(x), f_2(x,y,z) = g(y), f_3(x,y,z) = g(z),$$
$$\text{in the case of tripled coincidence points;}$$

(6) obtain results regarding the existence and uniqueness for coupled coincidence point of mixed-g-monotone mappings and (ψ, ϕ)-weakly contractive mappings in G-metric spaces, starting from the work of Chandok, Mustafa and Postolache in [**47**] and Aydi, Postolache and Shatanawi in [**15**], by replacing the relation of order with a reflexive relation;

(7) replacing assumptions (vii) from the hypothesis of Theorems 5.3.157, 5.3.159, 5.3.164, i.e. f and g commute, with a weaker condition;

(8) study coupled and tripled coincidence and fixed points in b-metric spaces endowed with a reflexive relation, starting from the work of Miculescu and Mihail in [**92**], Bota, Petruşel, Petruşel and Samet in [**43**], Mustafa, Roshan and Parvaneh in [**93**], Petruşel, Petruşel, Samet and Yao in [**107**] and[**105**], Sintunavarat, Plubtieng and Katchang in [**136**];

(9) study the existence and uniqueness of the solution of a Riccati equation.

Bibliography

[1] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., **87** (2008), no. 1, 109–116.

[2] R. P. Agarwal, E. Karapinar, D. O'Regan, and A. F. Roldán-López-de Hierro, *Fixed point theory in metric type spaces*, Springer, 2015.

[3] A. Aghajani, M. Abbas, and E. Pourhadi Kallehbasti, *Coupled fixed point theorems in partially ordered metric spaces and application*, Math. Commun., **17** (2012), no. 2, 497–509.

[4] S. A. Al-Mezel, H. H. Alsulami, E. Karapinar, and A.-F. Roldán-López-de Hierro, *Discussion on "multidimensional coincidence points" via recent publications*, Abstr. Appl. Anal., **2014** (2014), 13 pages, Art. ID 287492.

[5] I. Altun and Ö. Acar, *Fixed point theorems for weak contractions in the sense of Berinde on partial metric spaces*, Topology Appl., **159** (2012), no. 10–11, 2642–2648.

[6] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., **18** (1976), no. 4, 620–709.

[7] A. Amini-Harandi, *Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem*, Math. Comput. Model., **57** (2013), no. 9, 2343–2348.

[8] A. Amini-Harandi and H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., **72** (2010), no. 5, 2238–2242.

[9] M. S. Asgari and B. Mousavi, *Coupled fixed point theorems with respect to binary relations in metric spaces*, J. Nonlinear Sci. Appl., **8** (2015), 153–162.

[10] M. S. Asgari and B. Mousavi, *Solving a class of nonlinear matrix equations via the coupled fixed point theorem*, Appl. Math. Comput., **259** (2015), 364–373.

[11] H. Aydi, *Some fixed point results in ordered partial metric spaces*, J. Nonlinear Sci. Appl., **4** (2011), no. 3, 210–217.

[12] H. Aydi and M. Abbas, *Tripled coincidence and fixed point results in partial metric spaces*, Appl. Gen. Topol., **13** (2012), no. 2, 193–206.

[13] H. Aydi, E. Karapinar, and M. Postolache, *Tripled coincidence point theorems for weak ϕ-contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2012** (2012), no. 1, 44 pages.

[14] H. Aydi, E. Karapinar, and W. A. Shatanawi, *Tripled fixed point results in generalized metric spaces*, J. Appl.Math., **2012:101** (2012), 22 pages.

[15] H. Aydi, M. Postolache, and W. Shatanawi, *Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl, **63** (2012), no. 1, 298–309.

[16] G. V. R. Babu, M. L. Sandhya, and M. V. R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. (2008), 8–12.

[17] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math. (1922), no. 3, 133–181.

[18] I. Beg and A. R. Butt, *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal., **71** (2009), no. 9, 3699–3704.

[19] H. Ben-El-Mechaiekh, *The Ran–Reurings fixed point theorem without partial order: A simple proof*, J. Fixed Point Theory Appl., **16** (2014), no. 1-2, 373–383.

[20] V. Berinde, *Iterative approximation of fixed points*, Efemeride, 2002.

[21] V. Berinde, *On the stability of some fixed point procedures. Dedicated to Costică Mustaţă on his 60th anniversary*, Bul. Ştiinţ. Univ. Baia Mare, Fasc. Mat.-Inf., **XVIII** (2002), no. 1, 7–14.

[22] V. Berinde, *Approximating fixed points of weak ϕ-contractions using the Picard iteration*, Fixed Point Theory, **4** (2003), no. 2, 131–142.

[23] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), no. 1, 7–22.

[24] V. Berinde, *Iterative approximation of fixed points*, second ed., Springer, 2007.

[25] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), no. 18, 7347–7355.

[26] V. Berinde, *Coupled coincidence point theorems for mixed monotone nonlinear operators*, Comput. Math. Appl., **64** (2012), no. 6, 1770–1777.

[27] V. Berinde, *Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **75** (2012), no. 6, 3218–3229 (English).

[28] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), no. 15, 4889–4897.

[29] V. Berinde and M. Păcurar, *A note on the paper "Remarks on fixed point theorems of Berinde"*, Nonlinear Anal. Forum, vol. 14, 2009, pp. 119–124.

[30] V. Berinde and M. Păcurar, *A constructive approach to coupled fixed point theorems in metric spaces*, Carpathian J. Math., **31** (2015), no. 3, 277–287.

[31] S. R. Bernfeld and V. Lakshmikantham, *An introduction to nonlinear boundary value problems*, Academic Press, Inc. [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, vol. 109, Elsevier, 1974.

[32] M. Berzig, *Solving a class of matrix equations via the Bhaskar–Lakshmikantham coupled fixed point theorem*, Appl. Math. Lett., **25** (2012), no. 11, 1638–1643.

[33] M. Berzig, X Duan, and B. Samet, *Positive definite solution of the matrix equation $X = Q - A^*X^{-1}A + B^*X^{-1}B$ via Bhaskar-Lakshmikantham fixed point theorem*, Math. Sci., **6:27** (2012), no. 1, 6 pages.

[34] M. Berzig and B. Samet, *Solving systems of nonlinear matrix equations involving Lipshitzian mappings*, Fixed Point Theory Appl., **2011** (2011), no. 2011:89, 10 pages.

[35] M. Berzig and B. Samet, *An extension of coupled fixed point's concept in higher dimension and applications*, Comput. Math. Appl., **63** (2012), no. 8, 1319–1334.

[36] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), no. 7, 1379–1393.

[37] M. Borcut, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), no. 14, 7339–7346.

[38] M. Borcut, *Tripled coincidence theorems for monotone mappings in partially ordered metric spaces*, Creative Math. Inform., **21** (2012), no. 2, 135–142.

[39] M. Borcut, *Tripled fixed point theorems for monotone mappings in partially ordered metric spaces*, Carpathian J. Math., **28** (2012), no. 2, 215–222.

[40] M. Borcut, *Tripled fixed points for operators in partially ordered metric spaces*, Ph.D. thesis, Technical University of Cluj-Napoca, North University Center at Baia Mare, 2012.

[41] M. Borcut, *Puncte triple fixe pentru operatori definiţi pe spaţii metrice parţial ordonate*, Risoprint, Cluj-Napoca, 2016.

[42] M. Borcut, M. Păcurar, and V. Berinde, *Tripled fixed point theorems for mixed monotone Kannan type contractive mappings*, J. Appl. Math. (2014), 8 pages, Art. ID 120203,. MR 3182362

[43] M. F. Bota, A. Petruşel, G. Petruşel, and B. Samet, *Coupled fixed point theorems for single-valued operators in b-metric spaces*, Fixed Point Theory Appl., **2015:231** (2015), no. 1, 15 pages.

[44] N. Bourbaki, *Sur le théorème de Zorn*, Arch. Math. (Basel), **2** (1949–1950), 434–437.

[45] T. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc., **124** (1996), no. 8, 2383–2390.

[46] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215** (1976), 241–251.

[47] S. Chandok, Z. Mustafa, and M. Postolache, *Coupled common fixed point results for mixed g-monotone mapps in partially ordered G-metric spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A, Appl. Math. Phys, **75** (2013), no. 4, 13–26.

[48] S .K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci., **25** (1972), 727–730.

[49] B. S. Choudhury, E. Karapinar, and A. Kundu, *Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces*, Int. J. Math. Math. Sci., **2012** (2012), 14 pages, Art. ID 329298.

[50] B. S. Choudhury, N. Metiya, and M. Postolache, *A generalized weak contraction principle with applications to coupled coincidence point problems*, Fixed Point Theory Appl., **2013** (2013), 21 pages.

[51] B.S. Choudhury and A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal., **73** (2010), no. 8, 2524–2531.

[52] L. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2008** (2009), 11 pages.

[53] L. Ćirić and V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stochastic Analysis and Applications, **27** (2009), no. 6, 1246–1259.

[54] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.

[55] M. I. Dobrican, *Fixed point theorems in ordered metric spaces*, SYNASC 2015-IAFP Workshop, 2015.

[56] M. I. Dobrican, *Coupled fixed point theorems in metric spaces endowed with a reflexive relation*, SYNASC 2016 - IAFP Workshop, 2016.

[57] M. I. Dobrican, *Coupled and tripled fixed point theorems on a metric space endowed with a binary relation*, Miskolc Math. Notes, **18** (2017), no. 1, 189–198.

[58] M. I. Dobrican, *Coupled coincidence point theorems in a metric space endowed with a reflexive relation*, An. Univ. Vest Timiş. Ser. Mat.-Inform. (2017), accepted.

[59] M. I. Dobrican, *Fixed point theorems for a pair of single-valued operators in a metric space endowed with a reflexive relation*, Carpathian J. Math., **33** (2017), no. 3, 303–312.

[60] M. I. Dobrican, *Tripled coincidence point theorems on a metric space endowed with a binary relation*, (2017), submitted.

[61] M. I. Dobrican, *Tripled fixed point theorems for mixed $g - R-$monotone operators in metric spaces endowed with a reflexive relation*, Creat. Math. Inform., **27** (2018), no. 1, 21–30.

[62] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., **12** (1961), 7–10.

[63] Dakjum Eshi, Pramod K Das, and Pradip Debnath, *Coupled coincidence and coupled common fixed point theorems on a metric space with a graph*, Fixed Point Theory Appl., **2016:37** (2016), 14 pages.

[64] K. Fan, *Extensions of two fixed point theorems of FE Browder*, Math. Z., **112** (1969), no. 3, 234–240.

[65] A.-D. Filip and A. Petruşel, *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory Appl., **2010** (2010), no. 1, 15 pages, Art. ID 281381.

[66] D. Gao, *A Fixed Point Theorem for Monotone Maps and Its Applications to Nonlinear Matrix Equations*, J. Math., **2015** (2015), 6 pages, Art. ID 167049.

[67] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, no. 28, Cambridge University Press, 1990.

[68] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics., Springer Science & Business Media, 2003.

[69] D. Guo, Y.l J. Cho, and J. Zhu, *Partial ordering methods in nonlinear problems*, Nova Publishers, 2004.

[70] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11** (1987), no. 5, 623–632.

[71] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, vol. 5, Academic Press, 2014.

[72] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201–206.

[73] J. Harjani and K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., **71** (2009), no. 7-8, 3403–3410.

[74] N. Hooda and M. Savitri, *On tripled fixed point theorem in partially ordered complete metric space*, Int. J. Pure Appl. Sci. Technol., **20** (2014), no. 1, 111–116.

[75] G. Jungck and B. E. Rhoades, *Some fixed point theorems for compatible maps*, Internat. J. Math. Math. Sci., **16** (1993), no. 3, 417–428.

[76] Z. Kadelburg, H. K. Nashine, and S. Radenovic, *Common coupled fixed point results in partially ordered G-metric spaces*, Bull. Math. Anal. Appl, **4** (2012), no. 2, 51–63.

[77] Z. Kadelburg, M. Pavlović, and S. Radenović, *Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Comput. Math. Appl., **59** (2010), no. 9, 3148–3159.

[78] Z. Kadelburg and S. Radenović, *Remarks on some recent M. Borcut's results in partially ordered metric spaces*, Int. J. Nonlinear Anal. Appl., **6** (2015), no. 1, 96–104.

[79] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **60** (1968), 71–76.

[80] E. Karapinar, *Quartet fixed point for nonlinear contraction*, J. Fixed Point Theory Appl., **6** (2011), no. 2, 119–135.

[81] E. Karapinar, *Remarks on coupled fixed point theorems in partially ordered metric spaces*, Bull. Math. Anal. Appl., **4** (2012), no. 3, 115–128.

[82] E. Karapinar, *Tripled fixed point theorems in partially ordered metric spaces*, Stud. Univ. Babeş-Bolyai Math., **58** (2013), no. 1, 75–85.

[83] E. Karapinar, W. Shatanawi, and Z. Mustafa, *Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces*, J. Appl. Math., **2012** (2012), 17 pages, Art. ID 951912.

[84] E. Karapinar and N. Van Luong, *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl., **64** (2012), no. 6, 1839–1848.

[85] S. Kasahara, *On some generalizations of the Banach contraction theorem*, Publ. Res. Inst. Math. Sci., **12** (1976/1977), no. 2, 427–437.

[86] W. Kirk and B. Sims, *Handbook of metric fixed point theory*, Springer Science & Business Media, 2013.

[87] M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Cambridge University Press, New York, 1964.

[88] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), no. 12, 4341–4349.

[89] H. Lee and S. Kim, *Multivariate coupled fixed point theorems on ordered partial metric spaces*, J. Korean Math. Soc, **51** (2014), no. 6, 1189–1207.

[90] J. H. Long, X.Y. Hu, and L. Zhang, *On the Hermitian positive defnite solution of the nonlinear matrix equation* $X + A^*X^{-1}A + B^*X^{-1}B = I$, Bull. Braz. Math. Soc. (N.S.), **39** (2008), no. 3, 371–386.

[91] M. G. Maia, *Un'osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Univ. Padova, **40** (1968), no. 1, 139–143.

[92] R. Miculescu and A. Mihail, *A generalization of Matkowski's fixed point theorem and Istrăţescu's fixed point theorem concerning convex contractions*, J. Fixed Point Theory Appl., **19** (2017), no. 2, 1525–1533.

[93] Z. Mustafa, J. R. Roshan, and V. Parvaneh, *Existence of a tripled coincidence point in ordered G b-metric spaces and applications to a system of integral equations*, J. Inequal. Appl., **2013:453** (2013), no. 1, 27 pages.

[94] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135** (2007), no. 8, 2505–2517. MR 2302571

[95] J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), no. 3, 223–239.

[96] J. J. Nieto and R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.), **23** (2007), no. 12, 2205–2212.

[97] V. I. Opoĭcev, *Dynamics of collective behavior. I. Heterogenic systems*, Automat. Remote Control, **35** (1974), no. 4, 656–666.

[98] V. I. Opoĭcev, *Dynamics of collective behaviour. II. Systems with limited interelement interaction*, Avtomat. i Telemekh, **35** (1974), no. 6, 987–998.

[99] V. I. Opoĭcev, *Heterogeneous and combined concave operators*, Sibirsk. Mat. Ž., **16** (1975), no. 4, 781–792.

[100] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., **341** (2008), no. 2, 1241–1252.

[101] M. Păcurar, *Iterative methods for fixed point approximation*, Risoprint, 2009.

[102] M. Paknazar, M. E. Gordji, M. De La Sen, and S. M. Vaezpour, *N-fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2013:111** (2013), 15 pages.

[103] V. Parvaneh, J. R. Roshan, and S. Radenović, *Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations*, Fixed Point Theory Appl., **2013:130** (2013), 19 pages.

[104] D. Pascali and S. Sburlan, *Nolinear Mappings of Monotone Type*, Martinus Nijhoff Publishers, The Hague; Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, 1978.

[105] A. Petruşel, G. Petruşel, B. Samet, and J. C. Yao, *Coupled fixed point theorems for symmetric multi-valued contractions in b-metric space with applications to systems of integral inclusions*, J. Nonlinear Convex Anal., **17** (2016), no. 7, 1265–1282, Special Issue: SI.

[106] A. Petruşel, G. Petruşel, and J. C. Yao, *A study of a system of operator inclusions via a fixed point approach and applications to functional-differential inclusions*, Carpathian J. Math., **32** (2016), no. 3, 349–361, Special Issue: SI.

[107] A. Petruşel, G. Petruşel, B. Samet, and J. C. Yao, *A study of the coupled fixed point problem for operators satisfying a max-symmetric condition in b-metric spaces with applications to a boundary value problem*, Miskolc Math. Notes, **17** (2016), no. 1, 501–516.

[108] A. Petruşel, G. Petruşel, and C. Urs, *Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators*, Fixed Point Theory Appl., **2013:218** (2013), 21 pages.

[109] A. Petruşel and I. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc, **134** (2006), no. 2, 411–418.

[110] M. Păcurar, V. Berinde, M. Borcut, and M. Petric, *Triple fixed point theorems for mixed monotone Prešic-Kannan and Prešic-Chatterjea mappings in partially ordered metric spaces*, J. Comput. Anal. Appl. (2015), 793–802.

[111] S. Radenović, *Bhaskar-Lakshmikantham type results for monotone mappings in partially ordered metric spaces*, Int. J.Nonlinear Anal. Appl., **5** (2014), no. 2, 37–49.

[112] S. Radenović, *Coupled fixed point theorems for monotone mappings in partially ordered metric spaces*, Kragujevac J. Math., **38** (2014), no. 2, 249–257.

[113] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), no. 5, 1435–1443. MR 2053350

[114] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull, **14** (1971), no. 1, 121–124.

[115] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math., **21** (1990), no. 1, 1–9.

[116] B.E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures, II*, Indian J. Pure Appl. Math., **24** (1993), no. 11, 691–703.

[117] A. Roldán, J. Martínez-Moreno, and C. Roldán, *Multidimensional fixed point theorems in partially ordered complete metric spaces*, J. Math. Anal. Appl., **396** (2012), no. 2, 536–545.

[118] A. Roldán, J. Martínez-Moreno, and C. Roldán, *Tripled fixed point theorem in fuzzy metric spaces and applications*, Fixed Point Theory Appl., **2013:29** (2013), no. 1, 13 pages.

[119] I. A. Rus, *Some fixed point theorems in metric spaces*, Rend. Ist. Mat. Univ. Trieste, **3** (1971), 169–172.

[120] I. A. Rus, *Metrical fixed point theorems*, University of Cluj-Napoca, 1979.

[121] I. A. Rus, *Principles and applications of the fixed point theory*, Editura Dacia, Cluj-Napoca, 1979.

[122] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, 2001.

[123] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed point theory 1950-2000: Romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[124] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed point theory*, Cluj University Press Cluj-Napoca,, 2008.

[125] M. D. Rus, *The fixed point problem for systems of coordinate-wise uniformly monotone operators and applications*, Mediterr. J. Math., **11** (2014), no. 1, 109–122.

[126] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal., **13** (2012), no. 2, 82–97.

[127] B. Samet and C. Vetro, *Coupled fixed point, F-invariant set and fixed point of N-order*, Ann. Funct. Anal., **1** (2010), no. 2, 46–56.

[128] Bessem Samet, Calogero Vetro, and Pasquale Vetro, *Fixed point theorems for α–ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), no. 4, 2154–2165.

[129] M. Sangurlu, A. Ansari, and D. Turkoglu, *Coupled fixed point theorems for mixed G-monotone mappings in partially ordered metric spaces via new functions*, Gazi Univ. J. Sci., **29** (2016), no. 1, 149–158.

[130] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math., **2** (1930), 171–180.

[131] N. Shahzad, E. Karapinar, and A.F. Roldán-López-de Hierro, *On some fixed point theorems under $(\alpha,\ \psi,\ \phi)$ -contractivity conditions in metric spaces endowed with transitive binary relations*, Fixed Point Theory Appl., **2015:124** (2015), 24 pages.

[132] S. Shakeri, L. Ćirić, and R. Saadati, *Common fixed point theorem in partially ordered-fuzzy metric spaces*, Fixed Point Theory Appl., **2010** (2010), 13 pages, Art. ID 125082.

[133] W. Shatanawi, *Coupled fixed point theorems in generalized metric spaces*, Hacet. J. Math. Stat., **40** (2011), no. 3, 441–447.

[134] W. Shatanawi, B. Samet, and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Model., **55** (2012), no. 3, 680–687.

[135] W. Sintunavarat, P. Kumam, and Y. J. Cho, *Coupled fixed point theorems for nonlinear contractions without mixed monotone property*, Fixed Point Theory Appl., **2012:170** (2012), 16 pages.

[136] W. Sintunavarat, S. Plubtieng, and P. Katchang, *Fixed point result and applications on a b-metric space endowed with an arbitrary binary relation.*, Fixed Point Theory Appl., **2013:296** (2013), 13 pages.

[137] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math., **5** (1955), no. 2, 285–309.

[138] A. Tarski and S. Givant, *A formalization of set theory without variables*, vol. 41, American Mathematical Society Colloquium Publications, 1987.

[139] J. Tiammee and S. Suantai, *Coincidence point theorems for graph-preserving multi-valued mappings*, Fixed Point Theory Appl., **2014:70** (2014), 11 pages.

[140] J. Tiammee and S. Suantai, *Coincidence point theorems for multi-valued mappings of reich-type on metric spaces endowed with a graph*, J. Nonlinear Convex Anal., **16** (2015), no. 2, 365–373.

[141] I. Timiş, *Stability of tripled fixed point iteration procedures for mixed monotone mappings*, Carpathian Math. Publ., **6** (2014), no. 2, 377–388.

[142] A. Tychonoff, *Ein fixpunktsatz*, Math. Ann., **111** (1935), no. 1, 767–776.

[143] C. Urs, *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Math. Notes, **14** (2013), no. 1, 323–333.

[144] C. Urs, *Coupled fixed point theorems for mixed monotone operators and applications*, Stud. Univ. Babeş-Bolyai Math., **59** (2014), no. 1, 113–122.

[145] C. Urs, *Fixed point theorems for a system of operator equations with applications*, Stud. Univ. Babeş-Bolyai Math., **61** (2016), no. 1, 87–94.

[146] N. Van Luong and N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., **74** (2011), no. 3, 983–992.

[147] C. Vladimirescu and C. Avramescu, *Applications of the Fixed Point Method to Ordinary Differential and Integral Equations on Noncompact Intervals*, Universitaria Press, 2006.

[148] J. Yin, *Tripled coincidence points for mixed comparable mappings in partially ordered metric spaces*, J. Comput. Anal. Appl., **16** (2014), no. 2, 329–337.

[149] J. Yin, Q. Leng, H. Zhu, and S. Li, *Some coincidence point theorems for g-monotone increasing multi-valued mappings in cone metric spaces over Banach algebras*, J. Nonlinear Sci. Appl., **9** (2016), no. 5, 3036–3047.

[150] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), no. 1, 292–298.