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Fixed Point Theorems for Almost Local Contractions with Applications

Student doctorand: **Mónika ZÁKÁNY**

Conducător ştiinţific: **Prof. Univ. Dr. Vasile BERINDE**

Comisia de evaluare a tezei de doctorat:

Preşedinte: Prof. univ. dr. Ioan RAŞA - Universitatea Tehnică din Cluj-Napoca;

Conducător ştiinţific: Prof. univ. dr. Vasile BERINDE - Universitatea Tehnică din Cluj-Napoca, Centrul Universitar Nord din Baia Mare;

Referenţi:

- Prof. univ. dr. Radu MICULESCU - Universitatea Transilvania din Braşov,
- Prof. univ. dr. Dorel MIHEŢ - Universitatea de Vest, Timişoara;
- Conf. dr. Dan BĂRBOSU - Universitatea Tehnică din Cluj-Napoca, Centrul Universitar Nord din Baia Mare.

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Contents

INTRODUCTION

1. Theoretical/conceptual framework: contractions, Picard operators and almost contractions

In the beginning, we recall some basic concepts in fixed point theory.

Definition 0.1.1. [**24**] *Let X be a nonempty set. The functional $d : X \times X \to \mathbb{R}_+$ is said to be a metric on X if the following conditions hold:*

(1) $d(x, y) = 0$ if $x = y$;
(2) $d(x, y) = d(y, x), \forall x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$.

The set X equipped with the metric d is called a metric space and is denoted by (X, d).

Definition 0.1.2. [**24**] *Let X be a nonempty set. The element $x \in X$ is called a fixed point of the mapping $T : X \to X$ if $T(x) = x$.*

Denote by $Fix(T)$ the set of all fixed points of the mapping T. Use the notation Tx instead of $T(x)$.

The Banach's Contraction Principle represents probably the most important tool in nonlinear analysis. In the setting of what we call now a Banach space and later extended to metric spaces by Cacciopolli [**8**], it was first established by Stefan Banach (1922) [**9**]. In the following, we present the form of that theorem given by Berinde in [**19**]:

Theorem 0.1.3. [**19**] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq a \cdot d(x, y), \forall x, y \in X, \tag{0.1}$$

where $0 \leq a < 1$ is constant. Then:

(i) T has a unique fixed point p in X;
(ii) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots \tag{0.2}$$

converges to p, for any $x_0 \in X$;

(iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, p) \leq \frac{a^n}{1-a} d(x_0, x_1), n = 0, 1, 2, ... \tag{0.3}$$

$$d(x_n, p) \leq \frac{a}{1-a} d(x_{n-1}, x_n), n = 1, 2, ... \tag{0.4}$$

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, p) \leq a \cdot d(x_{n-1}, p), n = 1, 2, ... \tag{0.5}$$

A mapping satisfying (i) and (ii) is said to be a *Picard operator*, see [**98**], [**101**],[**102**].

A mapping satisfying (0.1) is usually called a *strict contraction* or an *a-contraction*. Hence, in essence, Theorem 0.1.3 shows that any contraction is a Picard operator.

A mapping satisfying condition (0.1) is always continuous. This fact lead researchers to look up for discontinuous classes of such kind of mappings for which conclusions of Theorem 0.1.3 still hold.

In 1968, R. Kannan [**39**] found a positive answer to this problem by proving a fixed point theorem for mappings that do not need to be continuous, by replacing condition (0.1) by the following one: there exists $0 \leq b < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq b \cdot [d(x, Tx) + d(y, Ty)], \forall x, y \in X. \tag{0.6}$$

2. Almost contractions

Definition 0.2.4. [**39**] *Let (X, d) be a complete metric space. A mapping $T : X \to X$ satisfying condition: there exists $0 \leq b < 1/2$ such that condition (0.6) holds, is called a Kannan mapping.*

Definition 0.2.5. [**40**] *Let (X, d) be a complete metric space. Any mapping $T : X \to X$ satisfying the contractive condition: there exists $0 \leq c < \dfrac{1}{2}$ such that*

$$d(Tx, Ty) \leq c \cdot [d(x, Ty) + d(y, Tx)], \forall x, y \in X,$$

is called Chatterjea contraction.

One of the most general contractive conditions related to Banach contractions has been introduced by Ćirić:

Definition 0.2.6. [**45**] *Let (X, d) be a complete metric space. The mapping $T : X \to X$ satisfying the contractive condition: there exists $0 \leq h < 1$ such that*

$$d(Tx, Ty) \leq h \cdot max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \forall x, y \in X \tag{0.7}$$

is called a quasi-contraction.

DEFINITION 0.2.7. [**19**] *Let (X,d) be a metric space. A mapping $T : X \to X$ is called weak (almost) contraction or (δ, L)-weak contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that*

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(y,Tx), \forall x,y \in X. \tag{0.8}$$

REMARK 0.2.8. *In* [**19**] *the author used the terminology of "weak contraction", then it was renamed as "almost contraction" beginning with* [**25**]. *In the present thesis we shall use the term "almost contraction".*
Note that any strict contraction, any Kannan mapping and any Chatterjea mapping are almost contractions with a unique fixed point.

Other examples of almost contractions are given in [**15**], [**16**], [**24**] and [**29**]. There are various other examples of contractive conditions which imply the almost contractiveness condition (0.8), see for example Taskovic [**99**] and Rus [**100**].

In the following, we present an existence result (Theorem 0.2.9), as well as an existence and uniqueness result (Theorem 0.2.10) for almost contractions, as they were established in [**15**], see also [**24**] and [**29**]. Their main merit is that they extend Banach's Contraction Principle [**9**], Kannan [**39**], Chatterjea [**40**], Zamfirescu's fixed point theorem [**127**] and many other related results and also provide a method for approximating the fixed points.

THEOREM 0.2.9. [**19**] *Let (X,d) be a complete metric space and $T : X \to X$ be a (δ, L)-almost contraction. Then*

(1) $Fix(T) = \{x \in X : \quad Tx = x\} \neq \phi$;
(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$, given by $x_{n+1} = Tx_n$, converges to some $x^* \in Fix(T)$;
(3) The following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0,1,2,... \tag{0.9}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1,2,... \tag{0.10}$$

hold, where δ is the constant appearing in (0.8).

THEOREM 0.2.10. [**19**] *Let (X,d) be a complete metric space and $T : X \to X$ be a (δ, L)-almost contraction for which there exist $\delta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L_1 \cdot d(x,Tx), \forall x,y \in X. \tag{0.11}$$

Then

(1) T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;
(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to x^*;

(3) The a priori and a posteriori error estimates

$$d(x_n, x^*) \le \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, ...$$

$$d(x_n, x^*) \le \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, ...$$

hold.

(4) The rate of convergence of the Picard iteration is given by

$$d(x_n, x^*) \le \delta \cdot d(x_{n-1}, x^*), \quad n = 1, 2, ... \tag{0.12}$$

REMARK 0.2.11. *(1) A large number of examples of almost contractions were given by Berinde in* [**19**] *and* [**24**]*. For example, it was proved that:*
- any Zamfirescu mapping [**127**] *is an almost contraction;*
- any quasi-contraction (see (0.7)*) with* $0 < h < \frac{1}{2}$ *is an almost contraction;*
(2) The fixed point x^* *of an almost contraction attained by the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *in Theorem 0.2.9 depends on the initial guess* x_0.

DEFINITION 0.2.12. [**100**], [**102**] *A map ping* $T : X \to X$ *is called a weakly Picard operator if the sequence* $\{T^n x_0\}_{n=0}^{\infty}$ *converges for any initial value* $x_0 \in X$*, and the limit of the sequence* $\{T^n x_0\}_{n=0}^{\infty}$ *is a fixed point of* T.

Therefore, the class of almost contractions provides a large class of weakly Picard operators. Condition (0.8) is related to another important concept in fixed point theory, that is, the so called *Banach orbital condition*

$$d(Tx, T^2x) \le a \cdot d(x, Tx), \forall x \in X, \quad 0 \le a < 1,$$

studied by various authors in the context of fixed point theorems, see for example Hicks and Rhoades [**61**], Ivanov [**64**], Rus [**98**] and Taskovic [**99**].

EXAMPLE 0.2.13. [**24**] *Let* $T : [0,1] \to [0,1]$ *be the identity mapping:* $Tx = x$*, for all* $x \in [0,1]$*. Then* T *is an almost contraction with* $\theta \in (0,1)$ *arbitrary and* $L \ge 1-\theta$ *and* $Fix(T) = [0,1]$.

An almost contraction is in general not continuous but, as shown by the next theorem, an almost contraction is continuous at any fixed point of it, see [**25**].

THEOREM 0.2.14. [**25**] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an almost contraction. Then* T *is continuous at* p*, for any* $p \in Fix(T)$.

From the various generalizations and extensions of almost contractions we mention the following one, due to Suzuki [**113**].

DEFINITION 0.2.15. [**113**] *Let T be a mapping on a metric space (X, d). Then T is called a generalized Berinde mapping if there exist a constant $r \in [0, 1)$ and a function $b : X \to [0, \infty)$ such that*

$$d(Tx, Ty) \leq r \cdot d(x, y) + b(y) \cdot d(y, Tx), \forall x, y \in X. \tag{0.13}$$

DEFINITION 0.2.16. [**45**] *Let (X, d) be a metric space. Any mapping $T : X \to X$ is called a Ćirić-Reich-Rus contraction if it satisfies condition:*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)], \forall x, y \in X, \tag{0.14}$$

where $\alpha, \beta \in \mathbb{R}_+$ and $\alpha + 2\beta < 1$.

3. Considerations about local contractions

The concept of local contraction was introduced by Rincón-Zapatero and Rodrigues-Palmero [**94**], in order to study discounted stochastic dynamic programming models with unbounded returns in the setting of metric spaces of functions with a metric constructed from a countable family of seminorms. Later on, the concept of local contraction has been developed by Martins da Rocha and Filipe Vailakis in [**76**]. They studied the existence and uniqueness of fixed points for local contractions in the setting of a semimetric space.

DEFINITION 0.3.17. [**129**], [**130**] *Let X be a nonempty set. The functional $d : X \times X \to \mathbb{R}_+$ is said to be a semimetric on X if the following conditions hold:*

(1) $d(x, y) = 0$ if $x = y$;
(2) $d(x, y) = d(y, x), \forall x, y \in X$.

Note that the triangle inequality is not necessarily satisfied in this case.

DEFINITION 0.3.18. [**76**] *Let F be a set. Denote by J a family of indices (which frequently can be considered as a subset of $\mathbb{N}$) and let $r : J \to J$. Let $\mathcal{D} = (d_j)_{j \in J}$ be a family of semimetrics defined on F. Let τ be the weak topology on F defined by the family $\mathcal{D}$. If A is a nonempty subset of F, then for each $h \in F$, we denote*

$$d_j(h, A) \equiv inf\{d_j(h, g) : g \in A\}, j \in J.$$

An operator $T : A \to A$ is called a local contraction with respect to $(\mathcal{D}, r)$ if, for every $j \in J$, there exists $\beta_j \in [0, 1)$ such that

$$d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g), \forall f, g \in A.$$

Following the contraction mapping principle, a lot of work was devoted to obtaining fixed point theorems in uniform spaces, amongst which we mention the work of

Gheorghiu [**53**], Gheorghiu and Rotaru [**54**]. In a recent paper, Vailakis and Martins-da-Rocha [**76**] have established new results involving local contractions in topological spaces whose topology is generated by a family of semidistances.

Instead of considering a self mapping $T : A \to A$, Vailakis and Martins-da-Rocha have considered non-self mappings $T : A \to F$, where $A \subset F$ is a nonempty, τ-bounded, sequentially τ-complete and T-invariant subset of F, and provided sufficient conditions to ensure the existence of a fixed point of T in A, as well as the uniqueness of the fixed point of T in A.

DEFINITION 0.3.19. [**76**] *The subset A is said to be τ-Hausdorff if for each pair $(f,g) \in A \times A$, $f \neq g$, there exists $j \in J$ such that $d_j(f,g) > 0$.*

The subset $A \subset F$ is called T-invariant if $T(A) \subset A$.

DEFINITION 0.3.20. [**76**] *Let F be a set and let J a family of indices. Let $\mathcal{D} = (d_j)_{j\in J}$ be a family of semimetrics defined on F. Let τ be the weak topology on F defined by the family $\mathcal{D}$.*

The sequence $\{x_n\}_{n\in\mathbb{N}^*}$ is τ-convergent to x^* if $d_j(x_n, x^*) \to 0$ as $n \to \infty$, $\forall j \in J$.

The sequence $\{x_n\}_{n\in\mathbb{N}^*}$ is said to be d_j-Cauchy if for each $j \in J$, $d_j(x_n, x_m) \to 0$ as $n, m \to \infty$.

The sequence $\{x_n\}_{n\in\mathbb{N}^*}$ is said to be τ-Cauchy if it is d_j-Cauchy, for all $j \in J$.

The subset A of F is said to be sequentially τ-complete if every τ-Cauchy sequence in F converges in F with respect to the τ-topology.

The subset $A \subset F$ is said to be τ-bounded if $diam_j(A) \equiv sup\{d_j(x,y) : x, y \in A\}$ is finite for every $j \in J$.

Now we state the main result in [**76**].

THEOREM 0.3.21. [**76**] *Assume that the space F is τ-Hausdorff and let $\mathcal{D} = (d_j)_{j\in J}$ be a family of semimetrics defined on F. Let τ be the weak topology on F defined by the family $\mathcal{D}$ and J be a family of indices Consider a function $r : J \to J$ and let $T : A \to A$ be a local contraction with respect to $(\mathcal{D}, r)$, with the coefficient of contraction $\beta_j \in [0,1), j \in J$. Denote $r^n(j) = r(r^{n-1}(j))$, for $n \geq 1$ and $r^0 = I_J$. Consider a nonempty, τ-bounded, sequentially τ-complete, and T-invariant subset $A \subset F$.*

E: *(existence): If condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} diam_{r^{n+1}(j)}(A) = 0 \tag{0.15}$$

is satisfied, then the operator T admits a fixed point f^*.

S: *(existence and uniqueness)*

Furthermore, if $h \in F$ satisfies

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0 \tag{0.16}$$

then the sequence $\{T^n h\}_{n\in\mathbb{N}}$ is τ-convergent to f^*.

The results obtained by Rincón-Zapatero and Rodrigues-Palmero [**95**] involving local contractions were successfully applied to the study of various applications in economics. Also, they considered metric spaces of functions with a metric constructed from a countable family of seminorms. That family is chosen as the supremum of continuous functions over the multitude of compact sets covering the set of consumption streams. The fixed-point results obtained by Martins da Rocha and Filipe Vailakis in [**76**] turned out to be also useful in solving recursive equations in economic dynamics. These results made easier the study of existence and uniqueness of solutions to recursive equations that arise in economic dynamics.

4. Motivation

The present thesis is intended to unify the two classes of contractive mappings that were presented previously and are important in fixed point theory:

1) the class of almost contractions, very extensively studied, presented in § 2;

2) the class of local contractions, with applications in economics, presented in §3.

Next, we summarize some facts that are at the source of our research work, i.e., the two types of contractions mentioned before.

The wide range of applications of Banach contraction mapping principle in nonlinear analysis has challenged researchers to obtain its conclusions under weaker assumptions than (0.1), which do not force the continuity of the operator T.

The first achievement in this respect has been stated by Kannan in 1968 [**39**], who obtained a fixed point theorem for discontinuous mappings. Chatterjea [**40**], Bianchini [**41**], Reich [**42**], Rus [**97**], Ćirić [**43**], Zamfirescu [**127**] and many other researchers continued this direction of research, see Rhoades [**92**] for a classification and comparison of various such contractive type mappings.

All the above quoted fixed point theorems ensure, based on specific assumptions, the following two conclusions for the contractive mapping $T : X \to X$

(1) $Fix(T) = \overline{x}$, i.e., T has a unique fixed point in X;
(2) if T^n stands for the n^{th} iterate of T, then $\lim_{n\to\infty} T^n x_0 = \overline{x}$, for any $x_0 \in X$, i.e., T is a Picard mapping.

More recently, Berinde [**19**] introduced a large class of contractive mappings, called weak contractions [**15**] and later named almost contractions (in [**25**] and afterwards).

The class of almost contractions includes Banach contractions, Kannan contractions, Chatterjea contractions, Zamfirescu contractions, Reich-Rus contractions, Bianchini contractions and, partially, the so called quasi-contractions, due to Ćirić [**44**]. But, unlikely the above mentioned classes of contractions, which admit a unique fixed point, an almost contraction may have two or more fixed points by simultaneously keeping almost all the other features of the Banach contraction mapping principle, including rate of convergence, error estimates, stability and so on.

On the other hand, Rincón-Zapatero and Rodrigues-Palmero [**94**], established another extension of the Banach contraction mapping principle, with various applications in economics: recursive utility is used in many contexts to model economic agents' preferences where current utility is expressed as a function (the aggregator) of current consumption and the utility of future consumption. Later on, Martins da Rocha and Filipe Vailakis in [**76**] have developed the idea of Rincón-Zapatero and Rodrigues-Palmero in [**95**].

The setting they are working is that of a set F endowed with a family $\mathcal{D} = (d_j)_{j\in J}$ of semidistances defined on F. They consider the weak topology τ defined by the family $\mathcal{D}$.

Let $A \subset F$ be a τ-bounded sequentially τ-complete and T-invariant subset of F. $T : A \to A$ is called a *local contraction* with respect to $(\mathcal{D}, r)$, where $r : J \to J$, if there exists $\beta_j \in [0, 1)$ such that

$$d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g), \forall f, g \in A.$$

The fixed point theorem of Martin da Rocha and Filipe Vailakis [**76**] essentially states that, if F is τ-Hausdorff, and,

$$\lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} diam_{r^{n+1}(j)}(A) = 0, \forall j \in J,$$

then T has a fixed point f^* in A. Moreover, if $h \in F$ satisfies

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} d_{r^{n+1}(j)}(h, A) = 0,$$

then the sequence $\{T^n h\}_{n\in\mathbb{N}}$ is τ-convergent to f^*.

The theoretical results in [**76**] were then applied to solve recursive equations in economic dynamics with various applications in dynamic programming. Both classes are extensions of the well known Banach contraction mapping principle, introduced by Stefan Banach in 1922 in his famous dissertation [**9**]. They represent the foundation of metrical fixed point theory, an extremely dynamic field of research starting with second half of the 20th century, see the monograph [**98**], for a selected list of reference books.

The two essentially different approaches presented in A and B, both emerging from Banach Contraction Mapping Principle, give rise to a very interesting and challenging problem: is it possible to unify almost contractions and local contractions to form a common class of contractive mappings that keep most of the features of the two sources?

The present thesis aims to answer this problem in the affirmative. We present a coherent theory of what we will call *local almost contractions* (often abbreviated as ALC-s), for which we state and prove various fixed point theorems, give illustrative examples, particular cases and indicate some relevant applications.

We plan to cover most of the classes of mappings studied in fixed point theory: single-valued self mappings, multivalued self- and non-self mappings, common fixed points and coincidence points of local almost contractions.

5. Structure of the thesis, main results

The thesis is organized into four chapters: Introduction, four chapters, Conclusions and a list of bibliographic references.

In **Introduction** we collect a non-exhaustive list of concepts, notions and basic results from the fixed point theory in metric and pseudometric spaces, respectively. The Introduction consists of six paragraphs, *Theoretical/ conceptual framework, Almost contractions, Considerations about local contractions, Motivation, Structure of the thesis, Acknowledgments.*

The first chapter, **Single valued self almost local contractions in uniform spaces**, consists of three paragraphs, *Preliminaries on almost local contractions, Data dependence of fixed points, New classes of almost local contractions.* This chapter contains the original contributions of the author, we present the concept of almost local contraction in pseudometric spaces, we prove existence as well as existence and uniqueness theorems for their fixed points. Further on, we present stability and data dependence of the fixed points. We introduce several classes of almost local contractions, we discuss about their approximate fixed points, almost local contractions in b-pseudometric spaces with a detailed study of their fixed points.

The personal contributions in the first chapter are:
Definitions 1.1.28, 1.1.31, 1.1.40, 1.2.44, 1.2.46, 1.3.54, 1.3.60, 1.3.70, 1.3.72, 1.3.75, 1.3.77, 1.3.79, 1.3.90, 1.3.101, 1.3.114, 1.3.135, 1.3.136, 1.3.138, 1.3.140, 1.3.142,
Theorems 1.1.33, 1.1.36, 1.1.38, 1.2.46, 1.2.48, 1.3.56, 1.3.65, 1.3.68, 1.3.69, 1.3.73, 1.3.74, 1.3.76, 1.3.78, 1.3.80, 1.3.95, 1.3.97, 1.3.99, 1.3.102, 1.3.122, 1.3.123, 1.3.125, 1.3.126, 1.3.128, 1.3.129, 1.3.130, 1.3.145, 1.3.146, 1.3.147, 1.3.148, 1.3.149, 1.3.150, 1.3.151, 1.3.152, 1.3.153, 1.3.162, 1.3.164, 1.3.165, 1.3.166, 1.3.167, 1.3.168, 1.3.169,
Corollaries 1.3.81, 1.3.82,
Lemmas 1.3.61, 1.3.63, 1.3.104, 1.3.115, 1.3.119, 1.3.121, 1.3.156, 1.3.158, 1.3.159, 1.3.161, 1.3.163,
Remarks 1.1.23, 1.1.32, 1.1.34, 1.1.37, 1.1.41, 1.1.42, 1.2.52, 1.3.55, 1.3.62, 1.3.64, 1.3.66, 1.3.86, 1.3.91, 1.3.93, 1.3.96, 1.3.100, 1.3.103, 1.3.108, 1.3.109, 1.3.111, 1.3.120, 1.3.124, 1.3.134, 1.3.137, 1.3.139, 1.3.141, 1.3.143, 1.3.157, 1.3.160,
Examples 1.1.24, 1.1.25, 1.1.35, 1.1.39, 1.2.49, 1.2.50, 1.2.51, 1.3.67, 1.3.71, 1.3.92, 1.3.94, 1.3.98, 1.3.127, 1.3.154.

In Chapter 2, **Multivalued almost local contractions**, we study multivalued self almost local contractions in pseudometric spaces, providing fixed point theorems

and approximate fixed points as well. Then, we investigate the non-self multivalued almost local contractions in pseudometric spaces. We state and prove two fixed point theorems related to the above-mentioned operators.
The personal contributions in Chapter 2 are:
Definitions 2.1.177, 2.1.181, 2.1.187, 2.2.193, 2.2.194,
Theorems 2.1.183, 2.1.184, 2.1.186, 2.1.189, 2.2.196, 2.2.199,
Lemmas 2.1.178, 2.1.182, 2.1.188,
Corollary 2.2.197,
Remarks 2.1.174, 2.2.195, 2.2.198.

In Chapter 3, **Non-self single valued almost local contractions**, we establish new fixed point theorems, both existence and uniqueness theorems. This chapter also contains the notion of α-graphic local contraction. Then we study various types of non-self almost local contractions, in order to establish if they are α-graphic local contractions.

The personal contributions in Chapter 3 are:
Definitions 3.1.204, 3.2.212, 3.2.216,
Theorems 3.2.208, 3.2.210, 3.2.213, 3.2.214, 3.2.217, 3.2.218, ,
Examples 3.1.205, 3.2.219,
Remarks 3.2.211, 3.2.215.

Chapter 4, **Application of almost local contractions in dynamic programming** focuses on the applicability of almost local contractions in economy, by using them in dynamic programming. First, we define the k-almost local contractions. Then, basic facts about Bellman operator and Bellman equation are introduced, having strong connections with the almost local contractions.
The personal contributions in the fourth chapter are:
Definition 4.2.229,
Propositions 4.2.230, 4.2.232,
Theorems 4.2.233, 4.2.243, 4.2.244, 4.2.245,
Remarks 4.1.223, 4.2.231, 4.2.234, 4.2.239, 4.2.246.

The thesis closes with a short part of **Conclusions**, where we resumed the original results from this thesis, also enumerating some future research possibilities and directions.

6. Acknowledgments

The main results of the thesis were presented to the mathematical community as follows:

My published papers:
- Zákány, M., *Fixed Point Theorems For Local Almost Contractions*, Miskolc Math. Notes, 18 (2017), No. 1, 499–506; ISI journal
- Zákány, M., *New classes of local almost contractions*, Acta Universitatis Sapientiae, Mathematica, 10 (2018), 378–394; ISI journal
- Zákány, M., *On the continuity of almost local contractions*, Creat. Math. Inform. 26 (2017), No. 2, 241–246.

Within the 17^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, Romania, 2015, I presented my research results "Fixed Point Theorems for Local Almost Contractions".

Within the 18^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, Romania, 2016, I presented my new research results "The Continuity of Almost Local Contractions".

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CHAPTER 1

SINGLE VALUED SELF ALMOST LOCAL CONTRACTIONS IN UNIFORM SPACES

1. Preliminaries on almost local contractions

The purpose of this chapter is to merge the concepts of almost contraction and local contraction, and thus to build a fixed point theory for what we will call *local almost contractions*.

Throughout this chapter, X is a Hausdorff topological space with its topology generated by a family $\{d_j\}_{j\in J}$ of pseudometrics on X.

DEFINITION 1.1.22. [**130**] *Let X be a nonempty set. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a pseudometric on X if:*

(1) $d(x,x) = 0, \forall x \in X$;
(2) $d(x,y) = d(y,x), \forall x, y \in X$;
(3) $d(x,y) \leq d(x,z) + d(z,y), \forall x, y, z \in X$.

REMARK 1.1.23. *Note that the axiom "$x = y \Leftrightarrow d(x,y) = 0$" from the case of a metric is no more assumed, which means that the distance between two different elements could be zero for a pseudometric, see Examples 1.1.24 and 1.1.25. Also, observe the difference between a pseudometric and a semimetric (Definitions 0.3.17 and 1.1.22).*

EXAMPLE 1.1.24. [**119**] *Let us condider $X = [0,n] \times [0,n] \subset \mathbb{R}^2, n \in \mathbb{N}^*$. Consider the family of pseudometrics $d_j : X \times X \to \mathbb{R}_+$,*

$$d_j\Big((x_1,y_1),(x_2,y_2)\Big) = |y_1 - y_2| \cdot e^{-j}, \forall j \in J, \forall (x_1,y_1),(x_2,y_2) \in X, \tag{1.17}$$

where J is a subset of $\mathbb{N}$ (or $\mathbb{R}_+$). Then d_j is a pseudometric but is not a metric, take for example: $d_j((1,4),(2,4)) = |4-4| \cdot e^{-j} = 0$, however $(1,4) \neq (2,4)$.

EXAMPLE 1.1.25. [**119**] *Let us consider $X = [0,n] \times [0,n] \subset \mathbb{R}^2, n \in \mathbb{N}^*$. Consider the pseudometric:*

$$d_j((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| \cdot e^{-j}, \forall j \in J, \forall (x_1,y_1),(x_2,y_2) \in X, \tag{1.18}$$

where J is a subset of $\mathbb{N}$. Similar to the previous example, d_j is a pseudometric, but is not a metric, take for example: $d_j((1,3),(1,5)) = |1-1| \cdot e^{-j} = 0$, however $(1,3) \neq (1,5)$.

A family of pseudometrics introduces on the set X a topology τ, named uniform space topology in some research papers (see [**128**]*).*

We recall some basic notions from the theory of uniform spaces, according to [**128**]:

DEFINITION 1.1.26. [**128**] *If X is a given set, then we call the subsets of $X \times X$ relations. The inverse relation of the relation U is*

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}.$$

The composition of two relations U and V is defined by:

$$U \circ V = \{(x, z) \in X \times X : (x, y) \in U \text{ and } (y, z) \in V, \forall y \in X\}.$$

Denote by $\Delta(X)$ the set of all ordered pairs $(x, x), \quad x \in A$.
Assign to every subset $A \subset X$ the set

$$U[A] = \{y : (x, y \in U) \text{ for some } x \in A\}.$$

Observe that $U[x] = U[\{x\}]$.
An uniformity (or uniform structure) on the set X is called a nonempty family U consisting of subsets $X \times X$ satisfying the following conditions:

(1) *every element $U \in \boldsymbol{U}$ contains $\Delta(X)$;*
(2) *if $U \in \boldsymbol{U}$ then $U^{-1} \in \boldsymbol{U}$;*
(3) *if $U \in \boldsymbol{U}$ then $V \circ V \subset U$ for some $V \in \mathbf{U}$;*
(4) *if U and $V \in \boldsymbol{U}$ then $U \cap V \in \boldsymbol{U}$;*
(5) *if $U \in \boldsymbol{U}$ and $U \subset V \subset X \times X$ then $V \in \boldsymbol{U}$.*

The ordered pair $(X, \boldsymbol{U})$ is called uniform space. All subsets $T \subset X$ such that for every $x \in T$ there exists $U \in \mathbf{U}$ for which $U[x] \subset T$, form a topology on X, called uniform topology.

The connection between the pseudometrics and the uniform spaces is given by the next result:

PROPOSITION 1.1.27. [**71**] *Every uniformity of X is generated by the family of all uniformly continuous pseudometrics on $X \times X$.*

Alternatively to Definition 1.1.26 stated by Weil, uniform spaces can be defined using systems of pseudometrics.

Bourbaki points out (see [**37**]) that any uniform structure $\mathbf{U}$ can be defined by a family of pseudometrics (even uncountable).

Definition 0.3.20 can be extended to the case of uniform spaces, as follows:

DEFINITION 1.1.28. *Let X be a uniform space. Denote by J a family of indices and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on X. Let τ be the weak topology on X defined by the family $\mathcal{D}$. A sequence $\{x_n\}_{n \in \mathbb{N}^*}$ is said to be $\tau - $Cauchy if it is*

d_j-Cauchy, $\forall j \in J$.
The subset A of X is said to be sequentially τ-complete if every τ-Cauchy sequence in A converges in A with respect to the τ-topology.
The subset $A \subset X$ is said to be τ-bounded if $diam_j(A) \equiv sup\{d_j(x,y) : x, y \in A\}$ is finite for every $j \in J$.

A more general global fixed point result has been obtained more than forty years before by Gheorghiu [**53**], see also Gheorghiu and Rotaru [**54**].

THEOREM 1.1.29. [**53**] *Let X be a uniform Hausdorff space, sequentially complete, and let $(d_i)_{i\in I}$ be a family of semimetrics defined on X. Let $f : X \to X$ be a mapping for which there exist $\varphi : I \to I, q : I \to \mathbb{R}_+$ such that*

$$d_i(f(x), f(y)) \leq q(i)d_{\varphi(i)}(x, y), \forall x, y \in X, \forall i \in I$$

and the series

$$\sum_{n=1}^{\infty} q(i)q(\varphi(i)) \cdots q(\varphi^n(i)) \cdot d_{\varphi^n(i)}(x, y)$$

is convergent for all $i \in I$ and for all $x, y \in X$.
Then f has a unique fixed point.

REMARK 1.1.30. *Note that if T satisfies condition* (0.15)*, then the series in the previous theorem is convergent. In* [**53**]*, Gheorghiu use the term "semimetric" for what we shall call "pseudometric" in the present thesis, according to the definition of Willard in* [**130**].

DEFINITION 1.1.31. [**118**] *Let X be a set and denote by J a family of indices. Let $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics defined on X and let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let the operator $T : A \to X$ and assume that the subset A is T-invariant. Let r be a function from J to J. T is called almost local contraction with regard to ($\mathcal{D}$,r) if there exist the constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + L \cdot d_{r(j)}(y, Tx), \quad \forall x, y \in A, \forall j \in J. \tag{1.19}$$

REMARK 1.1.32. *1) By taking a metric $d : X \times X \to \mathbb{R}_+$ instead of the pseudometrics d_j and r the identity function ($r(j) = j$), we can easily conclude that the almost contractions represent a particular case of almost local contractions.*
2) As the pseudometric possesses the property of symmetry, the almost local contraction condition (1.19) *implies the following dual condition:*

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + L \cdot d_{r(j)}(x, Ty), \quad \forall x, y \in A, \forall j \in J. \tag{1.20}$$

The next theorem states an existence result of the fixed points of almost local contractions in uniform spaces, and it appeared in [**118**].

THEOREM 1.1.33. [**118**] *Let X be a uniform Hausdorff space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J and let $T : A \to X$ be an almost local contraction with respect to ($\mathcal{D}$,r), with the constants $\theta \in (0,1)$ and $L \geq 0$. Assume the subset $A \subset X$ is T-invariant. If condition*

$$\lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = 0, \quad \forall j \in J \tag{1.21}$$

is satisfied, then T admits a fixed point x^* in A.

Proof: Let $x_0 \in A$ be arbitrary and let $\{\mathrm{x}_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}.$$

Take $x := x_{n-1}, y := x_n$ in (1.19) to obtain

$$d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n),$$

which means

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n), \forall j \in J, \forall n \geq 1. \tag{1.22}$$

By using (1.22), we inductively obtain :

$$d_j(x_n, x_{n+1}) \leq \theta^n \cdot d_{r^n(j)}(x_0, x_1), \quad n = 0, 1, 2, \cdots, \tag{1.23}$$

where $r^0(j) = j, \forall j \in J$. According to the triangle inequality, by (1.23) we get:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq \theta^n(1 + \theta + \cdots + \theta^{p-1}) d_{r^n(j)}(x_0, x_1) = \\ &= \frac{\theta^n}{1-\theta}(1 - \theta^p) \cdot d_{r^n(j)}(x_0, x_1) \leq \\ &\leq \frac{1}{1-\theta}\theta^n \cdot diam_{r^n(j)}(A), \quad n, p \in \mathbb{N}, p \neq 0. \end{aligned}$$

These relations show us that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$. As the subset A is assumed to be sequentially τ-complete, there exists x^* in A such that $\{x_n\}_{n\in\mathbb{N}}$ is τ-convergent to x^*. Besides, the sequence $\{x_n\}$ converges with respect to the topology τ to x^*, which implies

$$d_j(Tx^*, x^*) = \lim_{n\to\infty} d_j(Tx^*, T^{n+1}x_0), \quad \forall j \in J.$$

The operator T is an almost local contraction with respect to ($\mathcal{D}$,r), thus we have

$$\begin{aligned} d_j(Tx^*, x^*) &= \lim_{n\to\infty} d_j(Tx^*, T^{n+1}x_0) \leq \\ &\leq \theta \underbrace{\lim_{n\to\infty} d_{r(j)}(x^*, T^n x_0)}_{=0} + L \underbrace{\lim_{n\to\infty} d_{r(j)}(x^*, T^{n+1}x_0)}_{=0}, \forall j \in J. \end{aligned}$$

As the convergence with respect to the τ-topology implies convergence for the pseudometric $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$. This proves that $Tx^* = x^*$, as the space is Hausdorff. $\square$

Remark 1.1.34. *For T verifying* (1.19) *with $L = 0$, by taking $\theta = \beta_j$ (from Definition 0.3.18), we can easily conclude that an almost local contraction is a particular local contraction.*
Further, in the case $d_j = d, \forall j \in J$, with d = metric on X, we obtain the well known Banach contraction mapping principle, with a unique fixed point.

Example 1.1.35. *Let A be the set of all nonnegative functions:*

$$A = \{f | f : [0, \infty) \to [0, \infty)\}.$$

Let us consider $d_j(f, g) = |f(0) - g(0)| \cdot e^{-j}, \quad \forall f, g \in A, r(j) = j + 4$, $\forall j \in J$, where J is a subset of $\mathbb{N}$.
Observe that d_j is a pseudometric, but it is not a metric, take for example $d_j(x, x^2) = 0$, however $x \neq x^2$.
Consider the mapping $Tf = |f|, \quad \forall f \in A$, and apply condition (1.19) *for ALC-s:*

$$|f(0) - g(0)| \cdot e^{-j} \leq \theta \cdot |f(0) - g(0)| \cdot e^{-(j+4)} + L \cdot |g(0) - f(0)| \cdot e^{-(j+4)}, \forall f, g \in A,$$

which is equivalent to: $e^4 \leq \theta + L$.
This inequality is true if $j > 0, \quad \theta = \frac{1}{4} \in (0, 1), \quad L = 80 > 0$.
Hence, T is an ALC. However, T is not a contraction, because the contractive condition (0.1) *leads us to: $1 \leq a$, which is a contradiction.*
The condition (1.21) for the existence of the fixed point is valid:

$$\lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = \lim_{n\to\infty} \left(\frac{1}{4}\right)^{n+1} \cdot \sup_{f,g\in A} \left\{ \left| \Big(f(0) - g(0)\Big) e^{-j} \right| \right\} = 0, \quad \forall j \in J.$$

The mapping T has an infinite set of fixed points, $Fix(T) = A$, since

$$|f(x)| = f(x), \forall f \in A, x \in [0, \infty).$$

The next theorem presents an existence and uniqueness result for ALC-s.

Theorem 1.1.36. [**119**] *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.1.33. If, additionally:*
(U) there exist two constants $\theta_1 \in (0, 1)$ and $L_1 \geq 0$, $\theta_1 + L_1 < 1$, such that

$$d_j(Tx, Ty) \leq \theta_1 \cdot d_{r(j)}(x, y) + L_1 \cdot d_{r(j)}(y, Tx), \quad \forall x, y \in A, \forall j \in J \tag{1.24}$$

and

$$\lim_{n\to\infty} (\theta_1 + L_1)^n d_{r^n(j)}(z, A) = 0, \quad \forall z \in A, \forall j \in J, \tag{1.25}$$

then the fixed point x^* of T is unique.

Proof: Suppose, by contradiction, that there are two different fixed points x^* and y^* of T. Then we have:

$$\begin{aligned} 0 < d_j(x^*,y^*) &= d_j(Tx^*,Ty^*) \leq \theta_1 d_{r(j)}(x^*,y^*) + L_1 d_{r(j)}(y^*,Tx^*) = \\ &= (\theta_1+L_1)\cdot d_{r(j)}(x^*,y^*) \leq \cdots \leq (\theta_1+L_1)^n \cdot d_{r^n(j)}(x^*,y^*) \leq \\ &\leq \underbrace{(\theta_1+L_1)^n}_{\searrow 0 \text{ as } n\to\infty} \cdot d_{r^n(j)}(z,A), \quad \forall z\in A, \forall j \in J. \end{aligned}$$

Now, letting $n\to\infty$, we obtain a contradiction (0<0), hence the fixed point is unique.

□

Remark 1.1.37. *The proof of Theorem 1.1.36 is quite similar to that of Vailakis* [**76**] *from the local contractions. The uniqueness of the fixed point in the case of almost local contractions can also be proved without assuming the additional property (U) above, based only on the monotonicity property of the pseudometric and the uniqueness condition from the case of almost contractions, as shown in the next theorem.*

Theorem 1.1.38. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.1.33. If we add a monotonicity condition for the pseudometrics, namely:*

$$d_{r(j)}(f,g) \leq d_j(f,g), \forall f,g \in A, \forall j \in J, \tag{1.26}$$

and also the uniqueness condition (U'):

$$d_j(Tx,Ty) \leq \theta_u \cdot d_{r(j)}(x,y) + L_u \cdot d_{r(j)}(x,Tx), \forall x,y\in A, \forall j \in J, \tag{1.27}$$

with the constants $\theta_u \in (0,1)$ and $L_u \geq 0$, then the fixed point f^* of T is unique.

Proof: Suppose, by contradiction, there are two distinct fixed points f^* and g^* of T. Then, by using (1.26), and condition (1.27) with $f := f^*, g := g^*$, we get:

$$\begin{aligned} d_j(f^*,g^*) &\leq \theta_u \cdot d_{r(j)}(f^*,g^*) + L_u d_{r(j)}(f^*,Tf^*) = \theta_u \cdot d_{r(j)}(f^*,g^*) \leq \theta_u \cdot d_j(f^*,g^*) \\ &\Rightarrow d_j(f^*,g^*) \leq \theta_u \cdot d_j(f^*,g^*) \Rightarrow (1-\theta_u)\cdot d_j(f^*,g^*) \leq 0, \forall j \in J., \end{aligned}$$

which is obviously a contradiction with $d_j(f^*,g^*) > 0$ and $\theta_u \in (0,1)$. □

Example 1.1.39. *Let us consider $X = [0,n]\times[0,n] \subset \mathbb{R}^2, n\in\mathbb{N}^*, \quad T : X \to X,$*

$$T(x,y) = \begin{cases} (\frac{x}{2},\frac{y}{2}) & \textit{if (x,y)} \neq (1,0) \\ (0,0) & \textit{if (x,y)} = (1,0) \end{cases}$$

Consider the pseudometric:

$$d_j\Big((x_1,y_1),(x_2,y_2)\Big) = |x_1-x_2|\cdot e^j, \forall j\in J, \forall (x_1,y_1),(x_2,y_2) \in X, \tag{1.28}$$

where the family of indices J is a subset of $\mathbb{N}$. Then d_j is a pseudometric, but is not a metric, take for example: $d_j\Big((1,4),(1,6)\Big) = |1-1|\cdot e^j = 0$, however $(1,4)\neq(1,6)$.

In this case, consider the function $r(j) = j + 2$. *By applying the inequality* (1.19) *to the mapping* T, *we get for all* $x = (x_1, y_1), y = (x_2, y_2) \in X$

$$\Big|\frac{x_1}{2} - \frac{x_2}{2}\Big| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^{j+2} + L \cdot \Big|x_2 - \frac{x_1}{2}\Big| \cdot e^{j+2},$$

for all $j \in J$, *which can be write as the equivalent form*

$$|x_1 - x_2| \cdot e^{-2} \leq 2\theta \cdot |x_1 - x_2| + L \cdot |2x_2 - x_1|.$$

From that, we obtain

$$|x_1 - x_2| \cdot (e^{-2} - 2\theta) \leq L \cdot |2x_2 - x_1|, \forall x_1, x_2 \in [0, n].$$

The last inequality is true if we take $\theta = \frac{1}{2} \in (0,1), L = 10 \geq 0$, *since the left-hand side is negative and the right-hand side is positive. Hence* T *is an ALC, without having the monotonicity property. The uniqueness condition (1.27) becomes:*

$$\Big|\frac{x_1}{2} - \frac{x_2}{2}\Big| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^{j+2} + L \cdot \Big|x_1 - \frac{x_1}{2}\Big| \cdot e^{j+2},$$

for all $j \in J$. *From that, we obtain*

$$\Big|\frac{x_1}{2} - \frac{x_2}{2}\Big|(e^{-2} - 2\theta) \leq L \cdot \frac{x_1}{2}, \quad \forall x_1, x_2 \in [0, n].$$

This inequality is valid for $\theta = \frac{1}{5} \in (0,1)$ *and* $L = 5 \geq 0$.

The operator T *possesses the unique fixed point:* $Fix(T) = \{(0,0)\}$, *because*

$$T(x,y) = (x,y) \Leftrightarrow \Big(\frac{x}{2}, \frac{y}{2}\Big) = (x,y) \Leftrightarrow x = 0, y = 0.$$

Having in view the results in [**83**], we introduce, similarly to the case of strict almost contractions, a new type of almost local contractions, namely the strict almost local contractions.

Definition 1.1.40. *Assume that* $X, J, \mathcal{D}, r, \tau$ *and* A *are as in Theorem 1.1.38, with the monotonicity property fulfilled for the pseudometrics. An operator* $T : A \to A$ *is called strict almost local contraction if it satisfies both conditions* (1.19) *and* (1.27), *with some real constants* $\theta, \theta_u \in (0,1)$ *and* $L, L_u \geq 0$, *respectively.*

Remark 1.1.41. *The constants* $\theta, \theta_u \in (0,1)$ *and* $L, L_u \geq 0$ *are independent, according to Example 1.1.39.*

Remark 1.1.42. *The strict almost contractions have a remarkable importance: they always have a unique fixed point, see* [**83**].

2. Data dependence of fixed points

2.1. Stability and data dependence of the fixed points for ALC-s

The aim of this section is to present some contributions to the study of stability and data dependence of almost local contractions. The concept of stability of a fixed point iterative procedure, was introduced by Harder and Hicks [**59**]. The key idea was to approximate the sequence $\{x_n\}_{n=0}^{\infty}$ defined by a given iterative method with a more practical sequence $\{y_n\}_{n=0}^{\infty}$, whose limit will properly approximate the fixed point of the initial mapping.

Definition 1.2.43. [**59**] *Let (X,d) be a metric space, $T : X \to X$ be a mapping, $x_0 \in X$ and suppose that the sequence of successive approximations defined by*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N} \text{ with } x_0 \in X,$$

converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and denote:

$$\varepsilon_n = d(y_{n+1}, Ty_n), n = 0, 1, 2, ... \tag{1.29}$$

We say that the fixed point iteration procedure $\{x_n\}_{n=0}^{\infty}$ is T-stable or stable with regard to T if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} y_n = p. \tag{1.30}$$

Definition 1.2.43 will be extended to the case of uniform space, where we deal with pseudometrics, as follows:

Definition 1.2.44. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a mapping $T : X \to X$, $x_0 \in X$ and suppose that the sequence of successive approximations defined by*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N} \text{ with } x_0 \in X,$$

converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and denote:

$$\varepsilon_n^{(j)} = d_j(y_{n+1}, Ty_n), \quad n = 0, 1, 2, ..., j \in J. \tag{1.31}$$

We say that the fixed point iteration procedure $\{x_n\}_{n=0}^{\infty}$ is T-stable or stable with regard to T if

$$\lim_{n\to\infty} \varepsilon_n^{(j)} = 0 \Leftrightarrow \lim_{n\to\infty} y_n = p, \quad \forall j \in J. \tag{1.32}$$

The following lemma will be useful in proving the main results in this section.

LEMMA 1.2.45. [**14**] *Consider* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$ *two sequences of positive real numbers and* $q \in (0,1)$ *such that:*

(*i*) $a_{n+1} \leq q a_n + b_n, n \geq 0$;

(*ii*) $b_n \to 0$ *as* $n \to \infty$.

Then

$$\lim_{n\to\infty} a_n = 0.$$

Now, it is our aim to prove that the Picard iteration is T-stable with respect to almost local contractions.

THEOREM 1.2.46. *Let* X *be a uniform space,* $\mathcal{D} = (d_j)_{j\in J}$ *be a family of pseudometrics on* X*, where* J *is a set of indices. Let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$*. Consider a nonempty,* τ*-bounded, sequentially* τ*-complete subset* $A \subset X$*. Let* r *be a function from* J *to* J *and let* $T : A \to X$ *be an almost local contraction with respect to (*$\mathcal{D}$*,*r*), with the constants* $\theta \in (0,1)$ *and* $L \geq 0$*. Assume the subset* $A \subset X$ *is* T*-invariant and the monotonicity property is assured for the family of pseudometrics, namely:*

$$d_{r(j)}(x,y) \leq d_j(x,y), \forall x,y \in A, \forall j \in J. \tag{1.33}$$

If, for some constants $\theta_u \in (0,1)$ *and* $L_u \geq 0$*, the uniqueness condition:*

$$d_j(Tx,Ty) \leq \theta_u \cdot d_{r(j)}(x,y) + L_u \cdot d_{r(j)}(x,Tx), \forall x,y \in A, \forall j \in J \tag{1.34}$$

holds, then the fixed point p *of* T *is unique. Consider the Picard iteration*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N} \text{ with } x_0 \in A.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* p *and is* T*-stable.*

Proof: Let $\{y_n\}_{n\geq 0}$ be an arbitrary sequence in the subset A and denote

$$\varepsilon_n^{(j)} = d_j(y_{n+1}, Ty_n), \quad n = 0,1,2,...,$$

for every $j \in J$. We apply the triangle inequality and (1.34), then, by using the monotonicity of the pseudometrics, we obtain

$$\begin{aligned} d_j(y_{n+1},p) &\leq d_j(y_{n+1},Ty_n) + d_j(Ty_n,p) \leq \\ &\leq \varepsilon_n^{(j)} + \theta_u d_{r(j)}(y_n,p) + L_u \cdot \underbrace{d_{r(j)}(p,Tp)}_{=0} \leq \theta_u d_j(y_n,p) + \varepsilon_n^{(j)}. \end{aligned}$$

i) First, assume that $\lim\limits_{n\to\infty} \varepsilon_n^{(j)} = 0$, for every $j \in J$. Having in view that $\theta_u \in (0,1)$, by using Lemma 1.2.45, we can conclude that $\lim\limits_{n\to\infty} y_n = p$.
On the other hand, by using the uniqueness condition 1.34 and the monotonicity property, we can write

$$d_j(x_{n+1},p) = d_j(Tx_n,Tp) \leq \theta_u \underbrace{d_{r(j)}(x_n,p)}_{\leq d_j(x_n,p)} + L_u \underbrace{d_{r(j)}(p,Tp)}_{=0}, \forall j \in J.$$

It results that

$$d_j(x_{n+1}, p) \leq \theta_u \cdot d_j(x_n, p), \quad \forall j \in J,$$

which means: $\lim\limits_{n\to\infty} x_n = p$.
ii) Conversely, assume that $\lim\limits_{n\to\infty} y_n = p$. Then

$$\begin{aligned} \varepsilon_n^{(j)} &= d_j(y_{n+1}, Ty_n) \leq \\ &\leq d_j(y_{n+1}, p) + d_j(Tp, Ty_n) \leq \\ &\leq d_j(y_{n+1}, p) + \theta_u d_{r(j)}(p, y_n) + L_u \underbrace{d_{r(j)}(p, Tp)}_{=0} \to 0 \end{aligned}$$

as $n \to \infty$. □

REMARK 1.2.47. *In uniform spaces, Theorem 1.2.46 represent generalization of results established in* [**83**] *regarding the data dependence of of the fixed points for almost contractions in metric spaces.*

2.2. Continuity of almost local contractions

In this section we present an extension to almost local contractions of the results of Berinde and Păcurar [**25**] about the continuity of almost contractions at their fixed points. The main result is presented in Theorem 1.2.48, which gives the answer about the continuity of almost local contractions at their fixed points.

THEOREM 1.2.48. [**119**] *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J and let $T : A \to A$ be an almost local contraction with respect to ($\mathcal{D}$,r), satisfying condition* (1.21). *Then T admits a fixed point and moreover, T is continuous at f, for any $f \in Fix(T)$.*

Proof: The mapping T is an almost local contraction, i.e., there exist the constants $\theta \in (0, 1)$ and some $L \geq 0$ such that

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in A, \forall j \in J. \tag{1.35}$$

For any sequence $\{y_n\}_{n=0}^{\infty}$ in A converging to $f \in Fix(T)$, we take $y := y_n, x := f$ in (1.35), and we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_{r(j)}(f, y_n) + L \cdot d_{r(j)}(y_n, Tf), n = 0, 1, 2, ... \tag{1.36}$$

Using $Tf = f$, since f is a fixed point of T, we obtain:

$$d_j(Ty_n, Tf) \leq \theta \cdot d_{r(j)}(f, y_n) + L \cdot d_{r(j)}(y_n, f), n = 0, 1, 2, ... \tag{1.37}$$

Now, by letting $n \to \infty$ in (1.37), we get $Ty_n \to Tf$, which shows that T is continuous at f.
The fixed point has been chosen arbitrarily, so the proof is complete. □

EXAMPLE 1.2.49. [**119**] *Let us consider* $X = [0, n] \times [0, n] \subset \mathbb{R}^2, n \in \mathbb{N}^*$. *Consider the pseudometric:*

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot e^{-j}, \forall j \in J, \forall (x_1, y_1), (x_2, y_2) \in X, \tag{1.38}$$

where J *is a subset of* $\mathbb{N}$. *Then* d_j *is a pseudometric, but is not a metric, take for example:* $d_j((1, 4), (1, 3)) = |1 - 1| \cdot e^{-j} = 0$, *however* $(1, 4) \neq (1, 3)$.
In this case, consider the function $r(j) = j + 1$, *where* $j \in J$ *and* $T : X \to X$,

$$T(x, y) = \begin{cases} (x, \frac{y}{5}) & \text{if } (x,y) \neq (1, 1) \\ (0, 0) & \text{if } (x,y) = (1, 1) \end{cases}$$

T *is not a contraction because the contractive condition:*

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y), \tag{1.39}$$

is not valid $\forall x, y \in X$, *and for any* $\theta \in (0, 1)$. *The inequality* (1.39) *is equivalent to:*

$$| x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-j}, \quad \forall j \in J.$$

The last inequality leads us to $1 \leq \theta$, *which is obviously false, considering* $\theta \in (0, 1)$.
However, T *becomes an almost local contraction if:*

$$|x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-(j+1)} + L \cdot |x_2 - x_1| \cdot e^{-(j+1)},$$

which is equivalent to : $e \leq \theta + L$.
For $\theta = \frac{1}{3} \in (0, 1)$, $L = 3 \geq 0$ *and* $j > 0$, *the last inequality becomes true, i.e.,* T *is an ALC with an infinite set of fixed points:*

$$Fix(T) = \{(x, 0) : x \in [0, n]\} \subset X.$$

Observe that the monotonicity of the pseudometrics is fulfilled, because

$$d_{r(j)}(x_1, x_2) \leq d_j(x_1, x_2) \Leftrightarrow |x_1 - x_2| \cdot e^{-(j+1)} \leq |x_1 - x_2| \cdot e^{-j},$$

for each $j \in J$ *and for all* $x, y \in X$. *In this case, we have:*

$$\lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(X) = \lim_{n\to\infty} (n + 1)\sqrt{2} \cdot \left(\frac{1}{3}\right)^{n+1} = 0, \quad \forall j \in J.$$

The existence of the fixed point is assured, according to condition (1.21) *from Theorem 1.1.33.*
The operator T *is continuous at* $(0, 0) \in Fix(T)$ *but we have a discontinuity at* $(1, 1)$, *which is not a fixed point of* T. *Therefore, Theorem 1.2.48 is valid.*

EXAMPLE 1.2.50. *With the assumptions from Example 1.1.39 and the pseudometric defined by*

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot 2^{-j}, \forall j \in J, \forall (x_1, y_1), (x_2, y_2) \in X, \tag{1.40}$$

and $r(j) = j+3, \forall j \in J$ *(J is a subset of* $\mathbb{N}$*), we get another example of ALC-s. Consider* $T : X \to X$,

$$T(x,y) = \begin{cases} (x, \frac{y}{2}) & \text{if } (x,y) \neq (1,1) \\ (0,0) & \text{if } (x,y) = (1,1) \end{cases}$$

T is not a contraction because the contractive condition:

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x,y) \tag{1.41}$$

is not valid $\forall x, y \in X$, *and for any* $\theta \in (0,1)$. *The inequality* (1.41) *is equivalent to*

$$| \, x_1 - x_2| \cdot 2^{-j} \leq \theta \cdot |x_1 - x_2| \cdot 2^{-j}, \quad \forall j \in J, \forall (x_1,y_1),(x_2,y_2) \in X.$$

The last inequality leads us to $1 \leq \theta$, *which is obviously false, considering* $\theta \in (0,1)$. *However, T becomes an almost local contraction if:*

$$|x_1 - x_2| \cdot 2^{-j} \leq \theta \cdot \underbrace{|x_1 - x_2| \cdot 2^{-(j+3)}}_{d_{r(j)}(x,y)} + L \cdot \underbrace{|x_2 - x_1| \cdot 2^{-(j+3)}}_{d_{r(j)}(y,Tx)},$$

which is equivalent to : $8 \leq \theta + L$.
For $\theta = \frac{1}{5} \in (0,1)$, $L = 9 \geq 0$ *and* $j \in J$, *the last inequality becomes true, i.e., T is an ALC with an infinite set of fixed points:* $Fix(T) = \{(x,0) : x \in [0,n]\}$.
In this case, we have:

$$\lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = \lim_{n\to\infty} \left(\frac{1}{5}\right)^{n+1} \cdot n\sqrt{2} = 0, \quad \forall j \in J.$$

The existence of the fixed point is assured, according to condition (1.19) *from Theorem 1.1.33.*
Observe the continuity of T at $(0,0) \in Fix(T)$, *but discontinuity at* $(1,1)$, *which is not a fixed point of T. Therefore, Theorem 1.2.48 is valid.*

Example 1.2.51. *Let us consider* $A = [0,n] \times [0,n] \subset \mathbb{R}^2, n \in \mathbb{N}^*, \quad T : A \to A$,

$$T(x,y) = \begin{cases} (\frac{x}{2}, \frac{y}{4}) & \text{if } (x,y) \neq (1,0) \\ (0,0) & \text{if } (x,y) = (1,0) \end{cases}$$

Consider the pseudometric:

$$d_j\Big((x_1,y_1),(x_2,y_2)\Big) = |x_1 - x_2| \cdot e^{2j}, \forall j \in J, \forall (x_1,y_1),(x_2,y_2) \in A, \tag{1.42}$$

where J is a subset of $\mathbb{Q}$. d_j *is a pseudometric, but is not a metric, take for example:* $d_j((1,5),(1,6)) = |1-1| \cdot e^{2j} = 0$, *however* $(1,5) \neq (1,6)$.
In this case, consider the function $r(j) = j + \frac{1}{2}$. *Observe that the pseudometric not possess the monotonicity property. By applying the inequality* (1.19) *to the mapping T, we get for all* $x = (x_1,y_1), y = (x_2,y_2) \in A$:

$$\Big|\frac{x_1}{2} - \frac{x_2}{2}\Big| \cdot e^{2j} \leq \theta \cdot |x_1 - x_2| \cdot e^{2(j+\frac{1}{2})} + L \cdot \Big|x_2 - \frac{x_1}{2}\Big| \cdot e^{2(j+\frac{1}{2})},$$

for all $j \in J$, which can be put in the equivalent form

$$|x_1 - x_2| \cdot e^{-1} \leq 2\theta \cdot |x_1 - x_2| + L \cdot |2x_2 - x_1|.$$

From that, we obtain:

$$|x_1 - x_2| \cdot (e^{-1} - 2\theta) \leq L \cdot |2x_2 - x_1|.$$

The last inequality is true if we take $\theta = \frac{1}{2} \in (0,1), L = 4 \geq 0$, thus we obtain

$$\underbrace{\Big(\frac{1}{e} - 1\Big) \cdot |x_1 - x_2|}_{\leq 0} \leq \underbrace{4 \cdot |2x_2 - x_1|}_{\geq 0}.$$

Hence T is an ALC, with the unique fixed point $(0,0)$.
According to Theorem 1.2.48, T is continuous at the fixed point, at $(0,0) \in Fix(T)$, but is not continuous at $(1,0) \notin Fix(T)$.

Remark 1.2.52. *According to Definition 1.1.31, the almost local contractions are defined locally, on a subset $A \subset X$. In the case $A = X$, an almost local contraction becomes an usual almost contraction.*

3. New classes of almost local contractions

In this section we present some extensions of almost local contractions on pseudometric spaces by following the corresponding results existing in literature for almost contractions.

a) **Generalized almost local contractions**

The starting point is represented by the almost contractions (Definition 1.1.31) and the generalized almost contractions introduced in [**6**].

Definition 1.3.53. [**6**] *A self mapping T on a metric space (X,d) is said to satisfy the condition (B) if there exist $\delta \geq 0$ and $L \geq 0$ such that $\delta + L < 1$ and for all $x, y \in X$ we have*

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Now, it is natural to extend the above definition to the more general case of uniform spaces and pseudometrics.

Definition 1.3.54. *Let F be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on F, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset F$. Let the function $r : J \to J$. The mapping $T : A \to A$ is called generalized almost local contraction if there exist a constant $\theta \in (0,1)$ and some $L \geq 0$ such that we have:*

$$\begin{aligned} d_j(Tx,Ty) \;&\leq\; \theta \cdot d_{r(j)}(x,y) + \\ &+ L \cdot min\{d_{r(j)}(x,Tx), d_{r(j)}(y,Ty), d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\}, \end{aligned} \tag{1.43}$$

for every $x, y \in A$, and for all $j \in J$.

REMARK 1.3.55. *It is obvious that an almost contraction is a generalized almost local contraction, by taking $r(j) = j$, for all $j \in J$.*

THEOREM 1.3.56. *Assume that $F, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.54. Let $T : A \to A$ be a generalized ALC, i.e., a mapping satisfying (1.43), and also verifying the uniqueness condition (1.27) with the constants $\theta_u \in (0,1)$ and $L_u \geq 0$. Let $Fix(T) = \{f\}$. Then T is continuous at f.*

Proof: Since T is a generalized ALC, there exist a constant $\theta_u \in (0,1)$ and some $L_u \geq 0$ such that (1.27) is satisfied. We know by Theorem 1.1.38 that T has a unique fixed point, say f.
Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in A converging to f. Then by taking

$$y := y_n, \quad x := f$$

in the generalized almost local contraction condition (1.43), we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_{r(j)}(f, y_n), \quad \forall j \in J, \quad n = 0, 1, 2, \cdots \tag{1.44}$$

Since f is a fixed point for T, we have

$$\min_{x,y \in A} \{d_{r(j)}(x, Tx), d_{r(j)}(y, Ty), d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\} = d_{r(j)}(f, Tf) = 0,$$

for every $j \in J$.
Now, by letting $n \to \infty$ in (1.44), we get $Ty_n \to Tf$ which shows that T is continuous at f. □

b) **Ćirić type strong almost local contractions**

DEFINITION 1.3.57. [**26**], [**27**] *Let (X, d) be a complete metric space. The mapping $T : X \to X$ is called Ćirić almost contraction if there exist a constant $\alpha \in (0,1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \alpha \cdot M_1(x, y) + L \cdot d(y, Tx), \text{ for all } x,y \in X, \tag{1.45}$$

where

$$M_1(x, y) = max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}.$$

From the above definition the following question arises: is it possible to extend it to the case of almost local contractions? First we need to remind two Lemmas of Ćirić [**45**], which will be essential in proving our main results in this section.

LEMMA 1.3.58. [**45**] *Let (X, d) be a metric space. Let T be a quasi-contraction on X with the coefficient $0 \leq h < 1$ and let n be any positive integer. Then, for each $x \in X$, and all $i, j \in \{1, 2, \cdots, n\}$, we get*

$$d(T^i x, T^j x) \leq h \cdot \delta[O(x, n)],$$

where we denoted

$$\delta(A) = \sup\{d(a,b) : a, b \in A\} \text{ for a subset } A \subset X$$

and

$$O(x,n) = \{x, Tx, \cdots, T^n x\}, \quad n = 1, 2, \cdots$$

$$O(x,\infty) = \{x, Tx, \cdots\}$$

LEMMA 1.3.59. [**45**] *Let T be a quasi-contraction on X with the coefficient of contraction $0 \le h < 1$. Then the inequality*

$$\delta[O(x,n)] \le \frac{1}{1-h} d(x, T^k x)$$

holds for all $x \in X, n \in \mathbb{N}, 1 \le k \le n$.

DEFINITION 1.3.60. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let the function $r : J \to J$.*
The mapping $T : A \to A$ is called Ćirić type strong ALC with regard to $(\mathcal{D}, r)$ if there exist the constants $\theta \in (0,1)$ and $L \ge 0$ such that

$$d_j(Tf, Tg) \le \theta \cdot M_{r(j)}(f,g) + L \cdot d_{r(j)}(g, Tf), \text{ for all } f,g \in A, \forall j \in J, \tag{1.46}$$

where

$$M_{r(j)}(f,g) = \max\Big\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), \frac{d_{r(j)}(f,Tg) + d_{r(j)}(g,Tf)}{2}\Big\}.$$

In the sequel, it is our aim to extend Lemmas 1.3.58 and 1.3.58 to the case of uniform spaces instead of metric spaces.

LEMMA 1.3.61. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.60. Let $T : A \to A$ be a Ćirić type strong ALC with regard to $(\mathcal{D}, r)$ with the coefficients $\theta \in (0,1)$ and $L \ge 0$ and let n be any positive integer. Assume the monotonicity property 1.26 fulfilled for the pseudometrics, for each $j \in J$. Then, for every $x \in X$, and for all $k, l \in \{1, 2, \cdots, n\}$, we get*

$$d_j(T^k x, T^l x) \le (\theta + L) \cdot \delta_j[O(x,n)],$$

where we denoted

$$\delta_j(A) = \sup\{d_j(a,b) : a, b \in A\} \text{ for a subset } A \subset X, \forall j \in J$$

and

$$O(x,n) = \{x, Tx, \cdots, T^n x\}, \quad n = 1, 2, \cdots$$

$$O(x,\infty) = \{x, Tx, T^2 x, \cdots\}.$$

Proof: Let $x \in X$ an arbitrary element and consider $k, l \in \{1, 2, \cdots, n\}$. Then it is obvious that $T^{k-1}x, T^k x, T^{l-1}x, T^l x \in O(x, n)$, where $T^0 x = x$. The monotonicity of the pseudometrics leads us to:

$$d_{r(j)}(T^k x, T^l x) \leq d_j(T^k x, T^l x) \Rightarrow M_{r(j)}(T^k x, T^l x) \leq M_j(T^k x, T^l x), \forall j \in J.$$

By using the definition of T, we can write for every $j \in J$:

$$\begin{aligned} d_j(T^k x, T^l x) &= d_j(TT^{k-1}x, TT^{l-1}x) \leq \\ &\leq \theta \cdot \underbrace{M_{r(j)}}_{\leq M_j}(T^{k-1}x, T^{l-1}x) + L \cdot \underbrace{d_{r(j)}}_{\leq d_j}(T^{l-1}x, T^k x) \leq \\ &\leq (\theta + L) \cdot \delta_j[O(x, n)], \end{aligned}$$

where

$$M_{r(j)}(T^{i-1}x, T^{j-1}x) = \max\Big\{d_{r(j)}(T^{i-1}x, T^{j-1}x), d_{r(j)}(T^{i-1}x, T^i x), d_{r(j)}(T^{j-1}x, T^j x), \frac{d_{r(j)}(T^{i-1}x, T^j x) + d_{r(j)}(T^{j-1}x, T^i x)}{2}\Big\}.$$

□

Remark 1.3.62. *Note that, by using Lemma 1.3.61, for each n, there exists $k \leq n$ such that*

$$d_j(x, T^k x) = d_j(T^0 x, T^k x) = \delta_j[O(x, n)], \quad \forall j \in J.$$

Lemma 1.3.63. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.60. Let $T : A \to A$ be a Ćirić type strong ALC with regard to $(\mathcal{D}, r)$ with the coefficients $\theta \in (0, 1)$, $L \geq 0$, $\theta + L < 1$ and let n be any positive integer, $1 \leq k \leq n$. Assume the monotonity property 1.26 fulfilled for the pseudometric.*
Then the inequality

$$\delta_j[O(x, n)] \leq \frac{1}{1 - \theta - L} d_j(x, T^k x)$$

holds for all $x \in X, n \in \mathbb{N}, 1 \leq k \leq n$.

Proof: Let $x \in X$ arbitrary. From the Remark 1.3.62, there exists $1 \leq k \leq n$ such that $T^k x \in O(x, n)$ and also $d_j(x, T^k x) = \delta_j[O(x, n)], \forall j \in J$. After applying the triangle inequality and Lemma 1.3.61, we obtain:

$$\begin{aligned} d_j(x, T^k x) &\leq d_j(x, Tx) + d_j(Tx, T^k x) \leq d_j(x, Tx) + (\theta + L) \cdot \delta_j[O(x, n)] = \\ &= d_j(x, Tx) + (\theta + L) \cdot d_j(x, T^k x), \quad \forall j \in J. \end{aligned}$$

From that, we conclude:

$$\delta[O(x, n)] = d_j(x, T^k x) \leq \frac{1}{1 - \theta - L} \cdot d_j(x, Tx), \quad \forall j \in J.$$

□

REMARK 1.3.64. *Although this class of Ćirić type strong almost local contractions is wider than the one of almost local contractions, similar conclusions can be stated, as it follows:*

THEOREM 1.3.65. [**119**] *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let the function $r : J \to J$ and let $T : A \to A$ be a Ćirić type strong ALC with respect to ($\mathcal{D}$,r) with the coefficients $\theta \in (0, 1)$, $L \geq 0$, $\theta + L < 1$. Assume the monotonicity property* (1.26) *fulfilled.*
Then

(1) *T has a fixed point, i.e., $Fix(T) = \{x \in A : Tx = x\} \neq \phi$;*
(2) *For any $x_0 = x \in A$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in Fix(T)$;*
(3) *The following a priori estimate is available:*

$$d_j(x_n, x^*) \leq \frac{\theta^n}{(1-\theta)^2} \cdot d_j(x, Tx), \quad \forall n = 1, 2, ..., \forall j \in J. \tag{1.47}$$

Proof: For the conclusion of the theorem, we have to prove that T has at least a fixed point in the subset $A \subset X$. To this end, let $x \in A$ be arbitrary, and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ with $x_0 = x$.
Take $f := x_{n-1}, g := x_n$ in (1.46) to obtain

$$d_j(x_n, x_{n+1}) = d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot M_{r(j)}(x_{n-1}, x_n) + L \cdot \underbrace{d_{r(j)}(x_n, Tx_{n-1})}_{=0}, \forall j \in J,$$

that is,

$$d_j(x_n, x_{n+1}) \leq \theta \max\Big\{d_{r(j)}(x_{n-1}, x_n), d_{r(j)}(x_n, x_{n+1}), \frac{d_{r(j)}(x_{n-1}, x_{n+1}) + 0}{2}\Big\},$$

taking into account that $d_{r(j)}(x_n, Tx_{n-1}) = 0$. After using the triangle inequality, we obtain

$$d_{r(j)}(x_{n-1}, x_{n+1}) \leq d_{r(j)}(x_{n-1}, x_n) + d_{r(j)}(x_n, x_{n+1}), \quad \forall j \in J.$$

At this point, we use the inequality $\frac{x+y}{2} \leq \max\{x, y\}, \forall x, y \in \mathbb{R}$, we distinguish two cases:

$$\max\Big\{d_{r(j)}(x_{n-1}, x_n), d_{r(j)}(x_n, x_{n+1}), \frac{d_{r(j)}(x_{n-1}, x_{n+1})}{2}\Big\} = d_{r(j)}(x_{n-1}, x_n) \tag{1.48}$$

or

$$\max\Big\{d_{r(j)}(x_{n-1}, x_n), d_{r(j)}(x_n, x_{n+1}), \frac{d_{r(j)}(x_{n-1}, x_{n+1})}{2}\Big\} = d_{r(j)}(x_n, x_{n+1}). \tag{1.49}$$

The case (1.49) leads to the contradiction:

$$d_j(x_n, x_{n+1}) \leq \theta d_{r(j)}(x_n, x_{n+1}) < d_{r(j)}(x_n, x_{n+1}) \underbrace{\leq}_{(1.26)} d_j(x_n, x_{n+1}),$$

therefore (1.48) is valid.

Continuing in this way, for $n \geq 1$, by Lemma 1.3.61, we have

$$d_j(T^n x, T^{n+1}x) = d_j(TT^{n-1}x, T^2T^{n-1}x) \leq (\theta + L) \cdot \delta_j[O(T^{n-1}x, 2)].$$

By using Remark 1.3.62, we can easily conclude: there exists a positive integer $k_1 \in \{1, 2\}$ such that

$$\delta_j[O(T^{n-1}x, 2)] = d_j(T^{n-1}x, T^{k_1}T^{n-1}x),$$

and therefore

$$d_j(x_n, x_{n+1}) \leq (\theta + L) \cdot d_j(T^{n-1}x, T^{k_1}T^{n-1}x), \quad \forall j \in J.$$

By applying once again Lemma 1.3.61, we obtain, for $n \geq 2$,

$$d_j(T^{n-1}x, T^{k_1}T^{n-1}x) = d_j(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \leq$$
$$\leq (\theta + L) \cdot \delta_j[O(T^{n-2}x, k_1 + 1)] \leq (\theta + L) \cdot \delta_j[O(T^{n-2}x, 3)], \forall j \in J.$$

Continuing in this way, we get

$$d_j(T^n x, T^{n+1}x) \leq (\theta + L) \cdot \delta_j[O(T^{n-1}x, 2)] \leq (\theta + L)^2 \cdot \delta_j[O(T^{n-2}x, 3)].$$

By applying repeatedly the last inequality, we get

$$d_j(T^n x, T^{n+1}x) \leq (\theta + L) \cdot \delta_j[O(T^{n-1}x, 2)] \leq \cdots \leq (\theta + L)^n \cdot \delta_j[O(x, n + 1)]. \tag{1.50}$$

At this point, by Lemma 1.3.63, we obtain

$$\delta_j[O(x, n + 1)] \leq \delta_j[O(x, \infty)] \leq \frac{1}{1 - \theta - L} \cdot d_j(x, Tx),$$

which, by (1.50), yields

$$d_j(T^n x, T^{n+1}x) \leq \frac{(\theta + L)^n}{1 - \theta - L} d_j(x, Tx). \tag{1.51}$$

The inequality (1.51) and the triangle inequality can be merged to obtain the following estimate:

$$\begin{aligned} d_j(T^n x, T^{n+p}x) &\leq \frac{(\theta + L)^n}{1 - \theta - L} \cdot \frac{1 - (\theta + L)^p}{1 - \theta - L} \cdot d_j(x, Tx) < \\ &< \frac{1}{(1 - \theta - L)^2} \cdot d_j(x, Tx) \cdot (\theta + L)^n, \end{aligned} \tag{1.52}$$

for every $n, p \in \mathbb{N}$, $\forall j \in J$ and every $x \in A$.

Let us remind the fact that $0 < \theta + L < 1$. Then, by using (1.52), we can conclude that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As the subset A is assumed to be sequentially τ-complete, there exists x^* in A such that $\{x_n\}$ is τ-convergent to x^*. After computations involving the triangular inequality and the Definition (1.3.60), we get

$$\begin{aligned} d_j(x^*, Tx^*) &\leq d_j(x^*, x_{n+1}) + d_j(x_{n+1}, Tx^*) = d_j(T^{n+1}x, x^*) + d_j(T^{n+1}x, Tx^*) \leq \\ &\leq d_j(T^{n+1}x, x^*) + \theta \max\Big\{d_{r(j)}(T^n x, x^*), d_{r(j)}(T^n x, T^{n+1}x), d_{r(j)}(x^*, Tx^*), \\ &, \quad \frac{d_{r(j)}(T^n x, Tx^*) + d_{r(j)}(T^{n+1}x, x^*)}{2}\Big\} + L \cdot d_{r(j)}(x^*, Tx_n). \end{aligned}$$

By applying the monotonicity of the pseudometrics, we get:

$$\begin{aligned} d_j(x^*,Tx^*) &\leq d_j(T^{n+1}x,x^*)+\theta\cdot\max\Big\{d_j(T^nx,x^*),d_j(T^nx,T^{n+1}x),d_j(x^*,Tx^*), \\ &, \quad \frac{d_j(T^nx,Tx^*)+d_j(T^{n+1}x,x^*)}{2}\Big\}+L\cdot d_j(x^*,Tx_n). \end{aligned}$$

From that, we obtain the following inequality:

$$\begin{aligned} d_j(x^*,Tx^*) &\leq d_j(T^{n+1}x,x^*)+\theta\cdot[d_j(Tx_n,x^*)+d_j(T^nx,T^{n+1}x)+ \\ &+ \quad d_j(x^*,Tx^*)+d_j(T^{n+1}x,x^*)]+L\cdot d_j(x^*,Tx_n). \end{aligned}$$

It follows immediately:

$$\begin{aligned} d_j(x^*,Tx^*) \leq \frac{1}{1-\theta}\cdot\Big[(1+\theta)d_j(T^{n+1}x,x^*)+ \\ +(\theta+L)d_j(x^*,Tx_n)+\theta d_j(T^nx,T^{n+1}x)\Big]. \end{aligned} \tag{1.53}$$

Letting $n\to\infty$ in (1.53), we obtain

$$d_j(x^*,Tx^*)=0,$$

which means that x^* is a fixed point of T. The estimate (1.47) can be obtained from (1.52) by letting $n\to\infty$. □

Remark 1.3.66. *1) Theorem 1.3.65 represents a very important extension of Banach's fixed point theorem, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Zamfirescu's fixed point theorem, as well as of many other related results obtained on the base of similar contractive conditions. These fixed point theorems mentioned before ensure the uniqueness of the fixed point, but the Ćirić type ALC need not have a unique fixed point, according to Example 1.3.67.*

2) The main merit of Theorem 1.3.65 is that it provides a very large class of weakly Picard operators. Obviously, the fixed point x^* attained by the Picard iteration depends on the initial guess $x_0\in X$. However, the error estimate (1.47) *obtained in Theorem 1.3.65 is weaker than the one in Banach's contraction mapping principle.*

Example 1.3.67. *Let A be the set of positive functions*
$A=\{f|f:[0,\infty)\to[0,\infty)\}$.
Consider the pseudometric $d_j(f,g)=|f(0)-g(0)|\cdot j,\quad\forall j\in J; J\subset\mathbb{N},\quad\forall f,g\in A$. On observe that d_j is a pseudometric, but is not a metric, take for example $d_j(x^3,x^2)=0$, but $x^3\neq x^2$.
Consider the mapping $Tf=|f|,\quad\forall f\in A$, $r(j)=j+1$. Note that the monotonicity condition (1.26) *also holds. By using condition* (1.19) *for ALC-s, we obtain:*

$$|f(0)-g(0)|\cdot j\leq\theta\cdot|f(0)-g(0)|\cdot(j+1)+L\cdot|g(0)-f(0)|\cdot(j+1),\forall f,g\in A,$$

which is equivalent to: $j \leq (\theta + L)(j+1), \forall j \in J.$
This inequality is true if $\theta = \frac{1}{3} \in (0,1), \quad L = 1 > 0.$
Hence, T *is an almost local contraction. However,* T *is not a contraction, because the contractive condition* (1.41) *leads us to the contradiction:* $1 \leq \theta$.
The mapping T *is Ćirić type ALC, because*

$$M_{r(j)}(f,g) = |f(0) - g(0)| \cdot (j+1), \forall f, g \in A, \forall j \in J.$$

From (1.46) *we have the equivalent form*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j+1) + L \cdot |f(0) - f(0)| \cdot (j+1).$$

Again, we obtain the inequality $j \leq (\theta + L)(j+1)$, *valid for* $\theta = \frac{1}{3} \in (0,1), L = 1 \geq 0.$
The mapping T *has an infinite set of fixed points:* $Fix(T) = \{f \in A\} = A$, *because:*

$$Tf = f \Leftrightarrow |f(x)| = f(x), \forall f \in A, \quad \forall x \in [0, \infty), \forall x \in [0, \infty).$$

The uniqueness condition (1.27) *is not valid, having in view the equivalent form:*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j+1) + L_1 \cdot |f(0) - f(0)| \cdot (j+1),$$

which leads us to the contradiction $j \leq \theta(j+1)$, *i.e., the mapping* T *does not satisfy the uniqueness condition.*

The uniqueness of the fixed point for a Ćirić type ALC can be assured by imposing an additional contractive condition, according to the next theorem.

THEOREM 1.3.68. *Assume that* $X, J, \mathcal{D}, r, \tau$ *and* A *are as in Theorem 1.3.65. Let* $T : A \to A$ *be a Ćirić type strong almost local contraction, with the family of the pseudometrics satisfying the monotonicity property 1.26. If the mapping* T *satisfies the uniqueness condition from the almost local contractions: there exist the constants* $\theta_1 \in (0,1)$ *and some* $L_1 \geq 0$ *such that*

$$d_j(Tf, Tg) \leq \theta_1 \cdot d_{r(j)}(f,g) + L_1 \cdot d_{r(j)}(f, Tf), \quad \forall j \in J, \text{ for all } f,g \in A, \tag{1.54}$$

then

(1) T *has a unique fixed point, i.e.,* $Fix(T) = \{f^*\}$;
(2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ *converges to* f^*, *for any* $x_0 \in A$;
(3) *The a priori error estimate* (1.47) *holds;*
(4) *The rate of the convergence of the Picard iteration is given by*

$$d_j(x_n, f^*) \leq \theta_1 \cdot d_{r(j)}(x_{n-1}, f^*), \quad n = 1, 2, ..., \forall j \in J. \tag{1.55}$$

Proof: 1) Suppose, by contradiction, there are two distinct fixed points f^* and g^* of T. Then, by using (1.54), and condition (1.26) for every fixed $j \in J$ with $f := f^*, g := g^*$, we get:

$$d_j(f^*, g^*) \leq \theta_1 \cdot d_{r(j)}(f^*, g^*) \leq \theta_1 \cdot d_j(f^*, g^*).$$

The last inequality is equivalent with $(1 - \theta_1) \cdot d_j(f^*, g^*) \leq 0$, which is obviously a contradiction with $d_j(f^*, g^*) > 0$.
So, we prove the uniqueness of the fixed point.
The proof for 2) and 3) is quite similar to the proof from the Theorem 1.3.65.
4) At this point, letting $g := x_n, f := f^*$ in (1.54), it results the rate of convergence given by (1.55). □

The contractive conditions (1.46) and (1.54) can be merged in order to assure the uniqueness of the fixed point, stated by the next theorem.

THEOREM 1.3.69. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.60. Let $T : A \to A$ be a mapping for which there exist the constants $\theta \in (0, 1)$ and some $L \geq 0$ such that we have:*

$$\begin{aligned} d_j(Tf, Tg) &\leq \theta \cdot M_{r(j)}(f, g) + \\ &+ L \cdot \min\{d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}, \end{aligned} \tag{1.56}$$

for all $f, g \in A$ and for every $j \in J$, where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Then

(1) T has a unique fixed point, i.e., $Fix(T) = \{f^*\}$;
(2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ converges to f^*, for any $x_0 \in A$;
(3) The a priori error estimate (1.47) holds.

Particular case 1.
The famous Ćirić' s fixed point theorem for single valued mappings given in [**45**] can be obtained from Theorems 1.3.65, 1.3.69, 1.3.68 by taking $L = L_1 = 0$ and considering r the identity mapping: $r(j) = j$. The Ćirić' s contractive condition represents one of the most general metrical condition that provide a unique fixed point by means of Picard iteration. Despite this observation, the contractive condition given for Ćirić type ALC (in (1.46)) admits a very high level of generalization. Note that the fixed point could be approximated by means of Picard iteration, just like in the case of Ćirić' s fixed point theorem, although the uniqueness of the fixed point is not ensured by using (1.46).

Particular case 2.
If the maximum from Theorem 1.3.69, for $r(j) = j$, becomes:

$$\max\left\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\right\} = d_{r(j)}(f,g), \forall j \in J,$$

for all $f, g \in A$, then Theorem 1.1.33 is obtained from Theorem 1.3.65.

After these existence and the uniqueness theorems of the fixed points for the Ćirić type almost local contractions, it is natural to extend them to the case of Ćirić type strict almost local contractions.

DEFINITION 1.3.70. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudo-metrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$.*
The operator $T : A \to A$ is called Ćirić type strict ALC with respect to $(\mathcal{D}, r)$ if it satisfy simultaneously conditions (1.19), (1.21), $(Ci - ALC)$, $(ALC - U)$, *with some real constants $\theta \in (0,1), L \geq 0$, $\theta_C \in (0,1)$, $L_C \geq 0$ and $\theta_u \in (0,1)$, $L_u \geq 0$, respectively,*

$$(Ci - ALC) \qquad d_j(Tf,Tg) \leq \theta_C \cdot M_{r(j)}(f,g) + L_C \cdot d_{r(j)}(g,Tf), \text{ for all } f,g \in A,$$

for every $j \in J$, where

$$M_{r(j)}(f,g) = \max\left\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\right\}.$$

$$(ALC - U) \quad d_j(Tf,Tg) \leq \theta_u \cdot d_{r(j)}(f,g) + L_u \cdot d_{r(j)}(f,Tf), \text{ for all } f,g \in A, \forall j \in J.$$

We present an illustrative example (1.3.71) for our results: Ćirić' type almost local contractions, without having unique fixed point.

EXAMPLE 1.3.71. *By taking the mapping from Example 1.1.35, with a small modification, that is: let X be the set of positive functions*

$$X = \{f | f : [0,\infty) \to [0,\infty)\}.$$

Fix $x_0 \in [0,\infty)$ and take the pseudometric $d_j(f,g) = |f(x_0) - g(x_0)| \cdot e^{-j}$,
$\forall f, g \in X$. Let us consider $r(j) = j + 2$, $\forall j \in \mathbb{N}$.
We can conclude in the same manner that T is also a Ćirić type ALC, i.e., it satisfies the contractive condition (1.46), *since we have $M_{r(j)}(f,g) = |f(x_0) - g(x_0)| \cdot e^{-(j+2)}$.*
The inequality (1.46) *is equivalent with*

$$|f(x_0) - g(x_0)| \cdot e^{-j} \leq \theta |f(x_0) - g(x_0)| \cdot e^{-(j+2)} + L|f(x_0) - g(x_0)| \cdot e^{-(j+2)}, \forall j \in J,$$

which is valid for $\theta = \frac{1}{4} \in (0,1)$ and $L = 10 \geq 0$.
The mapping T has an infinite set of fixed points: $Fix(T) = \{f \in A\} = A$, by taking:

$$|f(x)| = f(x), \forall f \in A, x \in [0,\infty).$$

By considering $L = 0$ in the definition 1.3.60 of Ćirić type ALC, we get a new type of ALC, that is, the quasi-ALC.

c) **Quasi-almost local contractions**

Starting from the concept of quasi-contraction introduced by Ćirić (Definition 0.2.6), we obtain a new type of contraction with the aim of studying the existence of the fixed points.

Definition 1.3.72. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$.*

The operator $T : A \to A$ is called quasi-almost local contraction with regard to $(\mathcal{D}, r)$ if there exists the constant $h \in (0, 1)$ such that

$$d_j(Tf, Tg) \leq h \cdot M_{r(j)}(f, g), \text{ for all } f,g \in A, \forall j \in J, \tag{1.57}$$

where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Theorem 1.3.73. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$. Let $T : A \to A$ be a quasi-ALC with regard to $(\mathcal{D}$,$r)$. Then*

(1) T has a fixed point , i.e., $Fix(T) = \{x \in X : Tx = x\} \neq \phi$;

(2) For any $x_0 = x \in A$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in Fix(T)$;

(3) The following a priori estimate is available:

$$d_j(x_n, x^*) \leq \frac{h^n}{(1-h)^2} d_j(x, Tx), \quad n = 1, 2, ... \tag{1.58}$$

Proof: Obviously, we have to follow the steps from the proof of Theorem 1.3.65, with the only difference that the constant $L = 0$, as in the case of quasi ALC-s. □

The uniqueness of the fixed point is also assured by imposing an additional condition, just like in the class of Ćirić type ALC, as it follows.

Theorem 1.3.74. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.72. Let $T : A \to A$ be a quasi-almost local contraction with the monotonicity property* (1.26) *fulfilled for the family of pseudometrics. If T satisfies the supplementary condition: there exist the constants*

$\theta \in (0, 1)$ and $L \geq 0$ such that

$$d_j(Tf, Tg) \leq \theta \cdot d_{r(j)}(f, g) + L \cdot d_{r(j)}(f, Tf), \text{ for all } f,g \in A, \forall j \in J, \tag{1.59}$$

then

(1) T has a unique fixed point, i.e., $Fix(T) = \{f^*\}$;

(2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ converges to f^*, for any $x_0 \in A$;

(3) The a priori error estimate (1.47) holds;

(4) The rate of the convergence of the Picard iteration is given by

$$d_j(x_n, f^*) \leq \theta \cdot d_{r(j)}(x_{n-1}, f^*), \quad \forall j \in J, n = 1, 2, ... \tag{1.60}$$

d) **Ćirić-Reich-Rus type almost local contractions**

The Ćirić-Reich-Rus type contractions (Definition 0.2.16) in metric spaces is the starting point of the cotractions considered in this subsection.

Definition 1.3.75. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$.*

The operator $T : A \to A$ is called Ćirić-Reich-Rus type almost local contraction regarding to $(\mathcal{D}, r)$ if there exist the constants $\delta, L \in \mathbb{R}_+$ with $\delta + 2L < 1$ such that

$$d_j(Tf, Tg) \leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)], \tag{1.61}$$

for all $f, g \in A$ and for each $j \in J$.

Theorem 1.3.76. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.75. If the pseudometrics d_j fulfills the 1.26 monotonicity property for every $j \in J$, then any Ćirić-Reich-Rus type almost local contraction, i.e., any mapping $T : A \to A$ satisfying condition (1.61) with $0 < L < 1$, $\delta > 0$ and $\frac{\delta+L}{1-L} \in (0, 1)$, is an almost local contraction.*

Proof: Using condition (1.61) and the triangle inequality, we get

$$d_j(Tf, Tg) \leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)] \leq$$
$$\leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(g, Tf) + \underbrace{d_{r(j)}(Tf, Tg)}_{\leq d_j(Tf,Tg)} + d_{r(j)}(f, g) + d_{r(j)}(g, Tf)].$$

The monotonicity condition for the pseudometrics leads us to:

$$d_{r(j)}(Tf, Tg) \leq d_j(Tf, Tg), \quad \forall j \in J, \forall f, g \in A.$$

We get after simple computations:

$$(1 - L) \cdot d_j(Tf, Tg) \leq (\delta + L) \cdot d_{r(j)}(f, g) + 2L \cdot d_{r(j)}(g, Tf). \tag{1.62}$$

From that, we obtain

$$d_j(Tf, Tg) \leq \frac{\delta + L}{1 - L} \cdot d_{r(j)}(f, g) + \frac{2L}{1 - L} \cdot d_{r(j)}(g, Tf), \forall f, g \in A, \forall j \in J. \tag{1.63}$$

Consider $\delta, L \in \mathbb{R}_+$ and $\delta + 2L < 1$, the inequality (1.19) holds, with $\frac{\delta+L}{1-L} \in (0,1)$ and $\frac{2L}{1-L} \geq 0$. Therefore, every Ćirić-Reich-Rus type almost local contraction with the monotonicity condition valid for the pseudometrics and the mentioned conditions for the coefficients δ and L fulfilled, is an almost local contraction. □

e) **Chatterjea type almost local contraction**

The Chatterjea contraction (Definition 0.2.5) has raised the interest to extend it to a more general case, in uniform spaces.

Definition 1.3.77. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$. The operator $T : A \to A$ is called Chatterjea type almost local contraction with regard to ($\mathcal{D}, r$) if there exists a constant $0 \leq c < \frac{1}{2}$ such that*

$$d_j(Tf, Tg) \leq c \cdot [d_{r(j)}(f, Tg) + d_{r(j)}(g, Tf)], \quad \forall f, g \in A, \forall j \in J. \tag{1.64}$$

Theorem 1.3.78. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Definition 1.3.77. If the pseudometrics d_j satisfies the 1.26 monotonicity property for every $j \in J$, then any Chatterjea type almost local contraction, i.e., any mapping $T : A \to A$ satisfying condition* (1.64) *with $0 \leq c < \frac{1}{2}$, is an almost local contraction.*

Proof: Using condition (1.64) and the triangle inequality, we get

$$d_j(Tf, Tg) \leq c \cdot [d_{r(j)}(f, Tg) + d_{r(j)}(g, Tf)] \leq$$
$$\leq c \cdot [d_{r(j)}(f, g) + d_{r(j)}(g, Tf) + d_{r(j)}(Tf, Tg)] + c \cdot d_{r(j)}(g, Tf).$$

The monotonicity of the pseudometrics leads us to:

$$d_{r(j)}(g, Tf) \leq d_j(g, Tf), \quad \forall j \in J, \forall f, g \in A. \tag{1.65}$$

It follows immediately:

$$d_j(Tf, Tg) \leq c[d_{r(j)}(f, g) + d_{r(j)}(g, Tf)] + c[d_{r(j)}(g, Tf) + d_j(Tf, Tg)] \tag{1.66}$$
$$(1 - c)d_j(Tf, Tg) \leq c \cdot d_{r(j)}(f, g) + 2c \cdot d_{r(j)}(g, Tf),$$

and which implies

$$d_j(Tf, Tg) \leq \frac{c}{1-c} \cdot d_j(f, g) + \frac{2c}{1-c} \cdot d_j(g, Tf), \forall f, g \in A. \tag{1.67}$$

Considering

$$0 \leq c < \frac{1}{2}, \tag{1.68}$$

the inequality (1.19) holds, with $\theta = \frac{c}{1-c} \in [0, 1)$ and $L = \frac{2c}{1-c} \geq 0$.

Therefore, any Chatterjea type almost local contraction is an almost local contraction if the monotonicity condition is valid for the pseudometrics and 1.68 condition is fulfilled for the contractive coefficient c. □

f) **Generalized Berinde type almost local contractions**

The generalized Berinde mapping (Definition 0.2.15) motivated us to examine the possibility to extend it in uniform spaces, thus obtaining the generalized Berinde type almost local contractions, as follows:

Definition 1.3.79. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$. The operator $T : A \to A$ is called generalized Berinde type almost local contraction regarding to $(\mathcal{D}, r)$ if there exist a constant $\theta \in (0, 1)$ and a function $b : A \to [0, \infty)$ such that*

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + b(y) \cdot d_{r(j)}(y, Tx), \forall x, y \in A, \forall j \in J. \tag{1.69}$$

Theorem 1.3.80. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$. Let $T : A \to A$ be a mapping and consider a function $b : A \to [0, \infty)$. Assume that there exists $\theta \in (0, 1)$ such that*

$$(1+\theta)^{-1} d_j(x, Tx) \leq d_j(x, y) \quad \textit{implies} \quad d_j(Tx, Ty) \leq \theta d_{r(j)}(x, y) + b(y) \cdot d_{r(j)}(y, Tx), \tag{1.70}$$

for all $x, y \in A$. Then, for every $x \in A$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Proof: Since $\theta \in (0, 1)$, we have the inequality

$$(1+\theta)^{-1} d_j(x, Tx) \leq d_j(x, Tx), \quad \forall j \in J, \tag{1.71}$$

and we get

$$d_j(Tx, T^2x) \leq \theta d_{r(j)}(x, Tx) + b(Tx) \cdot d_{r(j)}(Tx, Tx) = \theta d_{r(j)}(x, Tx), \forall x \in A. \tag{1.72}$$

Let $u \in A$. Then from (1.72) we have

$$d_j(T^n u, T^{n+1} u) \leq \theta^n d_{r^n(j)}(u, Tu), \quad \forall j \in J, \forall n \in \mathbb{N}.$$

If the pseudometrics from the right-hand side are bounded, which means: there exists a constant C such that $d_{r^n(j)}(u, Tu) < C$, $\forall j \in J$, therefore:

$$\sum_{n=1}^{\infty} d_j(T^n u, T^{n+1} u) < \infty, \text{ for every } j \in J.$$

Thus, the sequence $\{T^n u\}$ is d_j-Cauchy for each $j \in J$. As the subset A is assumed to be sequentially τ-complete, there exists z in A such that $(T^n x)_{n\in\mathbb{N}}$ is τ-convergent to

$z \in A$, for every $x \in A$.

Apply (1.70), with $x := T^{f(n)}u, \quad y := z$. We can find a subsequence $\{f(n)\}$ of the sequence $\{n\}$ such that

$$(1+\theta)^{-1} d_j(T^{f(n)}u, T^{f(n)+1}u) \leq d_j(T^{f(n)}u, z)$$

implies

$$d_j(T^{f(n)+1}u, Tz) \leq \theta d_{r(j)}(T^{f(n)}u, z) + b(z) \cdot d_{r(j)}(z, T^{f(n)+1}u). \tag{1.73}$$

By (1.73), we have

$$\begin{aligned} d_j(z, Tz) &= \lim_{n\to\infty} d_j(T^{f(n)+1}u, Tz) \leq \\ &\leq \lim_{n\to\infty} \left(\theta d_{r(j)}(T^{f(n)}u, z) + b(z) \cdot d_{r(j)}(T^{f(n)+1}u, z)\right) = \\ &= \theta d_{r(j)}(z, z) + b(z) \cdot d_{r(j)}(z, z) = 0. \end{aligned}$$

Therefore, z is a fixed point of T. $\square$

Corollar 1.3.81. *If T is a generalized ALC on the subset $A \subset X$, then for every $x \in A$, the sequence $\{T^n x\}$ converges to a fixed point of T.*

Corollar 1.3.82. *Let T be a generalized ALC on $A \subset X$. Assume that there exist $\theta \in (0, 1)$ and some $B \in [0, \infty)$ such that*

$$(1+\theta)^{-1} d_j(x, Tx) \leq d_j(x, y) \quad \textit{implies } d_j(Tx, Ty) \leq \theta d_{r(j)}(x, y) + B \cdot d_{r(j)}(Tx, y),$$

for all $x, y \in A$. Then for every $x \in A$, the sequence $\{T^n x\}$ converges to a fixed point of T.

3.1. Almost local φ-contractions

We now extend the class of almost local contractions to the more general class of almost local φ-contractions. The aim of this subsection is to study the existence of fixed points for this new type of almost local contractions. First, let us recall some results and notions introduced by Berinde in [**29**].

Definition 1.3.83. [**101**] *A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies:*

(i_φ) *φ is monotone increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;*

(ii_φ) *the sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero, for all $t \in \mathbb{R}_+$, where φ^n stands for the n^{th} iterate of φ.*

If φ satisfies (i_φ) and

(iii_φ) *the series*

$$\sum_{k=0}^{\infty} \varphi^k(t)$$

converges for all $t \in \mathbb{R}_+$, then φ is said to be a (c)-comparison function.

The following lemma was proved in [**14**], it contains a few properties of (c)-comparison functions.

LEMMA 1.3.84. [**14**] *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a c-comparison function, then the following conditions hold:*

(i) *φ is a comparison function;*
(ii) *$\varphi(t) < t$, for any $t \in \mathbb{R}_+$;*
(iii) *φ is continuous at zero;*
(iv) *the series $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$.*

According to Berinde's work (see [**14**]), φ satisfies (iii_φ) if there exist $0 < c < 1$ and a convergent series of positive terms, $\sum\limits_{n=0}^{\infty} u_n$ such that

$$\varphi^{k+1}(t) \le c \cdot \varphi^k(t) + u_k, \text{ for all } t \in \mathbb{R}_+ \text{ and } k \ge k_0 \text{ (fixed)} .$$

Also, it was proved: if φ is a (c)- comparison function, then the sum of the comparison series, that is,

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), t \in \mathbb{R}_+ \tag{1.74}$$

is monotone increasing and continuous at zero, hence

$$\varphi^k(t) \to 0 \text{ as } k \to \infty, \quad \forall t \in \mathbb{R}_+, \tag{1.75}$$

and also: any (c)-comparison function is a comparison function.

The concept of (c)-comparison function was reformulated in [**13**] to that of b-comparison function, as it follows:

DEFINITION 1.3.85. [**13**] *Let $b \ge 1$ be a real number. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called b-comparison function if it satisfies:*

(i_φ) *φ is increasing;*
(ii_φ) *there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of non-negative terms $\sum\limits_{n=0}^{\infty} v_k$ such that*

$$b^{k+1}\varphi^{k+1}(t) \le ab^k\varphi^k(t) + v_k, \tag{1.76}$$

for $k \ge k_0$ and any $t \in \mathbb{R}_+$.

REMARK 1.3.86. *Obviously, for $b = 1$, the concept of b-comparison function reduces to that of (c)-comparison function.*

LEMMA 1.3.87. [**12**] *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a b-comparison function, then :*

(i) *the series $\sum\limits_{k=0}^{\infty} b^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;*

(ii) The function $s_b : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$s_b(t) = \sum_{k=0}^{\infty} b^k \varphi^k(t), t \in \mathbb{R}_+ \tag{1.77}$$

is increasing and continuous at zero.

LEMMA 1.3.88. [**84**] *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a b-comparison function, with constant* $b \geq 1$ *and* $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ *such that* $a_n \to 0$ *as* $n \to \infty$.
Then

$$\sum_{k=0}^{n} b^{n-k} \varphi^{n-k}(a_k) \to 0 \text{ as } n \to \infty. \tag{1.78}$$

Almost φ-contractions were first introduced by V. Berinde in [**17**]. It is our aim to extend almost φ-contractions to the more general case of almost local φ-contractions.

DEFINITION 1.3.89. [**17**] *Let* (X, d) *be a metric space. A self operator* $T : X \to X$ *is said to be an almost* φ*-contraction or* (φ, L)*-almost contraction, provided that there exist a comparison function* φ *and some* $L \geq 0$, *such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + L \cdot d(y, Tx), \text{ for all } x, y \in X. \tag{1.79}$$

At this point, we are able to extend the almost φ-contractions to the almost local φ-contractions.

DEFINITION 1.3.90. *Let* X *be a uniform space,* $\mathcal{D} = (d_j)_{j \in J}$ *be a family of pseudometrics on* X, *where* J *is a set of indices. Let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$. *Consider a nonempty,* τ*-bounded, sequentially* τ*-complete subset* $A \subset X$ *and let the function* $r : J \to J$. *The mapping* $T : A \to A$ *is called almost local* φ*-contraction if there exist a comparison function* φ *and some* $L \geq 0$ *such that we have:*

$$d_j(Tx, Ty) \leq \varphi(d_{r(j)}(x, y)) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in A, \forall j \in J. \tag{1.80}$$

REMARK 1.3.91. *It is obvious that any almost local* φ*-contraction becomes an almost local contraction if we take* $\varphi(t) = \theta t, t \in \mathbb{R}_+$ *and* $0 < \theta < 1$.

EXAMPLE 1.3.92. *Let us consider* $\varphi(t) = \dfrac{t}{t+1}$, $\quad t \in \mathbb{R}_+$, *with* $r(j) = j$ *and let* T *an almost local* φ*-contraction, i.e., a mapping which satisfies* (1.80).
Then φ *is a nonlinear comparison function, but it does not verify the condition for the (c)-comparison function. In this case,* T *is an almost local* φ*-contraction without being an almost local contraction.*

REMARK 1.3.93. *1) Similar to the case of ALC-s, the fact that* T *satisfies* (1.80), *for all* $x, y \in A$, *implies that the following dual inequality*

$$d_j(Tx, Ty) \leq \varphi(d_{r(j)}(x, y)) + L \cdot d_{r(j)}(x, Ty), \forall x, y \in A, \forall j \in J \tag{1.81}$$

obtained from (1.80) *by replacing x with y and vice versa, is also valid.*
2) The class of almost local φ-contractions includes a wide class of mappings, see Example 1.3.94, which contains a mapping with not one fixed point, but an infinite set of fixed points.

EXAMPLE 1.3.94. *Let us consider* $[0,1]$ *the unit interval with the Euclidean metric and the operator* $T:[0,1]\to[0,1]$ *the identity mapping, i.e.,* $Tx=x$*, for all* $x\in[0,1]$*. By taking* $\varphi(t)=a\cdot t,\quad t\in\mathbb{R},\quad 0<a<1,\quad \theta=a,\quad r(j)=j$ *and* $L\geq 1-a$*, condition* (1.80) *leads to*

$$|x-y|\leq a\cdot|x-y|+L\cdot|y-x|,$$

which is valid for all $x,y\in[0,1]$*. Note that the mapping* T *has an infinite set of fixed points:*

$$Fix(T)=\{x\in[0,1]:Tx=x\}=[0,1].$$

The next two theorems represent an existence theorem and, respectively, a uniqueness theorem for the fixed points of almost local φ-contractions.

THEOREM 1.3.95. *Let X be a uniform space, $\mathcal{D}=(d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A\subset X$ and let the function $r:J\to J$. Let $T:A\to A$ be an almost local φ-contraction with the (c)-comparison function φ. Assume that the 1.26 monotonicity property is fulfilled for the pseudometrics d_j, for every $j\in J$.*
Then

(1) $Fix(T)=\{x\in A:Tx=x\}\neq\phi$;

(2) For any $x_0\in A$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by* $x_0\in A$ *and*

$$x_{n+1}=Tx_n,\quad n=0,1,2,...;\tag{1.82}$$

converges to a fixed point $x^*\in Fix(T)$;

(3) The a posteriori estimate

$$d_j(x_n,x^*)\leq s(d_j(x_n,x_{n+1})),\quad n=0,1,2...,\forall x,y\in A,\forall j\in J\tag{1.83}$$

holds, where $s(t)$ *is given by* (1.74).

Proof: We will prove that the set of fixed points of T is nonempty. Using the fact that T is an almost local φ-contraction, there exist a (c)-comparison function φ and some $L\geq 0$, such that

$$d_j(Tx,Ty)\leq\varphi(d_{r(j)}(x,y))+L\cdot d_{r(j)}(y,Tx),\forall x,y\in A,\forall j\in J\tag{1.84}$$

holds. Let $x_0\in A$ be arbitrary and $\{\mathrm{x}_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1}=Tx_n,\quad n\in\mathbb{N}.$$

Take $x := x_{n-1}, y := x_n$ in (1.84) to obtain

$$d_j(Tx_{n-1}, Tx_n) \leq \varphi(d_{r(j)}(x_{n-1}, x_n)) + L \cdot \underbrace{d_{r(j)}(x_n, Tx_{n-1})}_{=0},$$

which yields

$$d_j(x_n, x_{n+1}) \leq \varphi(d_{r(j)}(x_{n-1}, x_n)), \quad \forall j \in J, \forall n = 1, 2, ... \tag{1.85}$$

Since φ is increasing, the monotonicity of the pseudometrics leads to:

$$d_j(x_{n+1}, x_{n+2}) \leq \varphi(d_{r(j)}(x_n, x_{n+1})) \leq \varphi(d_j(x_n, x_{n+1})), \forall x, y \in A, \forall j \in J. \tag{1.86}$$

From that, we inductively obtain:

$$d_j(x_{n+k}, x_{n+k+1}) \leq \varphi^k(d_j(x_n, x_{n+1})), \forall x, y \in A, \forall j \in J. \tag{1.87}$$

According to the triangle inequality, we get:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq d_j(x_n, x_{n+1}) + d_j(x_{n+1}, x_{n+2}) + ... + d_j(x_{n+p-1}, x_{n+p}) \\ &\leq r + \varphi(r) + ... + \varphi^{n+p-1}(r), \forall x, y \in A, \forall j \in J, \end{aligned} \tag{1.88}$$

where we denoted $r = d_j(x_n, x_{n+1})$. Again, by using (1.86), we conclude

$$d_j(x_n, x_{n+1}) \leq \varphi^n(d_j(x_0, x_1)), n = 0, 1, 2, ... \tag{1.89}$$

which, by property (ii_φ) from the definition 1.3.83 of a comparison function, implies

$$\lim_{n\to\infty} d_j(x_n, x_{n+1}) = 0. \tag{1.90}$$

Having in view that φ is positive, it is obvious that

$$r + \varphi(r) + ... + \varphi^{n+p-1}(r) < s(r), \tag{1.91}$$

where $s(t)$ is the sum of the series

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(r), \quad t \in \mathbb{R}_+.$$

Then, by (1.88) and (1.91), we get

$$d_j(x_n, x_{n+p}) \leq s(d_j(x_n, x_{n+1})), n \in \mathbb{N}, p \in \mathbb{N}, \forall x, y \in A, \forall j \in J. \tag{1.92}$$

Since s is continuous at zero, (1.90) and (1.91) imply that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$. As the subset A is assumed to be sequentially τ-complete, there exists x^* in A such that the sequence $\{x_n\}$ is τ-convergent to x^*.
We prove that x^* is a fixed point of T. From the triangle inequality, we have:

$$\begin{aligned} d_j(x^*, Tx^*) &\leq d_j(x^*, x_{n+1}) + d_j(x_{n+1}, Tx^*) = \\ &= d_j(x_{n+1}, x^*) + d_j(Tx_n, Tx^*), \forall x, y \in A, \forall j \in J, \forall n \in \mathbb{N}. \end{aligned}$$

By (1.80), after applying the monotonicity of the pseudometrics and the monotonicity of the map φ, we obtain

$$\begin{aligned} d_j(Tx_n, Tx^*) &\leq \varphi(d_{r(j)}(x_n, x^*)) + L \cdot d_{r(j)}(x^*, Tx_n) \leq \\ &\leq \varphi(d_j(x_n, x^*)) + L \cdot d_j(x^*, Tx_n), \end{aligned}$$

for all $x, y \in A$, for every $j \in J$, $\forall n \in \mathbb{N}$. At this point, we merge the last two inequalities in order to obtain:

$$d_j(x^*, Tx^*) \leq (1 + L)d_j(x_{n+1}, x^*) + \varphi(d_j(x_n, x^*)), \tag{1.93}$$

valid for all $n = 0, 1, 2, ...$
Now letting $n \to \infty$ in (1.93) and using the continuity of φ at zero, we conclude

$$d_j(x^*, Tx^*) = 0,$$

which means that x^* is a fixed point of T. It follows that the Picard iteration converges to a fixed point $x^* \in Fix(T)$.
The estimate (1.83) follows from (1.91) by taking $p \to \infty$. □

Remark 1.3.96. *1) The a posteriori error estimates* (1.83) *and* (1.89) *lead us to the a priori estimate for the Picard iteration* $\{x_n\}_{n=0}^{\infty}$.
2) An almost local φ-contraction may have more than one fixed point, as shown by Example 1.3.94. In Theorem 1.3.95, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ provides the fixed point x^*, but it generally depends on the initial guess x_0.
For the uniqueness of the fixed point of T we need an additional condition, as shown in Theorem 1.3.97.

Theorem 1.3.97. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.3.95. Assume the monotonicity property 1.26 for the pseudometrics is fulfilled. Suppose T also satisfies the following condition: there exist a comparison function Υ and some $L_1 \geqslant 0$ such that*

$$d_j(Tx, Ty) \leq \Upsilon(d_{r(j)}(x, y)) + L_1 \cdot d_{r(j)}(x, Tx) \tag{1.94}$$

is valid for all $x, y \in A, \forall j \in J$.
Then

(1) T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;
(2) The a posteriori error estimate

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \quad n = 0, 1, 2, ...$$

holds, where $s(t)$ is given by (1.74)*;*
(3) The rate of the convergence of the Picard iteration is given by

$$d_j(x_n, x^*) \leq \Upsilon(d_j(x_{n-1}, x^*)), \quad n = 1, 2, ... \tag{1.95}$$

Proof: Suppose, by contradiction, there are two different fixed points x^* and y^* of T. Then from (1.94), by taking $x := x^*$ and $y := y^*$, we obtain

$$d_j(x^*, y^*) \le \Upsilon \underbrace{(d_{r(j)}(x^*, y^*))}_{\le d_j(x^*,y^*))} \le \Upsilon(d_j(x^*, y^*)), \forall x, y \in A, \forall j \in J,$$

which by induction with respect to n, yields

$$d_j(x^*, y^*) \le \Upsilon^n(d_j(x^*, y^*)), \quad \forall j \in J, \forall n = 1, 2, ... \tag{1.96}$$

Letting $n \to \infty$ in (1.96), we get

$$d_j(x^*, y^*) = 0, \quad \forall j \in J,$$

which means that $x^* = y^*$, a contradiction.
In this way, we proved that the fixed point is unique.
For the proof of (1.95), all we have to do is to change $x := x^*$ and $y := x_n$ in the inequality (1.94). □

Example 1.3.98. *Let us consider $X = [0, n] \times [0, n] \subset \mathbb{R}^2, n \in \mathbb{N}^*$. Consider the pseudometric:*

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot e^{-j}, \forall j \in J, \tag{1.97}$$

where J is a subset of $\mathbb{N}$. Consider $r(j) = j + 1$ and the mapping $T : X \to X$,

$$T(x, y) = \begin{cases} (x, \frac{y}{3}) & \text{if } (x,y) \neq (1, 1) \\ (0, 0) & \text{if } (x,y) = (1, 1) \end{cases}$$

Let the (c)-comparison function: $\varphi(t) = \frac{1}{4}t$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \forall t \in \mathbb{R}_+$.
T is an almost local φ-contraction if:

$$|x_1 - x_2| \cdot e^{-j} \le \frac{1}{4} \cdot |x_1 - x_2| \cdot e^{-(j+1)} + L \cdot |x_2 - x_1| \cdot e^{-(j+1)},$$

which is equivalent to : $e \le \frac{1}{4} + L$.
For $L = 3 \ge 0$, the last inequality becomes true, i.e., T is an almost local φ-contraction with an infinite set of fixed points: $Fix(T) = \{(x, 0) : x \in [0, n]\}$.

Condition (1.94) could be reformulated by using another pseudometric, providing a more general result.

Theorem 1.3.99. *Let X be a uniform Hausdorff space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J. Consider d_j, ϱ_j two pseudometrics on A such that (X, d_j) is τ-complete for every $j \in J$.*
Let $T : A \to A$ be a self operator satisfying

(*i*) *There exist a* (c)*-comparison function* φ *and* $L \geq 0$ *such that*

$$d_j(Tx, Ty) \leq \varphi(d_{r(j)}(x, y)) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in A, \forall j \in J;$$

(*ii*) *There exist a comparison function* Υ *and some* $L_1 \geq 0$ *such that*

$$\varrho_j(Tx, Ty) \leq \Upsilon(\varrho_{r(j)}(x, y)) + L_1 \cdot \rho_{r(j)}(x, Tx), \forall x, y \in A, \forall j \in J.$$

Assume that the subset A *is* T*-invariant. Then*

(1) T *has a unique fixed point, i.e.,* $Fix(T) = \{x^*\}$*;*

(2) For any $x_0 \in A$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by* $x_0 \in A$ *and*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, ...$$

converges to a fixed point $x^* \in Fix(T)$*;*

(3) The a posteriori error estimate

$$d_{r(j)}(x_n, x^*) \leq s(d_{r(j)}(x_n, x_{n+1})), \quad n = 0, 1, 2..., \forall x, y \in A, \forall j \in J$$

holds, where $s(t)$ *is given by* (1.74)*;*

(4) The rate of the convergence of the Picard iteration is given by

$$\varrho_j(x_n, x^*) \leq \Upsilon(\varrho_j(x_{n-1}, x^*)), \quad \forall x, y \in A, \forall j \in J, \forall n = 1, 2...$$

REMARK 1.3.100. *We obtain Theorem 1.3.97 as a particular case of Theorem 1.3.99 if we set* $d_j \equiv \varrho_j$.

In order to extend the class of almost local φ-contractions, we begin with the quasi φ-contractions, as shown below.

DEFINITION 1.3.101. *Let* X *be a uniform Hausdorff space,* $\mathcal{D} = (d_j)_{j \in J}$ *be a family of pseudometrics on* X*, where* J *is a set of indices. Let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$*. Consider a nonempty,* τ*-bounded, sequentially* τ*-complete subset* $A \subset X$*. Let* r *be a function from* J *to* J*. The mapping* $T : A \to A$ *is called quasi* φ*-contraction if there exists a comparison function* φ *such that:*

$$d_j(Tx, Ty) \leq \varphi(d_{r(j)}(x, y)), \forall x, y \in A, \forall j \in J. \tag{1.98}$$

THEOREM 1.3.102. *Assume that* $X, J, \mathcal{D}, r, \tau$ *and* A *are as in Definition 1.3.101, let* $T : A \to A$ *be a quasi* φ*-contraction: a mapping which satisfies* (1.98) *with the comparison function* φ*.*
If there exists $x_0 \in A$ *such that the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

is bounded, then T *has a unique fixed point.*

Proof: If the sequence of the successive approximations $\{\mathrm{x}_n\}_{n=0}^{\infty}$ is bounded, for a certain $x_0 \in A$, it means that there exist a constant $k > 0$ and an element $u \in A$ such that

$$d_j(x_n, u) \leq k, \quad \forall n \in \mathbb{N}; \forall j \in J. \tag{1.99}$$

By choosing $m, n \in \mathbb{N}$, we have from the triangle inequality:

$$d_j(x_n, x_m) \leq d_j(x_n, u) + d_j(u, x_m) \leq 2k, \quad \forall j \in J.$$

By replacing $j := r(j)$, we obtain from that inequality:

$$d_{r(j)}(x_n, x_m) \leq 2k, \quad \forall j \in J.$$

Note that the comparison function φ is monotone increasing, so we get

$$\varphi(d_{r(j)}(x_n, x_m)) \leq \varphi(2k), m, n \in \mathbb{N}, \forall j \in J.$$

By applying repeatedly the function φ, we inductively obtain:

$$\varphi^{n-1}(d_{r^{n-1}(j)}(x_n, x_m)) \leq \varphi^{n-1}(2k), \quad m, n \in \mathbb{N}, \forall j \in J.$$

From that, we replace $n := 1, m := p + 1$ to obtain

$$\varphi^{n-1}(d_{r^{n-1}(j)}(x_1, x_{p+1})) \leq \varphi^{n-1}(2k), n, p \in \mathbb{N}, \quad j \in J.$$

After using (1.98) and the definition of the Picard iteration, we inductively obtain

$$\begin{aligned} d_j(x_n, x_{n+p}) = d_j(Tx_{n-1}, Tx_{n+p-1}) \leq \varphi(d_{r(j)}(x_{n-1}, x_{n+p-1})) \leq \\ \leq \cdots \leq \varphi^{n-1}(d_{r^{n-1}(j)}(x_1, x_{p+1})) \leq \varphi^{n-1}(2k), \quad n, p \in \mathbb{N}, \end{aligned}$$

for every $j \in J$. Note that φ is a comparison function and according to (1.75), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$.
As the subset A is assumed to be sequentially τ-complete, there exists x^* in A such that $\{x_n\}_{n\in\mathbb{N}}$ is τ-convergent to x^*. Then

$$\lim_{n\to\infty} d_j(x_n, x^*) = 0.$$

By using again the triangle inequality, we obtain

$$\begin{aligned} 0 \leq d_j(Tx^*, x^*) \leq d_j(Tx^*, Tx_n) + d_j(Tx_n, x^*) \leq \\ \leq \varphi\Big(d_{r(j)}(x^*, x_n)\Big) + d_j(x_{n+1}, x^*). \end{aligned}$$

This means

$$d_j(Tx^*, x^*) = 0,$$

because the comparison function φ is continuous at zero. It follows that x^* is a fixed point of T, hence $x^* \in Fix(T)$. □

3.2. (B)-almost local contractions

In this subsection, our goal is to eliminate the inconvenience of having non-symmetric relations for the definition of almost local contractions, that is,

$$ALC_1 \qquad d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in A$$

and, respectively,

$$ALC_2 \qquad d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(x,Ty), \forall x,y \in A,$$

under the assumptions of Definition 1.1.31 given for almost local contractions, following the idea of Păcurar M. [**83**].

Remark 1.3.103. *If ALC_1 holds for some $\theta \in (0,1)$ and $L \geq 0$ for any $x,y \in A$, then its dual is valid for all $x,y \in A$ as well, and vice versa. That means, (ALC_1) and (ALC_2) are equivalent, since each inequality can be obtained from the other one by replacing $x := y$ or $y := x$.*

This inconvenient non-symmetry between conditions (ALC_1) and (ALC_2) can be eliminated by introducing the following results:

Lemma 1.3.104. *Let X be a uniform Hausdorff space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J. The operator $T : A \to A$ fulfills condition ALC_1 with respect to $(\mathcal{D}, r)$ if it satisfies condition*

$$(m - ALC) \qquad d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot min\{d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\},$$

for all $x,y \in A$ and for all $j \in J$.

Proof: Without loss of generality, we can assume that $L > 0$, as $L = 0$ would lead to the trivial case of Banach contractions.
" $\Rightarrow$ ": In the beginning, suppose that T satisfies (ALC_1). By using Remark 1.3.103, it also satisfies (ALC_2).
The inequality (ALC_1) becomes:

$$(1.100) \qquad \frac{1}{L}[d_j(Tx,Ty) - \theta \cdot d_{r(j)}(x,y)] \leq d_{r(j)}(y,Tx), \forall x,y \in A, \forall j \in J.$$

The other condition, ALC_2 take the equivalent form:

$$(1.101) \qquad \frac{1}{L}[d_j(Tx,Ty) - \theta \cdot d_{r(j)}(x,y)] \leq d_{r(j)}(x,Ty), \forall x,y \in A, \forall j \in J.$$

The inequalities (1.100) and (1.101) can be merged to obtain

$$\frac{1}{L}[d_j(Tx,Ty) - \theta \cdot d_{r(j)}(x,y)] \leq \min\{d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\}, \forall x,y \in A,$$

which is equivalent to $(m - ALC)$:

$$d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) + L \cdot min\{d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\},$$

for any $x, y \in A$ and $\forall j \in J$.
" $\Leftarrow$ " : Secondly, for the proof of the reciprocal assessment, suppose that T satisfies $(m - ALC)$. Then, obviously it also fulfills (ALC_1).
Hence, conditions (ALC_1) and $(m - ALC)$ are equivalent. $\square$

3.3. Approximate fixed points

The ε-fixed points of operators took a very constructive and practical approach of fixed point problems, since, in real situations, sometimes it is enough to obtain an approximation of the solution. So, the existence of fixed points is not strictly required, but the proximity of fixed points is of interest to researchers. This approximation is also used when the conditions imposed for the existence of the fixed points are too strong.

It is a natural consequence to introduce the concepts of *ε-fixed point* (or *approximate fixed point*), which, in fact, represents a proximity fixed point. After that, we introduce the function with the *approximate fixed point property* in order to establish qualitative and quantitative theorems for various types of ALC-s.

In this section, the starting point is represented by the article of Tijs, Torre and Branzei (see [**116**]) and also the paper of M. Berinde (see [**22**]).
Note that we consider operators on pseudometric spaces, not on metric or complete metric spaces, the usual framework for fixed point problems.
The following definition is very useful for the study of approximate fixed points. It was published in [**58**], followed by the work of Granas, Dugundji [**55**]:

DEFINITION 1.3.105. [**55**] *Let (E, τ) be a topological space, α an open covering of E and $f : E \to E$ an operator. Then $x \in E$ is called α-fixed point of f if there exist $U \in \alpha$ such that x and $f(x)$ are in U.*

The following definition was given by V.R. Klee [**73**] and it was mentioned also by van der Walt [**117**]:

DEFINITION 1.3.106. [**117**] *Let (E, τ) be a topological space, (X, d) be a metric space, $f : E \to X$ an operator, and $\varepsilon > 0$. Then f is called ε-continuous if each $x \in E$ has a neighborhood U such that*

$$\delta(f(U)) \leq \varepsilon,$$

where $\delta(f(U))$ represents the diameter of the set $f(U)$.

DEFINITION 1.3.107. [**116**] *Let (X, d) be a metric space.*
Let $f : X \to X, \quad \varepsilon > 0, x_0 \in X$.

Then x_0 *is an* ε*-fixed point (approximate fixed point) of* f *if*

$$d(f(x_0), x_0) < \varepsilon.$$

REMARK 1.3.108. *The set of all* ε*-fixed points of* f*, for a given* ε*, will be denoted by*

$$Fix_\varepsilon(f) = \{x \in X| \quad x \text{ is an } \varepsilon\text{-fixed point of } f\}.$$

REMARK 1.3.109. *Any fixed point of* f *is also an* ε*-fixed point of* f*:*

$$x \in Fix(f) \Rightarrow x \in Fix_\varepsilon(f),$$

but the converse is not always true, as shown in Example 3.1.1 *from* [**83**].

DEFINITION 1.3.110. [**116**] *Let* (X, d) *be a metric space. Let* $f : X \to X$*. Then* f *has the approximate fixed point property if*

$$\forall \varepsilon > 0, \quad Fix_\varepsilon(f) \neq \varnothing.$$

REMARK 1.3.111. *The concept of asymptotically regular operator was first introduced in* [**36**] *in metric spaces, see also* [**72**] *and* [**101**].

DEFINITION 1.3.112. [**36**] *Let* (X, d) *be a metric space.*
The operator $f : X \to X$ *is called asymptotic regular if*

$$d(f^n(x), f^{n+1}(x)) \to 0 \text{ as } \quad n \to \infty, \forall x \in X.$$

LEMMA 1.3.113. [**22**] *Let* (X, d) *be a metric space,* $f : X \to X$ *is an asymptotic regular operator. Then* f *has the approximate fixed point property.*

In order to obtain our main results of this section, Definition 1.3.107 will be extended to the case of pseudometrics in uniform spaces, as follows:

DEFINITION 1.3.114. *Let* X *be a uniform space.*
Let the operator $f : X \to X, \quad \varepsilon > 0, x_0 \in X$ *and consider* J *a family of indices.*
Then x_0 *is an* ε*-fixed point (approximate fixed point) of* f *if*

$$d_j(f(x_0), x_0) < \varepsilon, \quad \forall j \in J.$$

Lemma 1.3.113 could be extended to a larger, more general space, such as the pseudometric space, as in the following result:

LEMMA 1.3.115. *Let* X *be a uniform Hausdorff space,* $\mathcal{D} = (d_j)_{j \in J}$ *be a family of pseudometrics on* X*, where* J *is a set of indices. Let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$*. Consider a nonempty,* τ*-bounded, sequentially* τ*-complete subset* $A \subset X$*. Let* r *be a function from* J *to* J *and let* $T : A \to A$ *be an asymptotic regular mapping. Then* T *has the approximate fixed point property.*

Proof: Fix an element $x_0 \in A$. Since T is asymptotic regular, and having in view the fact that the convergence with respect to the τ- topology imply convergence for the pseudometrics d_j, we have

$$d_j(T^n(x_0), T^{n+1}(x_0)) \to 0 \text{ as } \quad n \to \infty,$$

which is equivalent with

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}^* \quad \text{such that} \quad \forall n \geq n_0(\varepsilon), d_j(T^n(x_0), \underbrace{T^{n+1}(x_0))}_{=T(T^n(x_0))} < \varepsilon.$$

Denote:

$$y_0^{(n)} = T^n(x_0).$$

It is easy to see that:

$$\forall \varepsilon > 0, \exists y_0^{(n)} \in A \quad \text{such that} \quad d_j(y_0^{(n)}, T(y_0^{(n)})) < \varepsilon, \quad \forall j \in J.$$

Thus, we have: for each $\varepsilon > 0$, there exists an ε-fixed point of T in the subset $A \subset X$, that is, $y_0^{(n)}$.
So, we prove that T has the approximate fixed point property. $\square$

Definition 1.3.116. [**86**] *Let the metric spaces X, Y and let $f : X \to Y$ be a mapping. Then f is called compact map at $x \in X$ if, for some ε-open ball $B(x, \varepsilon)$, $(\varepsilon > 0)$, $f(B(x, \varepsilon))$ is totally bounded in Y. The map f is called compact in X if f is compact at every point in X.*

Remark 1.3.117. *There is an equivalence between the existence of fixed points for a given mapping and the approximate fixed points of it, stated by the following result:*

Proposition 1.3.118. [**55**] *Let A be a closed subset of a metric space (X, d) and $T : A \to X$ be a compact map. Then T has a fixed point if and only if it has the approximate fixed point property.*

We will denote by $\delta(A)$ the diameter of the nonempty set A, namely:

$$\delta(A) = \sup\{d(x, y) | x, y \in A\}.$$

Lemma 1.3.119. *Let X be a uniform space, J be a set of indices, let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, with the monotonicity property* (1.26) *fulfilled. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a subset $A \subset X$ and let r be a function from J to J. Let $T : A \to A$ be an operator and $\varepsilon > 0$. Suppose that:*

(i) $Fix_\varepsilon(T) \neq \varnothing$;
(ii) $\forall \eta > 0, \quad \exists \varphi(\eta) > 0$ *such that*

$$d_{r(j)}(x, y) - d_{r(j)}(T(x), T(y)) \leq \eta \Rightarrow d_j(x, y) \leq \varphi(\eta), \forall x, y \in Fix_\varepsilon(T), \forall j \in J.$$

Then:

$$\delta(Fix_\varepsilon(T)) \leq \varphi(2\varepsilon).$$

Proof: Fix $\varepsilon > 0$ and $x, y \in Fix_\varepsilon(T)$. By means of definition 1.3.114, we have:

$$d_j(x, T(x)) < \varepsilon, \quad d_j(y, T(y)) < \varepsilon, \quad \forall j \in J.$$

By using the triangle inequality, we obtain:

$$d_{r(j)}(x, y) \le \underbrace{d_{r(j)}(x, T(x))}_{\le d_j(x, T(x))} + d_{r(j)}(T(x), T(y)) + \underbrace{d_{r(j)}(y, T(y))}_{\le d_j(y, T(y))} \le d_{r(j)}(T(x), T(y)) + 2\varepsilon.$$

This implies:

$$d_{r(j)}(x, y) - d_{r(j)}(T(x), T(y)) \le 2\varepsilon.$$

At this point, it follows from (ii):

$$d_j(x, y) \le \varphi(2\varepsilon),$$

which means that

$$\delta(Fix_\varepsilon(T)) \le \varphi(2\varepsilon).$$

□

Remark 1.3.120. *According to Lemma 1.3.115, condition (i) from Lemma 1.3.119 can be replaced by the asymptotic regularity condition.*

Lemma 1.3.119 can be reformulated as follows:

Lemma 1.3.121. *Let X be a uniform space, J be a set of indices, let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, with the monotonicity property* (1.26) *fulfilled. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a subset $A \subset X$ and let r be a function from J to J. Let $T : A \to A$ be an operator such that for $\varepsilon > 0$ the following conditions hold:*

(i)

$$d_j(T^n(x), T^{n+1}(x)) \to 0 \text{ as } \quad n \to \infty, \forall x \in A, \forall j \in J;$$

(ii) $\forall \eta > 0, \quad \exists \varphi(\eta) > 0$ *such that for every $j \in J$ we have:*

$$d_{r(j)}(x, y) - d_{r(j)}(T(x), T(y)) \le \eta \Rightarrow d_j(x, y) \le \varphi(\eta), \forall x, y \in Fix_\varepsilon(T).$$

Then:

$$\delta(Fix_\varepsilon(T)) \le \varphi(2\varepsilon).$$

A. Qualitative results for mappings in pseudometric spaces

Our first goal is to state and prove, using Lemma 1.3.115, qualitative results for a large class of operators defined on pseudometric spaces. Most importantly, we are interested in establishing conditions under which the approximate fixed point property is fulfilled for the mappings considered.

THEOREM 1.3.122. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Assume that the* (1.26) *monotonicity property is fulfilled for the pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J and let $T : A \to A$ be a quasi-almost local contraction (see Definition 1.3.72).*
Then:

$$\forall \varepsilon > 0, \quad Fix_\varepsilon(T) \neq \varnothing.$$

Proof: Take $\varepsilon > 0$ and $x \in A$.

$$d_j(T^n(x), T^{n+1}(x)) = d_j(T(T^{n-1}(x), T(T^n(x))) \leq$$
$$\leq \theta \cdot M_{r(j)}(T^{n-1}(x), T^n(x)) \leq \cdots \leq \theta^n M_{r^n(j)}(x, T(x)), \quad \forall n \in \mathbb{N},$$

where

$$M_{r^n(j)}(x, T(x)) = \Big\{d_{r^n(j)}(x, T(x)), d_{r^n(j)}(x, T(x)),$$
$$d_{r^n(j)}(T(x), T^2(x)), d_{r^n(j)}(x, T^2(x)), d_{r^n(j)}(T(x), T(x))\Big\}.$$

Having in view the monotonicity of the pseudometrics, we can write:

$$d_j(x, T(x)) \geq d_{r(j)}(x, T(x)) \geq \cdots \geq d_{r^n(j)}(x, T(x)), \forall j \in J,$$

which means that $M_{r^n(j)}(x, T(x))$ is finite, therefore

$$\lim_{n\to\infty} \theta^n \cdot M_{r^n(j)}(x, T(x)) = 0, \quad \forall j \in J.$$

Note that $\theta \in (0, 1)$, which implies $\theta^n \to 0$ as $n \to \infty$.
Hence, we have:

$$d_j(T^n(x), T^{n+1}(x)) \to 0, \text{ as } n \to \infty, \forall x \in A, \forall j \in J.$$

According to Lemma 1.3.115, it follows that $Fix_\varepsilon(T) \neq \varnothing$, $\forall \varepsilon > 0$. $\square$

THEOREM 1.3.123. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, with the monotonicity property fulfilled for the pseudometrics. Denote by J a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J. Let $T : A \to A$ be a β-local contraction: a mapping for which there exists $\beta \in (0, 1)$ such that:*

$$d_j(Tx, Ty) \leq \beta \cdot d_{r(j)}(x, y), \text{ for all } x,y \in A, \forall j \in J. \tag{1.102}$$

Then:

$$\forall \varepsilon > 0, \quad Fix_\varepsilon(T) \neq \varnothing.$$

Proof: Let $\varepsilon > 0$ and $x \in A$. By using the definition of a local contraction, we obtain

$$\begin{aligned} d_j(T^n(x), T^{n+1}(x)) &= d_j(T(T^{n-1}(x), T(T^n(x))) \leq \\ &\leq \beta \cdot d_{r(j)}(T^{n-1}(x), T^n(x)) \leq \cdots \leq \beta^n \cdot \underbrace{d_{r^n(j)}(x, T(x))}_{finite}. \end{aligned}$$

We use the same method as used in Theorem 1.3.122 in order to show that $d_{r^n(j)}(x, T(x))$ is finite. Note that $\beta \in (0, 1)$, which implies that we have:

$$d_j(T^n(x), T^{n+1}(x)) \to 0, \text{ as } n \to \infty, \forall x \in A, \forall j \in J.$$

According to Lemma 1.3.115, it follows that $Fix_\varepsilon(T) \neq \varnothing$, $\forall \varepsilon > 0$. □

Remark 1.3.124. *In fact, Theorem 1.3.123 is a corollary of Theorem 1.3.122, by considering*

$$d_j(Tx, Ty) \leq \beta \cdot d_{r(j)}(x, y) \leq \beta \cdot M_{r(j)}(x, y), \textit{for all } x,y \in A, \forall j \in J.$$

Therefore, Theorem 1.3.122 can be applyed to obtain the conclusion of Theorem 1.3.123.

Theorem 1.3.125. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, with the monotonicity property fulfilled for the pseudometrics. Denote by J a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J. Let $T : A \to A$ be a Chatterjea type ALC (see Definition 1.3.77) with the constant $0 \leq c < \frac{1}{2}$.*
Then:

$$\forall \varepsilon > 0, \quad Fix_\varepsilon(T) \neq \varnothing.$$

Proof: Let $\varepsilon > 0$ and $x \in A$.
By using the monotonicity (1.26) of the pseudometrics, we obtain:

$$\begin{aligned} & d_j(T^n(x), T^{n+1}(x)) = d_j(T(T^{n-1}(x), T(T^n(x))) \leq \\ \leq \;& c \cdot [d_{r(j)}(T^{n-1}(x), T(T^n(x))) + d_{r(j)}(T^n(x), T(T^{n-1}(x)))] = \\ = \;& c \cdot [d_{r(j)}(T^{n-1}(x), T^{n+1}(x)) + \underbrace{d_{r(j)}(T^n(x), T^n(x))}_{=0}] = \\ = \;& c \cdot d_{r(j)}(T^{n-1}(x), T^{n+1}(x)). \end{aligned}$$

By using the triangle inequality and the monotonicity property, we get:

$$\begin{aligned} d_{r(j)}(T^{n-1}(x), T^{n+1}(x)) &\leq d_{r(j)}(T^{n-1}(x), T^n(x)) + d_{r(j)}(T^n(x), T^{n+1}(x)) \leq \\ &\leq d_{r(j)}(T^{n-1}(x), T^n(x)) + d_j(T^n(x), T^{n+1}(x)) \Rightarrow \\ &\Rightarrow (1 - c) d_j(T^n(x), T^{n+1}(x)) \leq c \cdot d_{r(j)}(T^{n-1}(x), T^n(x)), \end{aligned}$$

which is equivalent to:

$$\begin{aligned} d_j(T^n(x), T^{n+1}(x)) &\leq \frac{c}{1-c} d_{r(j)}(T^{n-1}(x), T^n(x)) \leq \cdots \leq \\ &\leq \Big(\frac{c}{1-c}\Big)^n \underbrace{d_{r^n(j)}(x, T(x))}_{finite}, \end{aligned}$$

by using again the arguments from Theorem 1.3.122. Taking into account that $c \in [0, \frac{1}{2})$, we have $\frac{c}{1-c} \in [0,1)$, which means: $\Big(\frac{c}{1-c}\Big)^n \to 0$ as $n \to \infty$.
Therefore,

$$\lim_{n\to\infty} \Big(\frac{c}{1-c}\Big)^n \cdot d_{r^n(j)}(x, T(x)) = 0, \quad \forall j \in J.$$

We obtain:

$$d_j(T^n(x), T^{n+1}(x)) \to 0, \text{ as } n \to \infty, \forall x \in A, \forall j \in J.$$

According to Lemma 1.3.115, it follows that $Fix_\varepsilon(T) \neq \varnothing, \quad \forall \varepsilon > 0.$ □

Theorem 1.3.126. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.3.125, let $T : A \to A$ be an almost local contraction (see Definition 1.1.31).*
Then:

$$\forall \varepsilon > 0, \quad Fix_\varepsilon(T) \neq \varnothing.$$

Proof: Let $\varepsilon > 0$ and $x \in A$.
By using the definition of the almost local contractions, we obtain:

$$\begin{aligned} & d_j(T^n(x), T^{n+1}(x)) = d_j(T(T^{n-1}(x), T(T^n(x))) \leq \\ \leq\ & \theta \cdot d_{r(j)}(T^{n-1}(x), T^n(x)) + L d_{r(j)}(T^n(x), T(T^{n-1}(x))) = \\ =\ & \theta \cdot d_{r(j)}(T^{n-1}(x), T^n(x)) \leq \cdots \leq \theta^n \cdot \underbrace{d_{r^n(j)}(x, T(x))}_{finite}, \forall j \in J. \end{aligned}$$

Let us remind that $\theta \in (0, 1)$, which implies that $\theta^n \to 0$ as $n \to \infty$. At this point, we conclude that we have:

$$d_j(T^n(x), T^{n+1}(x)) \to 0, \text{ as } n \to \infty, \forall x \in A, \forall j \in J.$$

According to Lemma 1.3.115, it follows that $Fix_\varepsilon(T) \neq \varnothing$, $\forall \varepsilon > 0$. □

Example 1.3.127. *Let us consider $X = [0, 1]$ with the pseudometrics:*

$$d_j(x, y) = |x - y| \cdot e^{-j}, \forall j \in J, \forall x, y \in X,$$

and let $T : [0, 1] \to \{0, 1\}$ defined by

$$T(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases}$$

Consider the function $r(j) = j + 1, \forall j \in J$, where J is a subset of $\mathbb{N}$.
Then:

(1) *$Fix(T) = \{0, 1\}$;*

(2) $Fix_\varepsilon(T) = [0, \varepsilon \cdot e^j]$, for any $\varepsilon > 0$ and some $j \in J$.

The mapping T is an ALC because:

- the ALC condition for every $x, y \in [0,1), x \neq y$ becomes:

$$|0-0| \cdot e^{-j} \leq (\theta \cdot |x-y| + L \cdot |y-0|) \cdot e^{-(j+1)},$$

which is obviously true;

- the ALC condition becomes for every $x \in [0,1), y = 1$:

$$|1-0| \cdot e^{-j} \leq (\theta \cdot |x-1| + L \cdot |1-0|) \cdot e^{-(j+1)},$$

which leads us to the conclusion

$$e \leq \theta \cdot |x-1| + L.$$

The last inequality holds for some $\theta \in (0,1)$ and for some $L \geq 3$.

Conclusion (001) is obvious, since:

- for $x \in [0,1)$, the fixed point condition $T(x) = x$ leads us to $x = 0$;

- for $x = 1$, $T(x) = x$ leads us to $x = 1$.

2) Let $\varepsilon > 0, j \in J$ fixed and select x an ε-fixed point of T, which means

$$|x - T(x)| \cdot e^{-j} \leq \varepsilon. \tag{1.103}$$

According to the definition of T, we have two cases:

a) If $x \in [0,1)$, in this case we have $T(x) = 0$, which implies from (1.103) :

$$|x-0| \cdot e^{-j} \leq \varepsilon \Leftrightarrow x \leq \varepsilon \cdot e^j \Leftrightarrow x \in [0, \varepsilon \cdot e^j].$$

b) If $x = 1$, by using (1.103), we get

$$|1-1| \leq \varepsilon \Leftrightarrow \varepsilon \geq 0.$$

The conditions from Theorem 1.3.126 are fulfilled, therefore can be applied. Thus, we have: $Fix_\varepsilon(T) = [0, \varepsilon \cdot e^j]$, for any $\varepsilon > 0$.

B. Quantitative results for mappings in pseudometric spaces

In the sequel, our main goal is to establish some quantitative results regarding the studied operators, using Lemma 1.3.119.

THEOREM 1.3.128. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X with the monotonicity property fulfilled for the pseudometrics, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let r be a function from J to J and let $T : A \to A$ be a β-local contraction (see (1.102)).*

Then:

$$\delta(Fix_\varepsilon(T)) \leq \frac{2\varepsilon}{1-\beta}, \quad \forall \varepsilon > 0.$$

Proof: Let $\varepsilon > 0$. Condition (i) from Lemma 1.3.119 was proved in Theorem 1.3.123.
It remain to show that (ii) holds for a-contractions.
Fix $\gamma > 0$ and $x, y \in Fix_\varepsilon(T)$. Assume that

$$d_j(x,y) - d_j(T(x),T(y)) \leq \gamma.$$

In order to obtain the conclusion, we will prove that there exists an $\varphi(\gamma) > 0$ such that $d_j(x,y) \leq \varphi(\gamma)$. It results, after using the monotonicity property:

$$d_j(x,y) \leq d_j(T(x),T(y)) + \gamma \leq \beta \cdot d_{r(j)}(x,y) + \gamma \leq \beta \cdot d_j(x,y) + \gamma.$$

Now, we can conclude that

$$(1-\beta)d_j(x,y) \leq \gamma,$$

which implies $d_j(x,y) \leq \frac{\gamma}{1-\beta}$.
Ergo, we have that: $\forall \gamma > 0, \exists \varphi(\gamma) = \frac{\gamma}{1-\beta}$ such that

$$d_j(x,y) - d_j(T(x),T(y)) \leq \gamma \Rightarrow d_j(x,y) \leq \varphi(\gamma).$$

Now, by Lemma 1.3.119, we have:

$$\delta(F_\varepsilon(T)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0.$$

The last inequality actually means that

$$\delta(Fix_\varepsilon(T)) \leq \frac{2\varepsilon}{1-a\beta}, \quad \forall \varepsilon > 0.$$

□

THEOREM 1.3.129. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X with the monotonicity property fulfilled for the pseudometrics, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $T : A \to A$ be a Chatterjea type ALC.*

Then:

$$\delta(Fix_\varepsilon(T)) \leq \frac{2\varepsilon(1+c)}{1-2c}, \quad \forall \varepsilon > 0.$$

Proof: Let $\varepsilon > 0$. Again, condition (i) from Lemma 1.3.119 is satisfied, according to the proof of Theorem 1.3.125.
It remain to show that (ii) holds for Chatterjea type ALC-s.
Let $\gamma > 0$ and $x, y \in Fix_\varepsilon(T)$. Assume that

$$d_j(x,y) - d_j(T(x),T(y)) \leq \gamma.$$

Then we have

$$\begin{aligned} d_j(x,y) &\leq \gamma + d_j(T(x),T(y)) \leq \gamma + cd_{r(j)}(x,T(y)) + cd_{r(j)}(y,T(x)) \leq \\ &\leq \gamma + cd_j(x,T(y)) + cd_j(y,T(x)) \leq \gamma + c[d_j(x,y) + d_j(y,T(y))] + \\ &+ c[d_j(y,x) + d_j(x,T(x))]. \end{aligned}$$

But since x, y are approximate fixed points of T, it results that

$$d_j(x,y) \leq 2c \cdot d_j(x,y) + 2\varepsilon \cdot c + \gamma, \quad \forall j \in J.$$

After rearranging the terms, we get

$$(1-2c)d_j(x,y) \leq 2\varepsilon \cdot c + \gamma,$$

and then we obtain

$$d_j(x,y) \leq \frac{\gamma + 2\varepsilon \cdot c}{1-2c}.$$

Hence for each $\gamma > 0$, there exists $\varphi(\gamma) = \frac{\gamma+2\varepsilon \cdot c}{1-2c}$ such that $d_j(x,y) - d_j(T(x),T(y)) \leq \gamma$, which implies $d_j(x,y) \leq \varphi(\gamma), \forall j \in J$.
Again, by Lemma 1.3.119 it results that

$$\delta(Fix_\varepsilon(T)) \leq \varphi(2\varepsilon), \quad \forall \varepsilon > 0.$$

The last inequality leads us to the conclusion:

$$\delta(Fix_\varepsilon(T)) \leq \frac{2\varepsilon(1+c)}{1-2c}, \quad \forall \varepsilon > 0.$$

□

THEOREM 1.3.130. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X with the monotonicity property fulfilled for the pseudometrics, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $T : A \to A$ be an almost local contraction with $\theta + L < 1$.*

Then:

$$\delta(Fix_\varepsilon(T)) \leq \frac{2+L}{1-\theta-L} \cdot \varepsilon, \quad \forall \varepsilon > 0.$$

Proof: Let $\varepsilon > 0$. Again, condition (i) from Lemma 1.3.119 holds.
It remain to show that (ii) holds for ALC-s.
Let $\gamma > 0$ and $x, y \in Fix_\varepsilon(T)$. Assume that

$$d_j(x,y) - d_j(T(x),T(y)) \leq \gamma.$$

By using the triangle inequality and the monotonicity property for the pseudometrics, we get

$$\begin{aligned} d_j(x,y) &\leq \gamma + d_j(T(x),T(y)) \leq \gamma + \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(y,T(x)) \leq \\ &\leq \gamma + \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(x,T(x)) \leq \\ &\leq \gamma + \theta \cdot d_j(x,y) + L \cdot d_j(x,y) + L \cdot d_j(x,T(x)) \leq \\ &\leq (\theta + L)d_j(x,y) + L\varepsilon + \gamma, \quad \forall j \in J. \end{aligned}$$

Thus, we have:

$$(1-\theta-L)d_j(x,y) \leq L\varepsilon + \gamma, \text{ which implies } d_j(x,y) \leq \frac{L\varepsilon+\gamma}{1-\theta-L}, \quad \forall j \in J.$$

Therefore, for all $\gamma > 0$, $\quad \exists \varphi(\gamma) = \frac{L\varepsilon+\gamma}{1-\theta-L}$ such that

$$d_j(x,y) - d_j(T(x),T(y)) \leq \gamma \Rightarrow d_j(x,y) \leq \varphi(\gamma), \quad \forall j \in J.$$

Again, by Lemma 1.3.119 it follows that

$$\delta(Fix_\varepsilon(T)) \leq \varphi(2\varepsilon), \quad \forall \varepsilon > 0.$$

The last inequality leads us to the conclusion:

$$\delta(Fix_\varepsilon(T)) \leq \frac{2+L}{1-\theta-L} \cdot \varepsilon, \quad \forall \varepsilon > 0.$$

□

REMARK 1.3.131. *Theorems 1.3.128, 1.3.129, 1.3.130 represent generalizations in uniform spaces of results established in* [**22**] *and* [**83**] *regarding the study of approximate fixed points in metric spaces.*

3.4. Almost local contractions in b-pseudometric spaces

In this subsection, the notion of almost local contraction in a b-pseudometric space is considered. In this framework some new fixed point results are given. A large number of generalizations for the concept of metric space were given by several authors, the most important of them: [**7**], [**37**], [**103**] and recent works, amongst which we mention [**34**], [**35**]. The concept of b-metric space was introduced by Czerwik in [**46**]. Since then several publications were devoted to the study of single valued and multivalued operators in b-metric spaces (see [**12**], [**65**], [**87**], [**111**]). The starting point of this part was the book of M. Păcurar [**83**], who researched thoroughly the question of fixed points in b-metric spaces, but only for the almost contractions.

DEFINITION 1.3.132. [**103**] *Let X be a nonempty set.*
A mapping $d_b : X \times X \to \mathbb{R}_+$ is called b-metric if the following conditions hold:

(1) $d_b(x,y) = 0$ if and only if $x = y$;
(2) $d_b(x,y) = d_b(y,x), \forall x,y \in X$;

(3) $d_b(x,z) \le b \cdot [d_b(x,y) + d_b(y,z)], \forall x,y,z \in X$,
where $b \ge 1$ *is a given real number.*

A nonempty set X *endowed with a b-metric* $d_b : X \times X \to \mathbb{R}_+$ *is called b-metric space.*

EXAMPLE 1.3.133. [**7**] *Let* $p \in (0,1)$ *and let* l_p *be the space of all real sequences* $\{x_n\}_{n\ge 0} \subset \mathbb{R}$ *such that*

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

Let $d : l_p \times l_p \to \mathbb{R}_+$ *defined by*

$$d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}},$$

for any $x = \{x_n\}_{n\ge 0}$, $y = \{y_n\}_{n\ge 0}$. *Then* d *is a b-metric on* l_p *with constant* $b = 2^{\frac{1}{p}} > 1$, *hence* (l_p, d) *is a b-metric space.*

REMARK 1.3.134. *In the sequel, our goal is to extend the b-metrics to the more general case of b-pseudometrics, as follows:*

DEFINITION 1.3.135. *Let* X *be a nonempty set and let* $b \ge 1$ *a given real number. A mapping* $d_b : X \times X \to \mathbb{R}_+$ *is called b-pseudometric if the following conditions hold:*

(1) $d_b(x,x) = 0, \forall x \in X$;
(2) $d_b(x,y) = d_b(y,x), \forall x,y \in X$;
(3) $d_b(x,z) \le b \cdot [d_b(x,y) + d_b(y,z)], \forall x,y,z \in X$.

A nonempty set X *endowed with a b-pseudometric* $d_b : X \times X \to \mathbb{R}_+$ *is called b-pseudometric space.*

In the beginning of this subsection, we will present a set of concepts which help us to establish a theory of some fixed point theorems, related to a various type of almost local contractions in b-pseudometric spaces.
The following concepts are given in metric spaces, they appear originally in [**104**], but we will present them in the framework of a b-pseudometric space.

DEFINITION 1.3.136. *Let* (X, d_b) *be a b-pseudometric space and* $f : X \to X$ *a (weakly) Picard operator. Then* f *is said to be a good (weakly) Picard operator if*

$$\sum_{n\in\mathbb{N}} d_b\Big(f^n(x), f^{n+1}(x)\Big) < \infty,$$

for any $x \in X$.

REMARK 1.3.137. *For the particular case* $b = 1$, *we get from Definition 1.3.136 the well-known definition of good (weakly) Picard operator in a metric space.*

Definition 1.3.138. *Let (X, d_b) be a b-pseudometric space and $f : X \to X$ a (weakly) Picard operator. Then f is said to be a special (weakly) Picard operator if*

$$\sum_{n\in\mathbb{N}} d_b\Big(f^n(x), f^\infty(x)\Big) < \infty,$$

for any $x \in X$, where

$$f^\infty : X \to X, f^\infty(x) = \lim_{n\to\infty} f^n(x), \quad \forall x \in X. \tag{1.104}$$

Moreover, let $(X, d_{b,j})$ a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a family of indices. Then $f : X \to X$ is said to be a special (weakly) Picard operator if

$$\sum_{n\in\mathbb{N}} d_{b,j}\Big(f^n(x), f^\infty(x)\Big) < \infty,$$

for any $x \in X, \forall j \in J$.

Remark 1.3.139. *For the particular case $b = 1$, from Definition 1.3.138, we get the well-known definition of a special (weakly) Picard operator in a metric space.*

The property of well-posedness for a fixed point problem for an operator proposed in [**49**] was studied also in [**77**], [**87**] and [**105**]. In a b-metric space was introduced in [**83**] by M.Păcurar. We introduce it in a b-pseudometric space as follows:

Definition 1.3.140. *Let (X, d_b) be a b-pseudometric space and let $f : X \to X$ a Picard operator with $Fix(f) = \{x^*\}$. Suppose there exist $z_n \in X, n \in \mathbb{N}$ such that*

$$d_b(z_n, f(z_n)) \to 0 \text{ as } n \to \infty.$$

If this implies

$$z_n \to x^* \text{ as } n \to \infty,$$

then we say that the fixed point problem for the operator f is well posed.
Moreover, let $(X, d_{b,j})$ a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a family of indices. Let $f : X \to X$ be a Picard operator with $Fix(f) = \{x^*\}$. Suppose there exist $z_n \in X, n \in \mathbb{N}$ such that

$$d_{b,j}(z_n, f(z_n)) \to 0 \text{ as } n \to \infty.$$

If this implies

$$z_n \to x^* \text{ as } n \to \infty,$$

then we say that the fixed point problem for the operator f is well posed.

Remark 1.3.141. *For the particular case $b = 1$, from Definition 1.3.140 we obtain the well-known definition of a well posed fixed point problem in a metric space.*

Definition 1.3.142. *Let (X, d_b) be a b-pseudometric space and $f : X \to X$ an operator. Suppose there exist $z_n \in X, n \in \mathbb{N}$ such that*

$$d_b(z_{n+1}, f(z_n)) \to 0 \text{ as } n \to \infty.$$

If there exists $x \in X$ such that

$$d_b(z_n, f^n(x)) \to 0 \text{ as } n \to \infty,$$

then we say that the operator f has the the limit shadowing property.
Moreover, let $(X, d_{b,j})$ a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a family of indices. Let $f : X \to X$ an operator. Suppose there exist $z_n \in X, n \in \mathbb{N}$ such that

$$d_{b,j}(z_{n+1}, f(z_n)) \to 0 \text{ as } n \to \infty.$$

If there exists $x \in X$ such that

$$d_{b,j}(z_n, f^n(x)) \to 0 \text{ as } n \to \infty,$$

then we say that the operator f has the limit shadowing property.

Remark 1.3.143. *For the particular case $b = 1$, from Definition 1.3.142 we obtain the usual definition of limit shadowing property for an operator on a metric space.*

Example 1.3.144. [**104**] *Let (X, d) be a complete metric space and $f : X \to X$ a Banach contraction with $\alpha \in [0, 1)$. Then f is a good Picard operator, the fixed point problem for the operator f is well posed and f has the limit shadowing property.*

3.5. φ-contractions in b-pseudometric spaces

The class of φ-contractions have been studied in uniform spaces, see section 3.1 in chapter 1. In the sequel, we intend to analyze them in the more general case of b-pseudometric spaces, as it follows:

Theorem 1.3.145. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property fulfilled for the b-pseudometrics, namely:*

$$d_{b,r(j)}(x, y) \leq d_{b,j}(x, y), \forall x, y \in X, \forall j \in J. \tag{1.105}$$

Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function and let $f : A \to A$ be a quasi φ-contraction, as in (1.98)*.*
Then f has a unique fixed point if and only if there exists $x_0 \in X$ such that the Picard iteration $\{x_n\}_{n\geq 0}$ given by (0.2) *is bounded.*

Proof: a) In order to simplify the proof, we denote the pseudometrics $d_{b,j}$ with d_j, for every $j \in J$.
" $\Longrightarrow$ " If f has a unique fixed point x^*, then by choosing $x_0 = x^*$, the sequence of successive approximations is bounded, being constant.
b)" $\Longleftarrow$ " Assume $\{x_n\}$ is bounded for a certain $x_0 \in X$. Consequently, there exist a constant $c > 0$ and an element $y \in X$ such that

$$d_j(x_n, y) \leq c, \quad \forall n \in \mathbb{N}, \forall j \in J.$$

For $m, n \in \mathbb{N}$, by using the definition of the b-pseudometric, we get

$$d_j(x_n, x_m) \leq b \cdot [d_j(x_n, y) + d_j(y, x_m)] \leq 2b \cdot c, \quad \forall j \in J.$$

Let us remind that φ is monotone increasing, then, by using the condition for the quasi φ-contraction, we obtain

$$\begin{aligned} d_j(x_n, x_{n+p}) &= d_j(f(x_{n-1}), f(x_{n+p-1})) \overset{mon.}{\leq} \varphi(d_{r(j)}(x_{n-1}, x_{n+p-1})) \leq \\ &\leq \cdots \leq \varphi^{n-1}(d_{r^n(j)}(x_1, x_{p+1})) \leq \varphi^{n-1}(2b \cdot c), \quad n, p \in \mathbb{N}, \end{aligned}$$

for all $j \in J$.

These relations show us that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$. As the subset A is assumed to be sequentially τ-complete, there exists x^* in A such that $\{f^n(x)\}_{n\in\mathbb{N}}$ is τ-convergent to x^*. The sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to x^*, which implies

$$x^* = \lim_{n\to\infty} x_n \Rightarrow \lim_{n\to\infty} d_j(x_n, x^*) = 0, \quad \forall j \in J.$$

By using the definition of a quasi φ-contraction, we obtain:

$$\begin{aligned} 0 &\leq d_j(f(x^*), x^*)) \leq b \cdot [d_j(f(x^*), f(x_n)) + d_j(f(x_n), x^*)] \leq \\ &\leq b \cdot [\varphi(d_{r(j)}(x^*, x_n)) + d_j(x_{n+1}, x^*)]. \end{aligned}$$

Having in view that φ is continuous at zero, we deduce: $d_j(f(x^*), x^*) = 0, \forall j \in J$. This actually means that x^* is a fixed point of the operator f. The uniqueness of the fixed point is based on reductio ad absurdum method. □

In the sequel, let us consider the real number $d \geq 1$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function for which there exists the real $0 \leq \alpha < 1$ and a series of positive terms $\sum\limits_{n=0}^{\infty} v_n$, which is convergent (see [**12**]) such that we have

$$b^{k+1}\varphi^{k+1}(t) \leq \alpha \cdot b^k \cdot \varphi^k(t) + v_k, \forall t \in \mathbb{R}, \tag{1.106}$$

for each $k \geq N$, with N fixed.
By using (1.106), the series

$$\sum_{k=0}^{\infty} b^k \varphi^k(t) \tag{1.107}$$

converges for each $t \in \mathbb{R}_+$ and its sum, denoted by $s_b(t)$ is monotone increasing and continuous at zero.

THEOREM 1.3.146. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let the function $r : J \to J$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function and let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition* (1.106). *If $x_0 \in A$ is chosen such that the sequence of successive approximations is bounded and $Fix(f) = \{x^*\}$, then we have*

$$d_j(x_n, x^*) \le b \cdot s_b(d_{r(j)}(x_n, x_{n+1})), \quad n \ge 0, \forall j \in J, \tag{1.108}$$

where $s_b(t)$ is the sum of the series (1.107).

Proof: In order to simplify the proof, we denote the pseudometrics $d_{b,j}$ with d_j for every $j \in J$. By means of contractive condition (1.98), we obtain for $n \ge 1$,

$$d_j(x_n, x_{n+1}) = d_j(f(x_{n-1}), f(x_n)) \le \varphi(d_{r(j)}(x_{n-1}, x_n)).$$

For $n \ge 2$, we can write

$$d_j(x_{n-1}, x_n) \le \varphi(d_{r(j)}(x_{n-2}, x_{n-1})). \tag{1.109}$$

Having in view that the comparison function φ is monotone increasing, it results

$$\begin{aligned} d_j(x_n, x_{n+p}) &\le b \cdot [d_j(x_n, x_{n+1}) + d_j(x_{n+1}, x_{n+p})] \le \\ &\le b \cdot d_j(x_n, x_{n+1}) + b^2[d_j(x_{n+1}, x_{n+2}) + d_j(x_{n+2}, x_{n+p})] \le \\ &\le \cdots \le b \cdot d_j(x_n, x_{n+1}) + b^2 d_j(x_{n+1}, x_{n+2}) + \cdots + b^p d_j(x_{n+p-1}, x_{n+p}), \end{aligned}$$

which yields:

$$d_j(x_n, x_{n+p}) \overset{(1.109)}{\le} b \sum_{k=0}^{p-1} b^k \varphi^k(d_{r(j)}(x_n, x_{n+1})). \tag{1.110}$$

For $k = 0$, the right-hand side term becomes: $b \cdot d_{r(j)}(x_n, x_{n+1})$.
Then, by taking $p \to \infty$ in (1.110), we get

$$d_j(x_n, x^*) \le b \cdot s_b(d_{r(j)}(x_n, x_{n+1})), \quad n \ge 0. \tag{1.111}$$

$\square$

THEOREM 1.3.147. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be*

a b-comparison function. Let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition (1.106)*.*
Then:

(1) f is a Picard operator;

(2) if $\{x_n\}_{n\in\mathbb{N}}$ denote the sequence of successive approximations, with

$$x^* = \lim_{n\to\infty} x_n,$$

the following a priori and a posteriori estimates are available:

$$d_j(x_n, x^*) \leq b \cdot s_b(\varphi^n(d_{r(j)}(x_0, x_1))), n \geq 0, \tag{1.112}$$

$$d_j(x_n, x^*) \leq b \cdot s_b(d_j(x_n, x_{n+1})), n \geq 0, \tag{1.113}$$

where $s_b(t)$ is the sum of the series (1.107)*.*

(3)

$$d_j(x, x^*) \leq b \cdot s_b(d_j(x, f(x))), \quad \forall x \in A, \forall j \in J. \tag{1.114}$$

Proof: In order to simplify the proof, we denote the pseudometrics d_{b_j} with d_j, for every $j \in J$.
1) Fix $x_0 \in A$ and consider the sequence of successive approximations $x_n = f(x_{n-1})$. We get:

$$d_j(x_n, x_{n+1}) = d_j(f(x_{n-1}), f(x_n)) \leq \varphi(d_{r(j)}(x_{n-1}, x_n)), \forall n \geq 1.$$

We inductively obtain the following inequality:

$$d_j(x_n, x_{n+1}) \leq \varphi^n(d_{r^n(j)}(x_0, x_1)). \tag{1.115}$$

Having in view that d_j is a b-pseudometric, for $n \geq 0$, $p \geq 1$, we get:

$$d_j(x_n, x_{n+p}) \leq bd_j(x_n, x_{n+1}) + b^2 d_j(x_{n+1}, x_{n+2}) + \cdots + b^p d_j(x_{n+p-1}, x_{n+p}). \tag{1.116}$$

By applying (1.115), we obtain:
(1.117)

$$d_j(x_n, x_{n+p}) \leq b\varphi^n(d_{r^n(j)}(x_0, x_1)) + b^2\varphi^{n+1}(d_{r^{n+1}(j)}(x_0, x_1) + \cdots + b^p\varphi^{n+p-1}(d_{r^{n+p-1}(j)}(x_0, x_1)),$$

which yields

$$d_j(x_n, x_{n+p}) \leq \frac{1}{b^{n-1}}[b^n\varphi^n(d_{r(j)}(x_0, x_1)) + \cdots + b^{n+p-1}\varphi^{n+p-1}(d_{r(j)}(x_0, x_1))]. \tag{1.118}$$

Denote

$$S_n = \sum_{k=0}^{n} b^k \varphi^k(d_{r^k(j)}(x_0, x_1), n \geq 1. \tag{1.119}$$

The inequality (1.118) becomes:

$$d_j(x_n, x_{n+p}) \leq \frac{1}{b^{n-1}}[S_{n+p-1} - S_{n-1}], \quad n \geq 1, p \geq 1. \tag{1.120}$$

But the series

$$\sum_{k=0}^{\infty} b^k \varphi^k(d_{r^k(j)}(x_0, x_1))$$

is convergent, hence there is

$$S = \lim_{n\to\infty} S_n \in \mathbb{R}_+. \tag{1.121}$$

Having in view that $b \geq 1$, by (1.120) we conclude that the sequence $\{x_n\}_{n\geq 0}$ is τ-Cauchy, which yields: there exists $x^* \in A$ such that

$$x^* = \lim_{n\to\infty} x_n.$$

Next, we have to prove that x^* is a fixed point of f. After simple computations, we get:

$$d_j(x_{n+1}, f(x^*)) = d_j(f(x_n), f(x^*)) \leq \varphi(d_{r(j)}(x_n, x^*)) \leq \varphi(d_j(x_n, x^*)), n \geq 0. \tag{1.122}$$

Remind that the comparison function φ is continuous at zero, and by taking $n \to \infty$, we obtain:

$$d_j(x^*, f(x^*)) = 0, \quad \forall j \in J,$$

which actually means that x^* is a fixed point for f.
Suppose, by contradiction, there exists another fixed point $y^* \in A$, different from x^*. By using the definition of a quasi φ-contraction, Lemma 1.3.84 and the monotonicity (1.105) of the b-pseudometrics, we have:

$$d_j(x^*, y^*) = d_j(f(x^*), f(y^*)) \leq \varphi(d_{r(j)}(x^*, y^*)) < d_{r(j)}(x^*, y^*) \overset{mon.}{\leq} d_j(x^*, y^*),$$

for all $j \in J$. The above inequality is obviously a contradiction. Ergo, the uniqueness of the fixed point is proved, which means that f is a Picard operator.
2) By using (1.117), it results that

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq b[\varphi^0(\varphi^n(d_{r^n(j)}(x_0, x_1))) + b\varphi(\varphi^n(d_{r^{n+1}(j)}(x_0, x_1))) + \cdots + \\ &+ b^{p-1}\varphi^{p-1}(\varphi^n(d_{r^{n+p-1}(j)}(x_0, x_1)))], \end{aligned} \tag{1.123}$$

where $n \geq 0, p \geq 1$.

At this point, we let $p \to \infty$ in (1.123), and we get the first required estimate:

$$d_j(x_n, x^*) \leq b \cdot s_b(\varphi^n(d_{r^{n+k}(j)}(x_0, x_1))), n \geq 0.$$

Secondly, for $n \geq 1, k \geq 0$, we obtain

$$d_j(x_{n+k}, x_{n+k+1}) = d_j(f(x_{n+k-1}), f(x_{n+k})) \leq \varphi(d_{r(j)}(x_{n+k-1}, x_{n+k})).$$

We obtain by induction the following inequality:

$$d_j(x_{n+k}, x_{n+k+1}) \leq \varphi^k(d_{r(j)}(x_n, x_{n+1})), n \geq 1, k \geq 0. \tag{1.124}$$

Now, by replacing (1.124) in (1.116), we get

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq b[d_j(x_n, x_{n+1}) + b\varphi(d_{r(j)}(x_n, x_{n+1})) + \\ &+ \cdots + b^{p-1}\varphi^{p-1}(d_{r(j)}(x_n, x_{n+1}))], \end{aligned} \tag{1.125}$$

for $n \geq 0, p \geq 1$. After using the monotonicity property for the pseudometrics d_j, we get:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq b[d_j(x_n, x_{n+1}) + b\varphi(d_j(x_n, x_{n+1})) + \\ &+ \cdots + b^{p-1}\varphi^{p-1}(d_j(x_n, x_{n+1}))]. \end{aligned} \tag{1.126}$$

Letting $p \to \infty$ in (1.126) we obtain the second required estimate:

$$d_j(x_n, x^*) \leq b \cdot s_b(d_j(x_n, x_{n+1})), n \geq 0.$$

3) Letting $x_n := x$, the a posteriori estimate (1.113) becomes for an arbitrary $x \in A$:

$$d_j(x, x^*) \leq b \cdot s_b(d_j(x, f(x))).$$

□

In the sequel, our main goal is to verify if a quasi φ-contraction fulfills the properties mentioned in the beginning of this subsection.

THEOREM 1.3.148. *Assume that $X, J, \mathcal{D}, r, \tau, \varphi$ and A are as in Theorem 1.3.147. Let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition* (1.106). *Then f is a good Picard operator.*

Proof: For a simpler writing, denote the b-pseudometric $d_{b,j}$ with d_j, for every $j \in J$. We may select $x_0 \in A$. By using (1.115) from the proof of Theorem 1.3.147, we get:

$$d_j(f^n(x_0), f^{n+1}(x_0)) = d_j(x_n, x_{n+1}) \leq \varphi^n(d_{r^n(j)}(x_0, x_1)), n \geq 0, \tag{1.127}$$

Next, we use the definition of a b-pseudometric and we obtain:

$$\sum_{n=0}^{\infty} d_j(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=0}^{\infty} b^n \varphi^n(d_{r^n(j)}(x_0, x_1)) = s_b(d_{r^n(j)}(x_0, x_1)).$$

Finally, according to Lemma 1.3.87 and the inequality $\sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) < \infty$, we conclude that f is a good Picard operator. □

THEOREM 1.3.149. *Assume that $X, J, \mathcal{D}, r, \tau, \varphi$ and A are as in Theorem 1.3.147. Let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition* (1.106). *Then the fixed point problem for f is well posed.*

Proof: For simplicity, denote $d_{b,j}$ with d_j, for all $j \in J$. Let the sequence $\{z_n\}_{n \in \mathbb{N}} \subset A$ such that

$$d_j(z_n, f(z_n)) \to 0 \text{ as } n \to \infty, \quad \forall j \in J. \tag{1.128}$$

By (1.114), if we choose $x = z_n, n \in \mathbb{N}$, we get:

$$d_j(z_n, x^*) \leq b \cdot s_b(d_j(z_n, f(z_n))), n \in \mathbb{N}, \quad \forall j \in J. \tag{1.129}$$

Let us remind that s_b is continuous at zero, according to Lemma 1.3.87. Letting $n \to \infty$ in (1.129), and combining it with (1.128), we obtain:

$$d_j(z_n, x^*) \to 0 \text{ as } n \to \infty, \quad \forall j \in J.$$

which means that the fixed point problem for f is well posed. □

THEOREM 1.3.150. *Assume that $X, J, \mathcal{D}, r, \tau, \varphi$ and A are as in Theorem 1.3.147. Let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition* (1.106)*. The monotonicity property is valid with respect to the b-pseudometrics. If the b-comparison function φ satisfies:*

$$\varphi(a_1 t_1 + a_2 t_2) \leq a_1 \varphi(t_1) + a_2 \varphi(t_2), \tag{1.130}$$

for any $a_1, a_2, t_1, t_2 \in \mathbb{R}_+$, then f has the limit shadowing property.

Proof: Let us denote d_j instead of $d_{b,j}$, for every $j \in J$. Let us consider the sequence $\{z_n\}_{n \in \mathbb{N}} \subset A$ such that

$$d_j(z_{n+1}, f(z_n)) \to 0 \text{ as } n \to \infty, \quad \forall j \in J. \tag{1.131}$$

Let x^* be the fixed point of f, thus $f(x^*) = x^*$. For $n \geq 0$, we obtain:

$$d_j(z_{n+1}, x^*) \leq b \cdot d_j(z_{n+1}, f(z_n)) + b \cdot d_j(f(z_n), f(x^*)), \tag{1.132}$$

for every $n \in \mathbb{N}$ and for all $j \in J$. Having in view that f is a φ-contraction, we obtain:

$$d_j(z_{n+1}, x^*) \leq b \cdot d_j(z_{n+1}, f(z_n)) + b\varphi(d_{r(j)}(z_n, x^*)), n \in \mathbb{N}, \forall j \in J. \tag{1.133}$$

The monotonicity of the pseudometrics leads to:

$$d_j(z_{n+1}, x^*) \leq b \cdot d_j(z_{n+1}, f(z_n)) + b\varphi(d_j(z_n, x^*)), n \in \mathbb{N}, \forall j \in J, \tag{1.134}$$

i.e., for $n := n - 1$ in (1.134), we obtain:

$$d_j(z_n, x^*) \leq b \cdot d_j(z_n, f(z_{n-1})) + b\varphi(d_j(z_{n-1}, x^*)), \quad n \geq 1, \forall j \in J,$$

which replaced in (1.134), by using (1.130) yields

$$d_j(z_{n+1}, x^*) \leq b \cdot d_j(z_{n+1}, f(z_n)) + b^2\varphi(d_j(z_n, f(z_{n-1}))) + b^2\varphi^2(d_j(z_{n-1}, x^*)), n \geq 1.$$

At this point, we inductively obtain:

$$\begin{gathered} d_j(z_{n+1}, x^*) \leq b \cdot d_j(z_{n+1}, f(z_n)) + b^2\varphi(d_j(z_n, f(z_{n-1}))) + \\ + \cdots + b^{n+1}\varphi^n(d_j(z_1, f(z_0))) + b^{n+2}\varphi^{n+1}(d_j(z_0, x^*)), \forall n \geq 1, \forall j \in J. \end{gathered}$$

Thus, we have:

$$d_j(z_{n+1}, x^*) \leq b\sum_{k=0}^{n} b^k\varphi^k(d_j(z_{n-k+1}, f(z_{n-k}))) + \tag{1.135}$$
$$+ \quad b^{n+2}\varphi^{n+1}(d_j(z_0, x^*)), \quad \forall n \geq 1, \forall j \in J.$$

Apply Lemma 1.3.88 with the substitution: $a_n = d_j(z_{n+1}, f(z_n))$. Consequently,

$$\sum_{k=0}^{n} b^k\varphi^k(d_j(z_{n-k+1}, f(z_{n-k}))) \to 0 \text{ as } n \to \infty. \tag{1.136}$$

For the sake of clarity, we divide the proof, as follows:
Case I: If $z_0 = x^*$, this means $b^n\varphi^n(d_j(z_0, x^*)) = 0 \to 0$ as $n \to \infty$.
Case II: If $z_0 \neq x^*$, we have that $b^n\varphi^n(d_j(z_0, x^*)) \to 0$ as $n \to \infty$, according to Lemma 1.3.87. At this point, we let $n \to \infty$ in (1.135) and it follows that

$$d_j(z_{n+1}, x^*) \to 0, n \to \infty. \tag{1.137}$$

By using Theorem 1.3.147, we conclude that for any $x \in A$ the Picard iteration $\{f^n(x)\}_{n\geq 0}$ converges to x^*. This means that the following inequalities holds:

$$d_j(z_{n+1}, f^n(x)) \leq d_j(z_{n+1}, x^*) + d_j(x^*, f^n(x)), n \geq 0 \tag{1.138}$$

for every $x \in A$. Taking the limit $n \to \infty$ in (1.138), we have that

$$d_j(z_{n+1}, f^n(x)) \to 0, \text{ as } n \to \infty,$$

which actually means that f has the limit shadowing property. □

The next theorem states and proves the data dependence of the fixed point for quasi φ-contractions on b-pseudometric spaces with φ a b-comparison function:

Theorem 1.3.151. *Assume that $X, J, \mathcal{D}, r, \tau, \varphi$ and A are as in Theorem 1.3.147. Let $f : A \to A$ be a quasi φ-contraction with φ satisfying condition* (1.106)*. Let x^* be the fixed point of f and let $g : A \to A$ be such that:*

(1) g has at least one fixed point, denoted by $y^* \in Fix(g)$;
(2) there exists $\eta > 0$ such that

$$d_j(f(x), g(x)) \leq \eta, \text{ for any } x \in A, \forall j \in J. \tag{1.139}$$

Then

$$d_j(x^*, y^*) \leq b \cdot s_b(\eta),$$

where s_b was introduced in Lemma 1.3.87.

Proof: For simplicity, denote by d_j the b-pseudometrics, instead of $d_{b,j}$, $\forall j \in J$. From 1.114 in Theorem 1.3.147, by using the substitution $x := y^*$, we obtain:

$$d_j(x^*, y^*) \leq b \cdot s_b(d_j(y^*, f(y^*))) = b \cdot s_b(d_j(g(y^*), f(y^*))).$$

Using again Lemma 1.3.87, it follows from that: s_b is increasing, which by (1.139) yields

$$d_j(x^*, y^*) \leq b \cdot s_b(\eta).$$

□

3.6. Fixed points of almost local contractions in b-pseudometric spaces

THEOREM 1.3.152. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j \in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be an almost local contraction with some constants $\theta \in [0, \frac{1}{b})$ and $L \geq 0$.*

Then:

(i) f is a weakly Picard operator;

(ii) If the b-pseudometrics are continuous $\forall j \in J$, then for any $x \in A$ the following error estimates hold:

$$d_j(f^n(x), f^\infty(x)) \leq \frac{b\theta^n}{1 - b\theta} d_j(x, f(x)), n \geq 1, \forall j \in J; \tag{1.140}$$

$$d_j(f^n(x), f^\infty(x)) \leq \frac{b\theta}{1 - b\theta} d_j(f^{n-1}(x), f^n(x)), n \geq 1, \forall j \in J, \tag{1.141}$$

where $f^\infty(x)$ has been defined in (1.104).

Proof: In order to simplify the proof, denote d_j instead of $d_{b,j}$, for every $j \in J$.
(i) In the beginning, we will prove that the operator f has at least one fixed point in X, i.e., the set of fixed points is nonempty. To this end, we let $x_0 \in A$ and $\{x_n\}_{n \geq 0}$ be the Picard iteration which starts from x_0. By using the definition of the Picard iteration and also the definition of ALC-s, we get:

$$d_j(x_n, x_{n+1}) = d_j(f(x_{n-1}), f(x_n)) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n) + L \underbrace{d_{r(j)}(x_n, x_n)}_{=0}, \tag{1.142}$$

for all $n \in \mathbb{N}$. From that, by using the monotonicity of the b-pseudometrics, we get: $d_j(x_n, x_{n+1}) \leq \theta \cdot d_j(x_{n-1}, x_n)$. We obtain by induction with respect to n:

$$d_j(x_n, x_{n+1}) \leq \theta^n \cdot d_j(x_0, x_1), \quad n = 1, 2, \cdots \tag{1.143}$$

For $n \geq 0$, $p \geq 1$ we can write:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq b[d_j(x_n, x_{n+1}) + d_j(x_{n+1}, x_{n+p})] = \\ &= bd_j(x_n, x_{n+1}) + bd_j(x_{n+1}, x_{n+p}) \leq \\ &\leq bd_j(x_n, x_{n+1}) + b^2[d_j(x_{n+1}, x_{n+2}) + d_j(x_{n+2}, x_{n+p}))] \leq \cdots \leq \\ &\leq bd_j(x_n, x_{n+1}) + b^2 d_j(x_{n+1}, x_{n+2}) + \cdots + b^p d_j(x_{n+p-1}, x_{n+p}). \end{aligned}$$

By (1.143) it follows that:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq b\theta^n d_{r(j)}(x_0, x_1) + b^2\theta^{n+1} d_{r(j)}(x_0, x_1) + \cdots + b^p\theta^{n+p-1} d_{r(j)}(x_0, x_1) = \\ &= b\theta^n d_{r(j)}(x_0, x_1)[1 + b\theta + (b\theta)^2 + \cdots + (b\theta)^{p-1}] = \\ &= b \cdot \frac{1-(b\theta)^p}{1-b\theta} d_{r(j)}(x_0, x_1) \cdot \theta^n, \end{aligned} \tag{1.144}$$

for $n \geq 0$, $p \geq 1$.

Remind that $\theta \in [0, \frac{1}{b})$, with $b \geq 1$, it is obvious that $0 \leq b\theta < 1$, which yields from (1.144) the conclusion that $\{x_n\}_{n\geq 0}$ is a d_j-Cauchy sequence in the b-pseudometric space. This means that it is convergent with its limit denoted by

$$x^* = \lim_{n\to\infty} x_n. \tag{1.145}$$

Applying the definition of the b-pseudometric, we get:

$$d_j(x^*, f(x^*)) \leq b[d_j(x^*, f(x_n)) + d_j(f(x_n), f(x^*))].$$

After using the definition of an almost local contraction and the monotonicity property (1.105), we obtain from the last inequality:

$$\begin{aligned} d_j(x^*, f(x^*)) &\leq bd_j(x^*, f(x_n)) + b\theta d_{r(j)}(x_n, x^*) + bLd_{r(j)}(x^*, f(x_n)) \leq \\ &\leq bd_j(x^*, f(x_n)) + b\theta d_j(x_n, x^*) + bLd_j(x^*, f(x_n)) = \\ &= b(1+L)d_j(x^*, x_{n+1}) + b\theta d_j(x_n, x^*), \quad \forall j \in J, \forall n \in \mathbb{N}. \end{aligned}$$

Having in view (1.145) and letting $n \to \infty$, it results that

$$d_j(x^*, f(x^*)) = 0, \quad \forall j \in J,$$

which means that x^* is a fixed point of f. Hence f is a weakly Picard operator.

(ii) In the sequel, remind that the b-pseudometric is continuous and $0 \leq b\theta < 1$. Letting $p \to \infty$ in (1.144), we get the a priori error estimate (1.140). By using the induction in (1.142), we obtain:

$$d_j(x_{n+k}, x_{n+k+1}) \leq \theta^{k+1} \cdot d_{r^{k+1}(j)}(x_{n-1}, x_n), \tag{1.146}$$

for any $n, k \in \mathbb{N}, n \geq 1$. The monotonicity of the b-pseudometrics leads to:

$$d_{r^k(j)}(x, y) \leq d_j(x, y), \forall x, y \in A, \forall j \in J, \forall k \in \mathbb{N}. \tag{1.147}$$

From that, we can write:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\leq bd_j(x_n, x_{n+1}) + b^2 d_j(x_{n+1}, x_{n+2}) + \cdots + b^p d_j(x_{n+p-1}, x_{n+p}) \leq \\ &\leq b\theta d_j(x_{n-1}, x_n) + (b\theta)^2 d_j(x_{n-1}, x_n) + \cdots + (b\theta)^p d_j(x_{n-1}, x_n) = \\ &= b\theta \cdot \frac{1-(b\theta)^p}{1-b\theta} \cdot d_j(x_{n-1}, x_n). \end{aligned} \tag{1.148}$$

By applying the continuity of the b-pseudometric and having in view the condition $0 \leq b\theta < 1$, it results the a posteriori error estimate if we let $p \to \infty$ in (1.148). □

In the sequel, our main goal is to extend the almost local contractions to the case of strict almost contractions, as a result of which we get an existence and uniqueness theorem:

THEOREM 1.3.153. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a strict almost local contraction with some constants $\theta \in [0, \frac{1}{b})$ and $L \geq 0$, and $\theta_u \in [0, \frac{1}{b})$, $L_u \geq 0$, respectively. Assume a uniqueness condition for the mapping f (see* [**118**]*), that is:*

$$d_{b,j}(Tx, Ty) \leq \theta_u \cdot d_{b,r(j)}(x, y) + L_u \cdot d_{b,r(j)}(x, T(x)), \forall x, y \in A, \forall j \in J. \tag{1.149}$$

Then:

(i) f is a Picard operator;

(ii) If the b-pseudometrics are continuous, then the following error estimates hold:

$$d_{b,j}(x_n, x^*) \leq \frac{b\theta^n}{1 - b\theta} d_{b,j}(x_0, x_1), n \geq 1, \forall j \in J; \tag{1.150}$$

$$d_{b,j}(x_n, x^*) \leq \frac{b\theta}{1 - b\theta} d_{b,j}(x_{n-1}, x_n), n \geq 1, \forall j \in J; \tag{1.151}$$

(iii) Assume the continuity of the b-pseudometric. The rate of convergence of the Picard iteration is given by

$$d_{b,j}(x_n, x^*) \leq \theta_u d_{b,j}(x_{n-1}, x^*), n \geq 1, \forall j \in J, \tag{1.152}$$

where $Fix(f) = \{x^*\}$.

Proof: In order to simplify the proof, denote d_j instead of $d_{b,j}$, for every $j \in J$. (i) The first part of the conclusion of the theorem, namely the existence of the fixed point is assured by Theorem 1.3.152. In order to prove the uniqueness of the fixed point, suppose that f has two different fixed points $x^*, y^* \in A$. Then, using the monotonicity condition (1.26) for the b-pseudometrics and the uniqueness condition (1.149), we can write that:

$$\begin{aligned} d_j(f(x^*), f(y^*)) &\leq \theta_u d_{r(j)}(x^*, y^*) + L_u d_{r(j)}(x^*, f(x^*)) \leq \\ &\leq \theta_u d_j(x^*, y^*) + L_u d_j(x^*, f(x^*)), \end{aligned}$$

which means that $d_j(x^*, y^*) \leq \theta_u d_j(x^*, y^*)$.
As $0 \leq \theta_u < 1$, we get the obvious contradiction $d_j(x^*, y^*) < d_j(x^*, y^*)$.
It results that $F_f = \{x^*\}$, hence f is a Picard operator.
(ii) The a priori and a posteriori estimates (1.150) and (1.151) follow by Theorem

1.3.152.

(iii) From (1.149) we obtain:

$$\begin{aligned}d_j(f(x^*), f(x_{n-1})) &\leq \theta_u d_{r(j)}(x^*, x_{n-1}) + L_u d_{r(j)}(x^*, f(x^*)) \\ &\leq \theta_u d_j(x^*, x_{n-1}) + L_u d_j(x^*, f(x^*)),\end{aligned}$$

which means:

$$d_j(x_n, x^*) \leq \theta_u d_j(x_{n-1}, x^*), n \geq 1.$$

□

We present an example of almost local contraction in b-pseudometric space with three fixed points:

EXAMPLE 1.3.154. *Let us consider* $X = \{-1, 0, 1\} \times \{-1, 0, 1\} \subset \mathbb{R}^2$.
Consider the b-pseudometrics:
$d_j((x_1, y_1), (x_2, y_2)) = 0$, *if* $x_1 = x_2$;
$d_j((x_1, y_1), (x_2, y_2)) = e^{-j}$, *if* $|x_1 - x_2|$=1;
$d_j((x_1, y_1), (x_2, y_2)) = b \cdot e^{-j}$, *if* $|x_1 - x_2| = 2$,
where $b \geq 2$ *and* J *is a subset of* $\mathbb{Q}$.
d_j *is a pseudometric, but is not a metric, take for example:*
$d_j((1, -1), (1, 0)) = 0$, *however* $(1, -1) \neq (1, 0)$.
In this case, consider the function $r(j) = j + \frac{1}{3}$, *where* $j \in J$.
Consider $T : X \to X$,

$$T(x, y) = \begin{cases} (x, -y) & \text{if } (x,y) \neq (1, 1) \\ (0, 0) & \text{if } (x,y) = (1, 1) \end{cases}$$

The mapping T *is an almost local contraction because:*
a) if $|x_1 - x_2|$=1 *or* $|x_1 - x_2|$=2, *then* (1.19) *becomes:* $e^{-j} \leq \theta \cdot e^{-(j+\frac{1}{3})} + L \cdot e^{-(j+\frac{1}{3})}$.
For $\theta = \frac{1}{4}$, $L = 2 \geq 0$ *and* $j \in J$, *the last inequality is true.*
b) If $|x_1 - x_2|$=0, *then* (1.19) *becomes* $0 \leq 0$.
Therefore, T *is an almost local b-contraction with three fixed points:*

$$Fix(T) = \{(-1, 0), (0, 0), (1, 0)\}.$$

Next, we will make a comparison to other type of contractive conditions in
b -pseudometric spaces.

REMARK 1.3.155. *In the beginning of Chapter 1, the various type of almost local contractions were introduced in a uniform space setting. But in sections 3.4-3.6 the framework is that of b-pseudometric spaces, starting from the work* [**104**]. *Uniform spaces and pseudometric spaces are intertwined, based on Proposition 1.1.27 and the subsequent explanations.*

The following two lemmas refers to the Ćirić-Reich-Rus almost local contractions.

LEMMA 1.3.156. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a Ćirić-Reich-Rus type almost local contraction, with constants $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + 2b\beta < 1$. Then f is an almost local contraction with $\theta = \frac{\alpha+b\beta}{1-b\beta}$ and $L = \frac{2b\beta}{1-b\beta}$.*

Proof: In order to simplify the writing, denote d_j instead of $d_{b,j}$, for every $j \in J$. Let $f : A \to A$ be a Ćirić-Reich-Rus type almost local contraction, where A is a subset of the b-pseudometric space X. Let $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2b\beta < 1$, be such that

$$d_j(f(x), f(y)) \leq \alpha d_{r(j)}(x, y) + \beta[d_{r(j)}(x, f(x)) + d_{r(j)}(y, f(y))], \tag{1.153}$$

for any $x, y \in A$, and for all $j \in J$.
At this point, after using the monotonicity property (1.105), we can write:

$$d_j(f(x), f(y)) \leq \alpha d_{r(j)}(x, y) + b\beta d_{r(j)}(x, y) + b\beta d_{r(j)}(y, f(x)) +$$
$$+ b\beta d_{r(j)}(y, f(x)) + b\beta d_j(f(x), f(y)), \forall j \in J,$$

which implies

$$d_j(f(x), f(y)) \leq \frac{\alpha + b\beta}{1 - b\beta} d_{r(j)}(x, y) + \frac{2b\beta}{1 - b\beta} d_{r(j)}(y, f(x)), \forall j \in J,$$

for every $x, y \in A$, i.e., f satisfies (1.19) with $\theta = \frac{\alpha+b\beta}{1-b\beta} \in [0, 1)$ and $L = \frac{2b\beta}{1-b\beta} \geq 0$. □

REMARK 1.3.157. *For $b = 1$ in Lemma 1.3.156, the values from Theorem 1.3.76 are obtained.*

LEMMA 1.3.158. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a Ćirić-Reich-Rus type almost local contraction, with constants $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + b(b + 1)\beta < 1$. Then f is a strict almost local contraction with $\theta = \frac{\alpha+b\beta}{1-b\beta}$ and $L = \frac{2b\beta}{1-b\beta} \geq 0$, and respectively, $\theta_u = \frac{\alpha+b^2\beta}{1-b\beta}$ and $L_u = \frac{\beta(b^2+1)}{1-b\beta}$.*

Proof: For a simpler writing, denote d_j instead of $d_{b,j}$, for every $j \in J$. As $b \geq 1$, assumption $\alpha + b(b+1)\beta < 1$ implies $\alpha + 2b\beta < 1$, therefore the conclusions of Lemma 1.3.156 holds. Furthermore, according to (1.153), we obtain:

$$d_j(f(x), f(y)) \leq \alpha d_{r(j)}(x, y) + \beta d_{r(j)}(x, f(x)) + \beta d_{r(j)}(f(y), y) \overset{mon.}{\leq}$$
$$\leq \alpha d_{r(j)}(x, y) + \beta d_{r(j)}(x, f(x)) + b\beta d_j(f(y), f(x)) + b^2\beta d_{r(j)}(f(x), x) + b^2\beta d_{r(j)}(x, y),$$

for every $j \in J$. Therefore

$$d_j(f(x), f(y)) \le \frac{\alpha + b^2\beta}{1-b\beta} d_{r(j)}(x,y) + \frac{\beta(b^2+1)}{1-b\beta} d_{r(j)}(x, f(x)), \forall j \in J,$$

for every $x, y \in A$, i.e., f satisfies (1.149) with $\theta_u = \frac{\alpha+b^2\beta}{1-b\beta} \in [0,1)$ and $L_u = \frac{\beta(b^2+1)}{1-b\beta}$, which means that f is a strict almost local contraction. $\square$

LEMMA 1.3.159. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a Chatterjea type almost local contraction with constant $c \in [0, \frac{1}{b(b+1)})$.*
Then f is an almost local contraction with $\theta = \frac{cb^2}{1-cb}$ and $L = \frac{c(b^2+1)}{1-cb}$.

Proof: For a simpler writing, denote d_j instead of $d_{b,j}$, for every $j \in J$. Let A a subset of the b-pseudometric space X and let $f : A \to A$ a Chatterjea type almost local contraction with $c \in [0, \frac{1}{b(b+1)})$ such that

$$d_j(f(x), f(y)) \le c[d_{r(j)}(x, f(y)) + d_{r(j)}(y, f(x))], \forall x, y \in A, \forall j \in J. \tag{1.154}$$

From that, we can write:

$$\begin{aligned} d_j(f(x), f(y)) \le c d_j(f(y), x) + c d_j(y, f(x)) &\overset{mon.}{\le} \\ \le cbd_j(f(y), f(x)) + cbd_{r(j)}(f(x), x) + cd_{r(j)}(y, f(x)) &\le \\ \le cbd_j(f(x), f(y)) + cb^2 d_{r(j)}(x, y) + cb^2 d_{r(j)}(y, f(x)) + cd_{r(j)}(y, f(x)), \end{aligned}$$

for every $j \in J$. We obtain:

$$d_j(f(x), f(y)) \le \frac{cb^2}{1-cb} d_{r(j)}(x,y) + \frac{c(b^2+1)}{1-cb} d_{r(j)}(y, f(x)), \forall j \in J,$$

for any $x, y \in A$, that is, f satisfies (1.19) with $\theta = \frac{cb^2}{1-cb} \in [0,1)$ and
$L = \frac{c(b^2+1)}{1-cb} \ge 0$. $\square$

REMARK 1.3.160. *For $b = 1$ in Lemma 1.3.159, we obtain the conclusion of Theorem 1.3.78.*

LEMMA 1.3.161. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a Chatterjea type almost local contraction with constant $c \in [0, \frac{1}{b(b+1)})$.*
Then f is a strict almost local contraction with $\theta = \frac{cb^2}{1-cb}$ and $L = \frac{c(b^2+1)}{1-cb}$ and, respectively, $\theta_u = \frac{cb}{1-cb}$ and $L_u = \frac{2cb}{1-cb}$.

Proof: In order to simplify the writing, denote d_j instead of $d_{b,j}$, for every $j \in J$. The conclusions of Lemma 1.3.159 hold and from (1.154) we have that:

$$d_j(f(x), f(y)) \leq cbd_{r(j)}(x, f(x)) + cbd_{r(j)}(f(x), f(y)) + cbd_{r(j)}(y, x) + cbd_{r(j)}(x, f(x)),$$

for every $j \in J$. After applying the monotonicity property for the pseudometrics, we get:

$$d_j(f(x), f(y)) \leq cbd_{r(j)}(x, f(x)) + cbd_j(f(x), f(y)) + cbd_{r(j)}(y, x) + cbd_{r(j)}(x, f(x)),$$

therefore

$$d_j(f(x), f(y)) \leq \frac{cb}{1-cb} d_{r(j)}(x, y) + \frac{2cb}{1-cb} d_{r(j)}(x, f(x)), \forall j \in J,$$

for any $x, y \in A$, that is, f satisfies (1.149) with $\theta_u = \frac{cb}{1-cb}$ and $L_u = \frac{2cb}{1-cb} \geq 0$.
Having in view that $c \in [0, \frac{1}{b(b+1)})$ and $b \geq 1$, hence, it results that $c < \frac{1}{2b}$, which means $\theta_u \in [0, 1)$. $\square$

THEOREM 1.3.162. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j \in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a local contraction with constant coefficient of contraction $\beta \in [0, 1)$, satisfying*

$$d_j(f(x), f(y)) \leq \beta d_{r(j)}(x, y), \quad \forall x, y \in A, \forall j \in J. \tag{1.155}$$

Then f is an almost local contraction with constants $\theta = \beta$ and $L = 0$.

Proof: For a simpler writing, denote d_j instead of $d_{b,j}$, for every $j \in J$. The contractive condition 1.155 can be written in the equivalent form:

$$d_j(f(x), f(y)) \leq \beta d_{r(j)}(x, y) + 0 \cdot d_{r(j)}(y, f(x)), \forall x, y \in A, \forall j \in J,$$

which means that f is an almost local contraction satisfying (1.19) with the coefficients $\theta = \beta \in [0, 1)$ and $L = 0 \geq 0$. $\square$

Our next goal is to study the case of quasi-almost local contractions.

LEMMA 1.3.163. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j \in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a quasi-ALC with constant $h \in [0, \frac{1}{b(b+1)})$.*
Then f is an ALC with constants $\theta = \frac{b^2 h}{1-bh}$ and $L = \frac{b^2 h}{1-bh}$.

Proof: Let $f : A \to A$ be a quasi-ALC with constant $h \in [0, \frac{1}{b(b+1)})$ such that
(1.156)
$d_j(f(x), f(y)) \leq h \max\{d_{r(j)}(x, y), d_{r(j)}(x, f(x)), d_{r(j)}(y, f(y)), d_{r(j)}(x, f(y)), d_{r(j)}(y, f(x))\}.$

for all $x, y \in A$ and for every $j \in J$.
For the sake of clarity, we divide the proof into five steps:

I. $M_{r(j)}(x, y) = d_j(x, y)$. We can write

$$d_j(f(x), f(y)) \leq hd_{r(j)}(x, y).$$

II. $M_{r(j)}(x, y) = d_j(x, f(x))$. Then

$$d_j(f(x), f(y)) \leq hd_{r(j)}(x, f(x)) \leq hbd_{r(j)}(x, y) + hbd_{r(j)}(y, f(x)),$$

therefore (1.19) is satisfied with constants $\theta = hb \in [0, 1)$ and $L = hb \geq 0$.
Then, for any $x, y \in A$ we can write:

$$d_j(f(x), f(y)) \leq hd_{r(j)}(x, f(x)) \leq hbd_{r(j)}(x, f(y)) + hbd_{r(j)}(f(y), f(x)).$$

After applying the monotonicity property for the family of pseudometrics, we obtain:

$$d_j(f(x), f(y)) \leq hbd_{r(j)}(x, f(y)) + hbd_j(f(x), f(y)).$$

Thus, we have:

$$d_j(f(x), f(y)) \leq \frac{hb}{1 - hb} d_{r(j)}(x, f(y)),$$

which means that (1.20) is valid with constants $\theta = 0$ and $L = \frac{hb}{1-hb}$.
From that, the almost local contraction condition (1.19) becomes:

$$d_j(f(x), f(y)) \leq \underbrace{0}_{\theta} \cdot d_{r(j)}(x, y) + \underbrace{\frac{hb}{1 - hb}}_{L} d_{r(j)}(x, f(y)), \forall x, y \in A, \forall j \in J.$$

is verified with:
$\theta = \max\{hb, 0\} = hb$ and $L = \max\{hb, \frac{hb}{1-hb}\} = \frac{hb}{1-hb}$.

III. If $M_{r(j)}(x, y) = d_j(y, f(y))$, in a similar manner to case II., it results that (1.19) is fulfilled with $\theta = hb$ and $L = \frac{hb}{1-hb}$.

IV. If $M_{r(j)}(x, y) = d_j(x, f(y))$, then we can write

$$d_j(f(x), f(y)) \leq hd_{r(j)}(x, f(y)).$$

This means that condition (1.20) holds by using the notations: $\theta = 0$ and $L = h$.
Applying the monotonicity of the b-pseudometrics, it results that

$$\begin{aligned} d_j(f(x), f(y)) &\leq hd_{r(j)}(x, f(y)) \leq \\ &\leq bhd_{r(j)}(f(y), f(x)) + b^2hd_{r(j)}(f(x), y) + b^2hd_{r(j)}(y, x) \overset{mon.}{\leq} \\ &\leq bhd_j(f(y), f(x)) + b^2hd_{r(j)}(f(x), y) + b^2hd_{r(j)}(y, x), \end{aligned}$$

for all $j \in J$, therefore

$$d_j(f(x), f(y)) \leq \frac{b^2 h}{1 - hb} d_{r(j)}(x, y) + \frac{b^2 h}{1 - hb} d_{r(j)}(y, f(x)).$$

From that, we may state that for the previously chosen x and y, the ALC (1.19) condition holds with $\theta = \max\{\frac{b^2 h}{1-bh}, 0\} = \frac{b^2 h}{1-bh} \in [0, 1)$ and
$L = \max\{h, \frac{b^2 h}{1-bh}\} = \frac{b^2 h}{1-bh}$.

V. $M_{r(j)}(x, y) = d_j(y, f(x))$. This case is quite similar to case IV.
These five cases lead us to the conclusion that for any $x, y \in A$, the ALC condition (1.19) is fulfilled with

$$\theta = \max\left\{h, hb, \frac{b^2 h}{1 - bh}\right\} = \frac{b^2 h}{1 - bh} \in [0, 1)$$

and

$$L = \max\left\{0, \frac{bh}{1 - bh}, \frac{b^2 h}{1 - bh}\right\} = \frac{b^2 h}{1 - bh} \geq 0.$$

□

In the sequel, we shall return to the class of strict ALC-s in b-pseudometric spaces, in order to study their stability.

THEOREM 1.3.164. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j \in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be a strict almost local contraction with constants $\theta \in [0, \frac{1}{b})$, $L \geq 0$ and $\theta_u \in [0, \frac{1}{b})$, $L_u \geq 0$, respectively.*

Then f is a good Picard operator.

Proof: For a simpler writing, denote d_j instead of $d_{b,j}$, for every $j \in J$.
Let $x_0 \in A$ and $\{x_n\}_{n \geq 0}$ be the Picard iteration which starts from x_0. If we let $x = x_0 \in A$ in the inequality (1.143), we obtain:

$$\sum_{n \geq 0} d_j(f^n(x), f^{n+1}(x)) = \sum_{n \geq 0} d_j(x_n, x_{n+1}) = \lim_{n \to \infty} \sum_{k=0}^{n} d_j(x_k, x_{k+1}) \leq$$
$$\leq \lim_{n \to \infty} (1 + \theta + \cdots + \theta^n) d_j(x_0, x_1) = \lim_{n \to \infty} \frac{1 - \theta^{n+1}}{1 - \theta} d_j(x_0, x_1).$$

From that, by using $\theta \in [0, 1)$ (since $b \geq 1$), we obtain

$$\sum_{n \geq 0} d_j(f^n(x), f^{n+1}(x)) \leq \frac{1}{1 - \theta} d_j(x_0, x_1).$$

Therefore, f is a good Picard operator, according to Definition 1.3.136. □

THEOREM 1.3.165. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.3.164. Let $f : A \to A$ be a strict almost local contraction with constants $\theta \in [0, \frac{1}{b})$ and $L \geq 0$, $\theta_u \in [0, \frac{1}{b})$ and $L_u \geq 0$.*
Then f is a special Picard operator.

Proof: We showed in Theorem 1.3.153 the following error estimate:

$$d_j(x_n, x^*) \leq \frac{b\theta^n}{1-b\theta} d_j(x_0, x_1), n \geq 1, \forall j \in J.$$

We let $x = x_0 \in A$ and we obtain

$$\begin{aligned}\sum_{n\geq 0} d_j(f^n(x), x^*) &= \lim_{n\to\infty} \sum_{k=0}^{n} d_j(x_k, x^*) \leq \\ &\leq \lim_{n\to\infty} \sum_{k=0}^{n} \frac{b\theta^k}{1-b\theta} d_j(x_0, x_1) = \frac{b d_j(x_0, x_1)}{1-b\theta} \lim_{n\to\infty} \sum_{k=0}^{n} \theta^k, \forall j \in J.\end{aligned}$$

Consequently,

$$\sum_{n\geq 0} d_j(f^n(x), x^*) = \frac{b}{(1-b\theta)(1-\theta)} d_j(x_0, x_1) < \infty, \quad \forall j \in J,$$

which means that f is a special Picard operator, having in view Definition 1.3.138. $\square$

THEOREM 1.3.166. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.3.164. Let $f : A \to A$ be a strict almost local contraction with constants $\theta \in [0, \frac{1}{b})$ and $L \geq 0$, $\theta_u \in [0, \frac{1}{b})$ and $L_u \geq 0$.*
Then the fixed point problem is well posed.

Proof: We may select $y_n \in A, n \in \mathbb{N}$ such that

$$d_j(y_n, f(y_n)) \to 0 \text{ as } n \to \infty, \quad \forall j \in J. \tag{1.157}$$

According to Theorem 1.3.153, the operator f admits a unique fixed point, denoted by $x^* \in A$.
Since f is an almost local contraction and having in view the definition of a b-pseudometric, we obtain:

$$\begin{aligned} d_j(y_n, x^*) &\leq b d_j(y_n, f(y_n)) + b d_j(f(y_n), x^*) = \\ &= b d_j(y_n, f(y_n)) + b d_j(f(x^*), f(y_n)) \leq \\ &\leq b d_j(y_n, f(y_n)) + b\theta_u d_j(x^*, y_n) + bL_u d_j(x^*, f(x^*)),\end{aligned}$$

for every $j \in J$. From that, we conclude:

$$d_j(y_n, x^*) \leq \frac{b}{1-b\theta_u} d_j(y_n, f(y_n)), n \geq 0.$$

Applying (1.157), we have:

$$y_n \to x^*, \text{ as } n \to \infty,$$

which means that the fixed point problem is well posed. $\square$

THEOREM 1.3.167. *Assume that $X, J, \mathcal{D}, r, \tau$ and A are as in Theorem 1.3.164. Let $f : A \to A$ be a strict almost local contraction with constants $\theta \in [0, \frac{1}{b})$ and $L \geq 0$, $\theta_u \in [0, \frac{1}{b})$ and $L_u \geq 0$.*
Then f has the limit shadowing property.

Proof: Select the point $y_n \in A, n \in \mathbb{N}$ such that

$$d_j(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty \tag{1.158}$$

According to Theorem 1.3.153, the operator f has a unique fixed point, denoted by $x^* \in A$. This is actually the limit of the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N},$$

for any $x_0 \in A$. By means of the b-pseudometric definition, we can write:

$$\begin{aligned} d_j(y_n, x^*) &\leq bd_j(y_n, f(y_{n-1})) + bd_j(f(y_{n-1}), x^*) = \\ &= bd_j(y_n, f(y_{n-1})) + bd_j(f(x^*), f(y_{n-1}), \quad \forall j \in J. \end{aligned}$$

By using the uniqueness condition for almost local contractions, we obtain:

$$d_j(y_n, x^*) \leq bd_j(y_n, f(y_{n-1})) + b\theta_u d_{r(j)}(y_{n-1}, x^*) + bL_u d_{r(j)}(x^*, f(x^*)),$$

for every $j \in J$. But x^* is a fixed point of the operator f. Thus, we have:

$$d_j(y_n, x^*) \leq bd_j(y_n, f(y_{n-1})) + b\theta_u d_{r(j)}(y_{n-1}, x^*), \quad \forall j \in J.$$

Take the following substitutions: $a_n = d_j(y_n, x^*), q = b\theta_u, b_n = bd_j(y_{n+1}, f(y_n)), \forall j \in J$. Note that $b\theta_u \in [0, 1)$. We get from (1.158) and by using Lemma 1.2.45:

$$d_j(y_n, x^*) \to 0, \text{ as } n \to \infty, \quad \forall j \in J. \tag{1.159}$$

Note that $x^* \in A$ is the limit of the Picard iteration:

$$x^* = \lim_{n\to\infty} f^n(x_0), \tag{1.160}$$

for any $x_0 \in A$. We obtain that

$$d_j(y_n, f^n(x_0)) \leq bd_j(y_n, x^*) + bd_j(x^*, f^n(x_0)), \quad \forall j \in J.$$

By using (1.159) and (1.160), the last inequality leads us to:

$$d_j(y_n, f^n(x_0)) \to 0, \text{ as } n \to \infty, \forall j \in J,$$

which means, according to Definition 1.3.142, that f has the limit shadowing property.

□

In the sequel, we propose to study the data dependence of the fixed point for the class of strict almost local contractions in b-pseudometric spaces.

THEOREM 1.3.168. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ be an operator as appears in Theorem 1.3.153 and $g : A \to A$ a mapping satisfying:*

(i) g has at least one fixed point, denoted by $x_g^* \in Fix(g)$;

(ii) there exists $\eta > 0$ such that

$$d_j(f(x), g(x)) \leq \eta, \quad \forall x \in A, \forall j \in J, \tag{1.161}$$

Then

$$d_j(x_f^*, x_g^*) \leq \frac{b\eta}{1 - b\theta_u}, \quad \forall j \in J,$$

where x_f^* is the unique fixed point of the operator f.

Proof: Theorem 1.3.153 assures a unique fixed point for the operator f. Using the b-pseudometric property, we get:

$$d_j(x_f^*, x_g^*) = d_j(f(x_f^*), g(x_g^*)) \leq bd_j(f(x_f^*), f(x_g^*)) + bd_j(f(x_g^*), g(x_g^*)), \forall j \in J.$$

By using the uniqueness condition for almost local contractions, the (1.161) inequality, and the monotonicity property (1.26), it results that

$$\begin{aligned} d_j(x_f^*, x_g^*) &\leq b\theta_u d_{r(j)}(x_f^*, x_g^*) + bL_u d_{r(j)}(x_f^*, f(x_f^*)) + b\eta \leq \\ &\leq b\theta_u d_j(x_f^*, x_g^*) + bL_u d_j(x_f^*, f(x_f^*)) + b\eta, \quad \forall j \in J. \end{aligned}$$

It follows that:

$$d_j(x_f^*, x_g^*) \leq \frac{b\eta}{1 - b\theta_u}, \quad \forall j \in J.$$

□

THEOREM 1.3.169. *Let $(X, d_{b,j})$ be a b-pseudometric space with $\mathcal{D} = (d_{b,j})_{j\in J}$ a family of b-pseudometrics defined on X, where J denote a set of indices. Assume the monotonicity property* (1.105) *fulfilled for the b-pseudometrics. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$ and let r be a function from J to J. Let $f : A \to A$ an operator as appears in Theorem 1.3.153 and the mappings $f_n : A \to A, n \in \mathbb{N}$ satisfying:*

(i) the operators f_n, $n \in \mathbb{N}$ are strict almost contractions with constants $a \in [0, \frac{1}{b})$, $K \geq 0$ and $a_u \in [0, \frac{1}{b}), K_u \geq 0$, respectively;

(ii) the sequence of functions $\{f_n\}$ converges uniformly to f as $n \to \infty$, that is, for every $\varepsilon > 0$, there exists a natural number N such that for all $n \geq N$ and $x \in A$

$$d_j(f_n(x), f(x)) < \varepsilon, \quad \forall j \in J.$$

In other words, $f_n \to f$ *uniformly if and only if* $\lim_{n\to\infty} d_j(f_n, f) = 0$.
Then $x_n^* \to x^*$ *as* $n \to \infty$*, where* $Fix(f_n) = \{x_n^*\}, n \in \mathbb{N}$ *and* $Fix(f) = \{x^*\}$.

Proof: Applying the definition of the b-pseudometric and the monotonicity property for the b-pseudometrics, we can write for any $n \in \mathbb{N}$:

$$\begin{aligned} d_j(x_n^*, x^*) &= d_j(f_n(x_n^*), f(x^*)) \le bd_j(f_n(x_n^*), f_n(x^*)) + bd_j(f_n(x^*), f(x^*)) \le \\ &\le b\theta_u d_{r(j)}(x_n^*, x^*) + bL_u d_{r(j)}(x_n^*, f_n(x_n^*)) + bd_j(f_n(x^*), f(x^*)) \le \\ &\le b\theta_u d_j(x_n^*, x^*) + bL_u d_j(x_n^*, f_n(x_n^*)) + bd_j(f_n(x^*), f(x^*)) = \\ &= b\theta_u d_j(x_n^*, x^*) + bd_j(f_n(x^*), f(x^*)), \quad \forall j \in J. \end{aligned}$$

From that, we deduce:

$$d_j(x_n^*, x^*) \le \frac{b}{1 - b\theta_u} d_j(f_n(x^*), f(x^*)), \quad \forall j \in J.$$

By using condition ii), letting $n \to \infty$, it results that:

$$d_j(x_n^*, x^*) \to 0, \text{ as } n \to \infty.$$

□

REMARK 1.3.170. *The theorems contained in this subsection, enunciated in b-pseudometric spaces, represent generalizations of results established in* [**83**] *regarding the fixed points in b-metric spaces.*

CHAPTER 2

MULTIVALUED ALMOST LOCAL CONTRACTIONS

1. Multivalued self almost local contractions

The notion of multivalued contraction was first introduced by Nadler in [**80**]. Let (X, d) be a complete metric space. Denote:

(1) $\mathcal{C}(X)$ the family of all nonempty closed subsets of X;
(2) $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X;
(3) $\mathcal{P}(X)$ the family of all nonempty subsets of X;
(4) $\mathcal{K}(X)$ the collection of all nonempty compact subsets of X.

For $A, B \in \mathcal{CB}(X)$ and $x \in X$, we consider the following functionals:
$D(x, A) = inf\{d(x, a) : a \in A\}$, the distance between x and A,
$D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$, the distance between A and B,
$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$, the diameter of A and B,
$H(A, B) = max\{sup\{D(a, B) : a \in A\}, sup\{D(b, A) : b \in B\}\}$, the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ induced by d. In fact, Hausdorff distance is the greatest of all the distances measured from a point in one set to some point in the other set We know that $\mathcal{CB}(X)$ is a metric space with the Pompeiu-Hausdorff distance function H. It is also known, that if (X, d) is a complete metric space, then $(\mathcal{CB}(X)$,$H)$ is a complete metric space, too. (Rus [**101**])
Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X and let $T : X \to \mathcal{P}(X)$ be a multivalued mapping.

Definition 2.1.171. [**80**] *Let $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping .*

(i) A point $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed point of f (resp. T) is denoted by $Fix(f)$, (resp. $Fix(T)$).
(ii) A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T will be denoted by $\mathcal{C}(f, T)$.
(iii) A point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f, T)$.

The following lemma can be found in Rus [**101**]. It is useful for the next theorem.

LEMMA 2.1.172. [**80**] *Let (X, d) be a metric space, let $A, B \subset X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq qH(A, B). \tag{2.162}$$

DEFINITION 2.1.173. [**23**] *Let (X, d) be a metric space and $T : X \to \mathcal{P}(X)$ be a multivalued operator. T is said to be a multivalued almost contraction or a multivalued (θ, L)-almost contraction if there exist two constants $\theta \in (0, 1), L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta \cdot d(x, y) + L \cdot D(y, Tx), \forall x, y \in X. \tag{2.163}$$

REMARK 2.1.174. *Because of the symmetry of the distance d and H, the almost contraction condition* (2.163) *includes the following dual one:*

$$H(Tx, Ty) \leq \theta \cdot d(x, y) + L \cdot D(x, Ty), \forall x, y \in X. \tag{2.164}$$

The following concept was published by Rus [**106**], respectively Rus et al. [**107**].

DEFINITION 2.1.175. [**106**] *Let (X, d) be a metric space and $T : X \to \mathcal{P}(X)$ be a multivalued operator. T is said to be a multivalued weakly Picard (breafly MWP) operator if for every $x \in X$ and for each $y \in T(x)$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that*

(i) $x_0 = x, x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, *for every* $n = 0, 1, 2, \cdots$;
(iii) the sequence $\{x_n\}_{n=0}^{\infty}$ *is convergent to a fixed point of* T.

If the sequence $\{x_n\}_{n=0}^{\infty}$ satisfy conditions (i) and (ii), then it is called a sequence of successive approximations of T with the starting point (x, y) or a Picard iteration of T or a (Picard) orbit of T at the initial point x_0.

THEOREM 2.1.176. [**23**] *Let (X, d) be a metric space and $T : X \to \mathcal{P}(X)$ be a (θ, L)- contraction. Then*
(1) $Fix(T) \neq \phi$,
(2) for any $x_0 \in X$, there exists the orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point u of T, for which the following estimates hold:

$$d(x_n, u) \leq \frac{h^n}{1-h} d(x_0, x_1), \quad n = 0, 1, 2, ... \tag{2.165}$$

$$d(x_n, u) \leq \frac{h}{1-h} d(x_{n-1}, x_n), \quad n = 1, 2, ... \tag{2.166}$$

for a certain constant $h < 1$.

It is our aim to extend the almost local contractions to the multivalued case. We need to introduce new concepts and notions in uniform spaces, as follows:

DEFINITION 2.1.177. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. For $A, B \in \mathcal{K}$, $x \in X$ denote*
$D_j(A,B) = inf\{d_j(a,b) : a \in A, b \in B, j \in J\}$
$\delta_j(A,B) = sup\{d_j(a,b) : a \in A, b \in B, j \in J\}$;
$D_j(x,A) = inf\{d_j(x,a) : a \in A, j \in J\}$;
$H_j(A,B) = max\{sup\{D_j(a,B) : a \in A\}, sup\{D_j(b,A) : b \in B\}\}$,
the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ induced by d_j, $\forall j \in J$.

LEMMA 2.1.178. *Assume that $X, J, \mathcal{D}, \tau$ are as in Definition 2.1.177.*
If $D_j(x,A) = 0$, then $x \in A$.

Proof: If $inf\{d_j(x,a) : a \in A\} = 0$, then for every $n \in \mathbb{N}$ there exists $a_n \in A$ such that $d_j(x,a_n) < \frac{1}{n}$. This means that the sequence $\{a_n\}$ converges to x. The subset A is assumed to be compact, which means it is closed. It follows that $x \in A$. □

DEFINITION 2.1.179. [**25**] *Let (X,d) be a metric space. A multivalued mapping $T : X \to C(X)$ is said to be continuous at the point p if*

$$\lim_{n\to\infty} d(x_n,p) = 0 \text{ implies } \lim_{n\to\infty} H(Tx_n,Tp) = 0.$$

Observe that in the work of Rhoades (see [**93**]) instead of H, the author used the functional D.

Approximate fixed points of multivalued contractions

DEFINITION 2.1.180. [**10**] *A multivalued mapping $T : X \to \mathcal{P}(X)$ is said to have the approximate fixed point property provided*

$$\inf_{x\in X} d(x,Tx) = 0, \tag{2.167}$$

or, equivalently, for any $\varepsilon > 0$, there exists $z \in X$ such that

$$d(z,Tz) \le \varepsilon, \tag{2.168}$$

or, equivalently, for any $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that

$$T(x_\varepsilon) \cap B_\varepsilon(x_\varepsilon) \neq \phi, \tag{2.169}$$

where $B_r(x)$ denotes a closed ball of radius r centered at x.

Main results

The purpose of the present section is to present the author's main contributions to the case of multivalued almost local contractions in uniform spaces, starting from the multivalued almost contractions. To this end, we will improve a lot the corresponding results in literature [**6**], [**42**], [**47**], [**52**], [**68**], [**69**], [**74**], [**79**] and many others.

DEFINITION 2.1.181. *Let X be a uniform Hausdorff space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. An operator $T : S \to \mathcal{P}(S)$ is called a multivalued almost local contraction or multivalued (θ, L)-almost local contraction related to $(\mathcal{D}, r)$ if there exist the constants $\theta \in (0,1)$ and $L \geq 0$ such that*

$$H_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot D_{r(j)}(y, Tx), \forall x, y \in S, \forall j \in J. \tag{2.170}$$

The following lemma is useful for the proof of future theorems.

LEMMA 2.1.182. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let $A, B \subset S$ and $q > 1$.*
Then, for every $j \in J$ and $a \in A$, there exists $b \in B$ such that

$$d_j(a,b) \leq q H_j(A,B). \tag{2.171}$$

Proof: We distinguish two cases:
Case I. If $H_j(A,B) = 0$, then for every $a \in A$, we have:

$$\underbrace{H_j(A,B)}_{=0} \geq D_j(a,B), \text{ which implies } D_j(a,B) = 0.$$

From that, we conclude: there exists $b \in B$ such that $d_j(a,b) = 0$.
The conclusion (2.171) is valid, since we obtain the obvious inequality $0 \leq 0$.
Case II. If $H_j(A,B) > 0$, then let us denote

$$\varepsilon \overset{not.}{=} \underbrace{(q-1)}_{>0} H_j(A,B) > 0. \tag{2.172}$$

Using the definition of $H_j(A,B)$ and $D_j(a,B)$, we conclude that for any $\varepsilon > 0$ there exists $b \in B$ such that

$$d_j(a,b) \leq q D_j(a,B) + \varepsilon \leq H_j(A,B) + \varepsilon. \tag{2.173}$$

The last two inequalities imply the conclusion of the lemma. $\square$

THEOREM 2.1.183. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Assume that the monotonicity property (1.26) for the pseudometrics is fulfilled. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. Let $T : S \to \mathcal{CB}(S)$ be a multivalued ALC.*
Then we have:

(1) $Fix(T) \neq \phi$;

(2) for any $x_0 \in S$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point u of T, for which the following estimates hold:

$$d_j(x_n, u) \leq \frac{h^n}{1-h} d_j(x_0, x_1), \quad n = 0, 1, 2, ... \tag{2.174}$$

$$d_j(x_n, u) \leq \frac{h}{1-h} d_j(x_{n-1}, x_n), \quad n = 1, 2, ... \tag{2.175}$$

for a certain constant $h < 1$ and for all $j \in J$.

Proof: We consider $q > 1$, let $x_0 \in X$ and $x_1 \in Tx_0$. We distinguish two cases:

Case I. If $H_j(Tx_0, Tx_1) = 0$, that means:

$$D_j(x_1, Tx_1) \leq \underbrace{H_j(Tx_0, Tx_1)}_{=0}, \tag{2.176}$$

and that is possible only if $D_j(x_1, Tx_1) = 0$. From here, we deduce $x_1 \in Tx_1$, which leads us to the conclusion $Fix(T) \neq \phi$.

Case II. Let $H_j(Tx_0, Tx_1) \neq 0$. According to Lemma 2.1.182, there exists $x_2 \in Tx_1$ such that

$$d_j(x_1, x_2) \leq qH_j(Tx_0, Tx_1). \tag{2.177}$$

By (2.170), we have

$$\begin{aligned} d_j(x_1, x_2) &\leq qH_j(Tx_0, Tx_1) \leq \\ &\leq q[\theta \cdot d_{r(j)}(x_0, x_1) + L \cdot \underbrace{D_{r(j)}(x_1, Tx_0)}_{=0}] = q\theta \cdot d_{r(j)}(x_0, x_1), \forall j \in J, \end{aligned}$$

since $x_1 \in Tx_0$.
We take $q > 1$ such that

$$h \overset{not.}{=} q\theta < 1,$$

and we obtain $d_j(x_1, x_2) < h \cdot d_{r(j)}(x_0, x_1)$. By applying the monotonicity of the pseudometrics, we get: $d_j(x_1, x_2) \leq h \cdot d_j(x_0, x_1), \quad \forall j \in J$.
If $H_j(Tx_1, Tx_2) = 0$ then $D_j(x_2, Tx_2) = 0$, that means $x_2 \in Tx_2$, by using Lemma 2.1.178. Let $H_j(Tx_1, Tx_2) \neq 0$. By using repeatedly Lemma 2.1.182, there exists $x_3 \in Tx_2$ such that

$$d_j(x_2, x_3) \leq h \cdot d_j(x_1, x_2), \quad \forall j \in J. \tag{2.178}$$

We obtain an orbit $\{\mathrm{x}_n\}_{n=0}^{\infty}$ of T at the point x_0 satisfying

$$d_j(x_n, x_{n+1}) \leq h \cdot d_j(x_{n-1}, x_n), \forall j \in J, \quad n = 1, 2, ... \tag{2.179}$$

By (2.179), we inductively obtain

$$d_j(x_n, x_{n+1}) \leq h^n d_j(x_0, x_1), \quad \forall j \in J, \tag{2.180}$$

and, respectively,

$$d_j(x_{n+k},x_{n+k+1}) \le h^{k+1} d_j(x_{n-1},x_n), \quad k \in \mathbb{N}, \forall j \in J. \tag{2.181}$$

Using the inequality (2.180), we obtain

$$d_j(x_n,x_{n+p}) \le \frac{h^n(1-h^p)}{1-h} d_j(x_0,x_1), \quad n,p \in \mathbb{N}, \forall j \in J. \tag{2.182}$$

Recall $0 < h < 1$, condition (2.182) show us that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each j, which shows that $\{\mathrm{x}_n\}_{n=0}^{\infty}$ is a Cauchy sequence. That means $\{\mathrm{x}_n\}_{n=0}^{\infty}$ is convergent with the limit u:

$$u = \lim_{n\to\infty} x_n. \tag{2.183}$$

By applying the definition of D_j, we get:

$$D_j(u,Tu) \le d_j(u,v) \le d_j(u,x_{n+1}) + \underbrace{d_j(x_{n+1},v)}_{\le H_j(Tx_n,Tu)}, \quad \forall v \in Tu, \forall j \in J.$$

The definition of H_j leads to:

$$D_j(u,Tu) \le d_j(u,x_{n+1}) + H_j(Tx_n,Tu),$$

which, by (2.170) yields

$$D_j(u,Tu) \le d_j(u,x_{n+1}) + \theta d_{r(j)}(x_n,u) + L \cdot \underbrace{D_{r(j)}(u,Tx_n)}_{\le d_{r(j)}(u,x_{n+1})}, \forall j \in J. \tag{2.184}$$

Letting $n \to \infty$ and using the fact that $x_{n+1} \in Tx_n$, we obtain $D_{r(j)}(u,Tx_n) \to 0$, as $n \to \infty$. By using inequality (2.184), we get:

$$D_j(u,Tu) = 0, \quad \forall j \in J.$$

Since Tu is closed, this means $u \in Tu$.
We let $p \to \infty$ in (2.182) to obtain (2.174). Using the inequality (2.181), we get

$$d_j(x_n,x_{n+p}) \le \frac{h(1-h^p)}{1-h} d_j(x_{n-1},x_n), \quad p \in \mathbb{N}, n \ge 1, \forall j \in J, \tag{2.185}$$

and letting $p \to \infty$ in (2.185), we obtain (2.175). □

The next theorem shows that any multivalued almost local contraction is continuous at the fixed point.

THEOREM 2.1.184. *Assume that $X, J, \mathcal{D}, r, \tau$ and S are as in Theorem 2.1.183 and let $T : S \to \mathcal{P}(S)$ be a multivalued almost local contraction.*
Then T is continuous at p, for any $p \in Fix(T)$.

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in the subset S converging to the fixed point p. Then by taking $y := y_n$ and $x := p$ in the multivalued almost local contraction condition (2.170), we get

$$H_j(Tp, Ty_n) \leq \theta \cdot d_{r(j)}(p, y_n) + L \cdot D_{r(j)}(y_n, Tp), n = 0, 1, 2, ..., \forall j \in J. \tag{2.186}$$

Using the definition of $D_j(y_n, Tp)$, we know that:

$$D_j(y_n, Tp) = inf\{d_j(y_n, a) : a \in Tp\} \leq H_j(Tp, Ty_n).$$

Take for example $a = p \in Tp$. Now, we have the following inequality:

$$D_{r(j)}(y_n, Tp) \leq d_{r(j)}(y_n, p), \quad \forall j \in J.$$

The last two inequalities together imply:

$$\begin{aligned} D_j(Ty_n, Tp) &\leq H_j(Tp, Ty_n) \leq \\ &\leq (\theta + L) \cdot d_{r(j)}(y_n, p), n = 0, 1, 2, ..., \forall j \in J. \end{aligned} \tag{2.187}$$

Now, by letting $n \to \infty$ in (2.187), we get $Ty_n \to Tp$ as $n \to \infty$.
Thus, T is continuous at p. □

Remark 2.1.185. *Theorems 2.1.183 and 2.1.184 provide generalizations in uniform spaces of the results contained in* [**23**] *and* [**25**]*, respectively.*

Theorem 2.1.186. *Assume that $X, J, \mathcal{D}, r, \tau$ and S are as in Definition 2.1.181 and let $T : S \to \mathcal{CB}(S)$ be a generalized multivalued almost local contraction, that is, a mapping for which there exist $\theta \in (0, 1)$ and some $L \geq 0$ such that*

$$H_j(Tx, Ty) \leq \theta d_{r(j)}(x, y) + L \min\{D_{r(j)}(x, Tx), D_{r(j)}(y, Ty), D_{r(j)}(x, Ty), D_{r(j)}(y, Tx)\},, \tag{2.188}$$

for all $j \in J$ and for every $x, y \in S$.

Then $Fix(T) \neq \phi$ and for any $p \in Fix(T)$, T is continuous at p.

Proof: The proof for the existence of the fixed point is very similar to the proof of Theorem 2.1.183, with minor differences, but it will be presented below:
We consider $q > 1$, let $x_0 \in X$ and $x_1 \in Tx_0$. If $H_j(Tx_0, Tx_1) = 0$, that means from the definition of D_j and H_j:

$$0 = H_j(Tx_0, Tx_1) \geq D_j(x_1, Tx_1), \tag{2.189}$$

and that is possible only if $D_j(x_1, Tx_1) = 0$. Thus, we obtain $x_1 \in Tx_1$, which leads us to the conclusion $Fix(T) \neq \phi$.
Let $H_j(Tx_0, Tx_1) \neq 0$. According to Lemma 2.1.182, there exists $x_2 \in Tx_1$ such that

$$d_j(x_1, x_2) \leq qH_j(Tx_0, Tx_1). \tag{2.190}$$

By (2.188) we have

$$d_j(x_1,x_2) \le q[\theta \cdot d_{r(j)}(x_0,x_1) + L \cdot D_{r(j)}(x_1,Tx_0)] = q\theta \cdot d_{r(j)}(x_0,x_1),$$

since we have:

$$\min\{D_{r(j)}(x_0,Tx_0), D_{r(j)}(x_1,Tx_1), D_{r(j)}(x_0,Tx_1), D_{r(j)}(x_1,Tx_0)\} = D_{r(j)}(x_1,Tx_0) = 0.$$

By applying the monotonicity of the pseudometrics, we get: $d_j(x_1,x_2) \le q\theta \cdot d_j(x_0,x_1)$. We take $q > 1$ such that

$$h \overset{not.}{=} q\theta < 1,$$

and we obtain $d_j(x_1,x_2) \le h \cdot d_j(x_0,x_1)$.
If $H_j(Tx_1,Tx_2) = 0$, then $D_j(x_2,Tx_2) = 0$ (as in the inequality (2.189)). Thus, $x_2 \in Tx_2$, by using Lemma 2.1.178.
Let $H_j(Tx_1,Tx_2) \neq 0$. Again, by using Lemma 2.1.182, there exists $x_3 \in Tx_2$ such that

$$d_j(x_2,x_3) \le h \cdot d_j(x_1,x_2), \forall j \in J. \tag{2.191}$$

We obtain an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 satisfying

$$d_j(x_n,x_{n+1}) \le h \cdot d_j(x_{n-1},x_n), \forall j \in J, \quad n = 1,2,... \tag{2.192}$$

By (2.192), we inductively obtain

$$d_j(x_n,x_{n+1}) \le h^n d_j(x_0,x_1), \forall j \in J, \tag{2.193}$$

and, respectively,

$$d_j(x_{n+k},x_{n+k+1}) \le h^{k+1} d_j(x_{n-1},x_n), \quad k \in \mathbb{N}, \forall j \in J. \tag{2.194}$$

Using the inequality (2.193), we obtain

$$d_j(x_n,x_{n+p}) \le \frac{h^n(1-h^p)}{1-h} d_j(x_0,x_1), \quad n,p \in \mathbb{N}, \forall j \in J. \tag{2.195}$$

Recalling $0 < h < 1$, condition (2.195) shows us that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each j, which means that $\{\mathrm{x}_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore, $\{x_n\}_{n=0}^{\infty}$ is convergent with the limit u:

$$u = \lim_{n\to\infty} x_n. \tag{2.196}$$

We get for all $j \in J$ and for $v \in Tu$:

$$D_j(u,Tu) \le d_j(u,v) \le d_j(u,x_{n+1}) + d_j(x_{n+1},v), \forall j \in J.$$

At this point, apply the Definition of the funtional H_j and we obtain:

$$D_j(u,Tu) \le d_j(u,x_{n+1}) + H_j(Tx_n,Tu), \forall j \in J,$$

which, by (2.188), yields

$$D_j(u,Tu) \le d_j(u,x_{n+1}) + \theta d_{r(j)}(x_n,u) +$$
$$+L \cdot \underbrace{\min\{D_{r(j)}(x_n,Tx_n), D_{r(j)}(u,Tu), D_{r(j)}(x_n,Tu), D_{r(j)}(u,Tx_n)\}}_{\le D_{r(j)}(u,Tx_n)}.$$

From that, we deduce:

$$D_j(u,Tu) \le d_j(u,x_{n+1}) + \theta d_{r(j)}(x_n,u) + L \cdot D_{r(j)}(u,Tx_n), \forall j \in J. \tag{2.197}$$

Letting $n \to \infty$ and using the fact that $x_{n+1} \in Tx_n$, we obtain $D_{r(j)}(u,Tx_n) \to 0$, as $n \to \infty$. We get

$$D_{r(j)}(u,Tu) = 0, \quad \forall j \in J.$$

Since Tu is closed, this implies $u \in Tu$.
Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in the subset S converging to $p \in Fix(T)$. Applying condition (2.188), by taking $y := y_n$ and $x := p$, we obtain:

$$D_j(Tp,Ty_n) \le H_j(Tp,Ty_n) \le \theta d_{r(j)}(p,y_n), \quad \forall j \in J, n = 0,1,2,\cdots,$$

since:

$$\min\{D_{r(j)}(p,Tp), D_{r(j)}(y_n,Ty_n), D_{r(j)}(p,Ty_n), D_{r(j)}(y_n,Tp)\} = D_{r(j)}(p,Tp) = 0.$$

We obtain from the last inequality :

$$D_j(Ty_n,Tp) \le \theta d_{r(j)}(y_n,p), \quad \forall j \in J, n = 0,1,2,\cdots \tag{2.198}$$

If we let $n \to \infty$ in (2.198), we obtain $Ty_n \to Tp$, which means that T is continuous at p. □

In the sequel, we extend the generalized multivalued almost contractions (see [**23**]) to the more general case of generalized multivalued almost local contractions.

Definition 2.1.187. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. An operator $T : S \to \mathcal{P}(S)$ is called a generalized multivalued (α, L) almost local contraction with respect to $(\mathcal{D}, r)$ if, for every $j \in J$, there exists a function $\alpha : [0,\infty) \to [0,1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0,\infty)$ such that*

$$H_j(Tx,Ty) \le \alpha(d_{r(j)}(x,y)) \cdot d_{r(j)}(x,y) + L \cdot \min\Big\{D_{r(j)}(x,Tx),$$
$$D_{r(j)}(y,Ty), D_{r(j)}(x,Ty), D_{r(j)}(y,Tx)\Big\}, \forall x,y \in S, \forall j \in J. \tag{2.199}$$

LEMMA 2.1.188. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Assume the pseudometrics satisfies the* (1.26) *monotonicity property. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. Consider the mapping $T : S \to \mathcal{C}(S)$. Then, for every $x \in S$ with $D_j(x, Tx) > 0$ and any $b \in (0, 1)$, there exists $y \in Tx, y \neq x$, such that*

$$bd_{r(j)}(x, y) \leq D_j(x, Tx), \quad \forall j \in J.$$

Proof: Having in view that Tx is nonempty and closed, the inequality $D_j(x, Tx) > 0$ implies that there exists $y \in Tx, y \neq x$. From the definition of $D_j(x, Tx)$ we know that, for every $\varepsilon > 0$, there exists $y \in Tx$ such that

$$d_{r(j)}(x, y) \leq D_j(x, Tx) + \varepsilon, \quad \forall j \in J.$$

At this point, by considering $\varepsilon = \left(\frac{1}{b} - 1\right) D_j(x, Tx) > 0$, we obtain the conclusion of this lemma. □

THEOREM 2.1.189. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. Consider the operator $T : S \to \mathcal{C}(S)$ and assume that the following conditions are fulfilled:*

(i) the mapping $f : S \to R_+ \quad f(x) = D_{r(j)}(x, Tx), x \in S$ is lower semi-continuous;
(ii) there exist $L \geq 0, b \in (0, 1)$ and $\varphi : (0, \infty) \to [0, b)$ such that for all $t \in (0, \infty)$,

$$\limsup_{r\to t^+} \varphi(r) < b, \tag{2.200}$$

and for all $x \in S, \exists y \in I_b^x$ such that

$$\begin{aligned} D_j(y, Ty) \leq \varphi(d_{r(j)}(x, y)) \cdot d_{r(j)}(x, y) + L \cdot \min\{&D_{r(j)}(x, Tx), \\ &D_{r(j)}(y, Ty), D_{r(j)}(x, Ty), D_{r(j)}(y, Tx)\}, \forall j \in J, \end{aligned} \tag{2.201}$$

where we denote $I_b^x = \{y \in Tx : bd_{r(j)}(x, y) \leq D_j(x, Tx)\}$, for all $j \in J$.
Then T has a fixed point.

Proof: If there exists $x \in S$ such that $D_{r(j)}(x, Tx) = 0$, for every $j \in J$, this means that $x \in Tx$. Therefore x is a fixed point of T.
Having in view that the range of T is closed, for each $b \in (0, 1)$ and any $x \in X$, satisfying $D_{r(j)}(x, Tx) > 0$, $\forall j \in J$. It results by Lemma 2.1.188 that there exists $y \in Tx$ such that $y \in I_b^x$, i.e.,

$$bd_{r(j)}(x, y) \leq D_j(x, Tx), \quad \forall j \in J. \tag{2.202}$$

Further, suppose that we have $y \in I_b^x, y \neq x$, otherwise $y = x \in Tx$ is actually a fixed point of T, which would complete the proof.

Take $x_1 \in S$ arbitrary but fixed with $D_{r(j)}(x_1, Tx_1) > 0$, for every $j \in J$. Combining (2.202) and condition (ii), there exists $x_2 \in Tx_1, x_2 \neq x_1$, satisfying:

$$bd_{r(j)}(x_1, x_2) \leq D_j(x_1, Tx_1), \quad \forall j \in J. \tag{2.203}$$

The last inequality can be written, using (2.201) in the equivalent form:

$$D_j(x_2, Tx_2) \leq \varphi\Big(d_{r(j)}(x_1, x_2)\Big) \cdot d_{r(j)}(x_1, x_2), \text{ where } \varphi(d_{r(j)}(x_1, x_2)) < b, \tag{2.204}$$

because $D_{r(j)}(x_2, Tx_1) = 0$.
At this point, (2.203) and (2.204) can be merged as:

$$\begin{aligned} D_j(x_1, Tx_1) - D_j(x_2, Tx_2) &\geq bd_{r(j)}(x_1, x_2) - \varphi(d_{r(j)}(x_1, x_2)) \cdot d_{r(j)}(x_1, x_2) = \\ &= [b - \varphi(d_{r(j)}(x_1, x_2))] \cdot d_{r(j)}(x_1, x_2) > 0, \end{aligned}$$

for every $j \in J$. For the obtained x_2, by continuing the construction of the sequence $\{x_n\}$, we claim there exists $x_3 \in Tx_2, x_3 \neq x_2$, satisfying

$$bd_{r(j)}(x_2, x_3) \leq D_j(x_2, Tx_2), \quad \forall j \in J, \tag{2.205}$$

such that

$$D_j(x_3, Tx_3) \leq \varphi\Big(d_{r(j)}(x_2, x_3)\Big) \cdot d_{r(j)}(x_2, x_3), \text{ where } \varphi(d_{r(j)}(x_2, x_3)\Big) < b. \tag{2.206}$$

Again, by (2.205) and (2.206), we get

$$\begin{aligned} D_j(x_2, Tx_2) - D_j(x_3, Tx_3) &\geq bd_{r(j)}(x_2, x_3) - \varphi(d_{r(j)}(x_2, x_3)) \cdot d_{r(j)}(x_2, x_3) = \\ &= [b - \varphi(d_{r(j)}(x_2, x_3))] \cdot d_{r(j)}(x_2, x_3) > 0, \forall j \in J. \end{aligned}$$

Thus, we obtain

$$d_{r(j)}(x_2, x_3) \leq \frac{1}{b} D_j(x_2, Tx_2) \leq \frac{1}{b} \varphi(d_{r(j)}(x_1, x_2)) \cdot d_{r(j)}(x_1, x_2) < d_{r(j)}(x_1, x_2),$$

for every $j \in J$. By induction with respect to $n > 1$, we conclude there exist $x_{n+1} \in Tx_n, x_n \neq x_{n+1}$, such that

$$bd_{r(j)}(x_n, x_{n+1}) \leq D_j(x_n, Tx_n), \quad \forall j \in J, \tag{2.207}$$

and, similar to (2.206), satisfying

$$D_j(x_{n+1}, Tx_{n+1}) \leq \varphi\Big(d_{r(j)}(x_n, x_{n+1})\Big) \cdot d_{r(j)}(x_n, x_{n+1}), \text{ where } \varphi(d_{r(j)}(x_n, x_{n+1})\Big) < b. \tag{2.208}$$

By using (2.207) and (2.208), we have

$$\begin{aligned} D_j(x_n, Tx_n) - D_j(x_{n+1}, Tx_{n+1}) &\geq bd_{r(j)}(x_n, x_{n+1}) - \\ &- \varphi(d_{r(j)}(x_n, x_{n+1})) \cdot d_{r(j)}(x_n, x_{n+1}) = \\ &= [b - \varphi(d_{r(j)}(x_n, x_{n+1}))] \cdot d_{r(j)}(x_n, x_{n+1}) > 0, \forall j \in J, \end{aligned} \tag{2.209}$$

and also, we get:

$$d_{r(j)}(x_n, x_{n+1}) < d_{r(j)}(x_n, x_{n-1}). \tag{2.210}$$

It is obvious that $\{D_j(x_n, Tx_n)\}_{n\in\mathbb{N}}$ and $\{d_{r(j)}(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ are both decreasing sequences of positive numbers, which means they are convergent.
By applying (2.200), it results that there exists $s \in [0, b)$ such that

$$\limsup_{n\to\infty} \varphi(d_{r(j)}(x_n, x_{n+1})) = s, \quad \forall j \in J. \tag{2.211}$$

From that, we deduce that there exists $n_0 \in \mathbb{N}$ such that for each $b_0 \in (q, b)$,

$$\varphi(d_{r(j)}(x_n, x_{n+1})) < b_0, \quad \forall n > n_0. \tag{2.212}$$

Denote $a = b - b_0$, we get from (2.209) the following inequality:

$$D_j(x_n, Tx_n) - D_j(x_{n+1}, Tx_{n+1}) \geq a \cdot d_{r(j)}(x_n, x_{n+1}), \forall n > n_0. \tag{2.213}$$

At this point, we use (2.207), (2.208), (2.212) and we obtain for all $n > n_0$:

$$\begin{aligned}
D_j(x_{n+1}, Tx_{n+1}) &\leq \varphi(d_{r(j)}(x_n, x_{n+1})) \leq \frac{\varphi(d_{r(j)}(x_n, x_{n+1}))}{b} \cdot D_{r(j)}(x_n, Tx_n) \leq \\
&\leq \ldots \leq \frac{\varphi(d_{r(j)}(x_n, x_{n+1})) \cdot \ldots \cdot \varphi(d_{r(j)}(x_1, x_2))}{b^n} \cdot D_{r(j)}(x_1, Tx_1) = \\
&= \frac{\varphi(d_{r(j)}(x_n, x_{n+1})) \cdot \ldots \cdot \varphi(d_{r(j)}(x_{n_0+1}, x_{n_0+2}))}{b^{n-n_0}} \cdot \\
&\cdot \frac{\varphi(d_{r(j)}(x_{n_0}, x_{n_0+1})) \cdot \ldots \cdot \varphi(d_{r(j)}(x_1, x_2))}{b^{n_0}} \cdot D_{r(j)}(x_1, Tx_1) < \\
&< \left(\frac{b_0}{b}\right)^{n-n_0} \cdot \frac{\varphi(d_{r(j)}(x_{n_0}, x_{n_0+1})) \cdot \ldots \cdot \varphi(d_{r(j)}(x_1, x_2))}{b^{n_0}} \cdot D_{r(j)}(x_1, Tx_1),
\end{aligned}$$

for every $j \in J$. Note that $b_0 < b$, therefore we have:

$$\lim_{n\to\infty} \left(\frac{b_0}{b}\right)^{n-n_0} = 0,$$

thus, it follows from the previous inequalities:

$$\lim_{n\to\infty} D_j(x_n, Tx_n) = 0, \quad \forall j \in J.$$

We claim that $\{x_n\}$ is a Cauchy sequence. In order to show that, apply the triangle inequality and (2.213), for $n, p \in \mathbb{N}, n, p > n_0$ and we get:

$$\begin{aligned}
d_{r(j)}(x_n, x_{n+p}) &\leq \sum_{j=n}^{n+p-1} d_{r(j)}(x_j, x_{j+1}) \leq \frac{1}{a} \sum_{j=n}^{n+p-1} [D_{r(j)}(x_j, Tx_j) - D_{r(j)}(x_{j+1}, Tx_{j+1})] = \\
&= \frac{1}{a}\Big(D_{r(j)}(x_n, Tx_n) - D_{r(j)}(x_{n+p}, Tx_{n+p})\Big), \qquad (2.214)
\end{aligned}$$

where we denote $a = b - b_0$. Having in view that the sequence of positive real numbers $\{D_j(x_n, Tx_n)\}_{n\in\mathbb{N}}$ is convergent, it is also a Cauchy sequence, which means that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$.
As the subset $S \subset X$ is assumed to be sequentially τ-complete, there exists x^* in S

such that $\{T^n x\}_{n\in\mathbb{N}}$ is τ-convergent to x^*. Besides, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges with respect to the topology τ to x^*, which implies

$$0 \leq D_j(x^*, Tx^*) \leq \liminf_{n\to\infty} D_j(x_n, Tx_n) = \lim_{n\to\infty} D_j(x_n, Tx_n) = 0, \quad \forall j \in J.$$

Using the fact that Tx^* is closed, we obtain $x^* \in Tx^*$, i.e., x^* is a fixed point of T. $\square$

2. Non-self multivalued almost local contractions

In [**23**], M. Berinde and V. Berinde introduce the non-self multivalued almost contractions in the case of a metric space.

Definition 2.2.190. [**23**] *The notations are the same as in the case of self multivalued almost local contractions. Let (X,d) be a metric space and K a nonempty subset of X.*
A mapping $T : K \to CB(X)$ is called a multivalued non-self almost contraction if there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta \cdot d(x,y) + L \cdot D(y, Tx), \quad \forall x, y \in K. \tag{2.215}$$

Proposition 2.2.191. [**4**] *Let K be a nonempty closed subset of a convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x,y) = d(x,z) + d(z,y). \tag{2.216}$$

Lemma 2.2.192. [**4**] *Let (X,d) be a metric space and $A, B \in CB(X)$. If $x \in A$, then for each positive number α, there exists $y \in B$, such that*

$$d(x,y) \leq H(A,B) + \alpha. \tag{2.217}$$

Fundamental results
We extend Definition 2.2.190 of a non-self multivalued almost contraction to the case of non-self multivalued almost local contractions in uniform spaces, in order to study the existence of their fixed points. The main result of this section is represented by two fixed point theorems for multivalued non-self almost local contractions.

Definition 2.2.193. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let r be a function from J to J. Let K a nonempty closed subset of X.*
An operator $T : K \to \mathcal{CB}(X)$ is called a non-self multivalued almost local contraction with respect to $(\mathcal{D}, r)$ if there exist the constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot D_{r(j)}(y, Tx), \forall x, y \in K, \forall j \in J. \tag{2.218}$$

DEFINITION 2.2.194. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. X is called uniform space with the property of convexity if for any two distinct points $x, y \in X$, there exists $z \in X$ such that $d_j(x,z) + d_j(z,y) = d_j(x,y), \quad \forall j \in J$.*

REMARK 2.2.195. *If K is a closed subset of the uniform space X with the property of convexity, then for every $x \in K, y \notin K$, there exists a point $z \in \partial K$ such that*

$$d_j(x,z) + d_j(z,y) = d_j(x,y), \quad \forall j \in J.$$

THEOREM 2.2.196. *Let X be a uniform space with the property of convexity, let $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $S \subset X$. Let $r : J \to J$ be a function and let K a nonempty closed subset of X.*
Suppose that $T : K \to CB(X)$ is a multivalued almost local contraction, that is,

$$H_j(Tx,Ty) \le \theta \cdot d_{r(j)}(x,y) + L \cdot D_{r(j)}(y,Tx), \forall x,y \in K, \forall j \in J, \tag{2.219}$$

with $\theta \in (0,1)$ and $L \ge 0$ such that $\theta(1+L) < 1$.
Assume the monotonicity property is valid for the pseudometrics:

$$d_{r(j)}(x,y) \le d_j(x,y), \quad \forall j \in J, \forall x,y \in K, \tag{2.220}$$

and also assume that

$$D_{r(j)}(y,Tx) \le D_j(y,Tx) \text{ for each } j \in J, \forall x,y \in K. \tag{2.221}$$

If T satisfies Rothe's type condition, that is,

$$x \in \partial K \Longrightarrow Tx \subset K, \tag{2.222}$$

then T has a fixed point in K.

Proof: Consider two sequences $\{x_n\}$ and $\{y_n\}$ by constructing them as it follows: Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, let $x_1 = y_1$. If $y_1 \notin K$, then, according to Remark 2.2.195, there exists $x_1 \in \partial K$ such that

$$d_j(x_0,x_1) + d_j(x_1,y_1) = d_j(x_0,y_1). \tag{2.223}$$

We have $x_1 \in K$ and using Lemma 2.2.192 with $\alpha = \theta$, we choose $y_2 \in Tx_1$ such that

$$d_j(y_1,y_2) \le H_j(Tx_0,Tx_1) + \theta. \tag{2.224}$$

Again, if $y_2 \in K$, take $x_2 = y_2$. If $y_2 \notin K$, then there exists $x_2 \in \partial K$ such that

$$d_j(x_1,x_2) + d_j(x_2,y_2) = d_j(x_1,y_2). \tag{2.225}$$

Therefore, $x_2 \in K$, and by Lemma 2.2.192 and $\alpha = \theta^2$, we may select $y_3 \in Tx_2$ such that

$$d_j(y_2, y_3) \leq H_j(Tx_1, Tx_2) + \theta^2. \tag{2.226}$$

Continuing in this way, we construct two sequences $\{\mathrm{x}_n\}$ and $\{\mathrm{y}_n\}$ such that the following conditions are fulfilled:

(i) $y_{n+1} \in Tx_n$,
(ii) $d_j(y_n, y_{n+1}) \leq H_j(Tx_{n-1}, Tx_n) + \theta^n$,
(iii) $y_n \in K \Rightarrow y_n = x_n$,
(iv) $y_n \neq x_n$ when $y_n \notin K$, and then $x_n \in \partial K$ satisfying the condition

$$d_j(x_{n-1}, x_n) + d_j(x_n, y_n) = d_j(x_{n-1}, y_n), \tag{2.227}$$

for each $n \geq 1$.
Our next goal is to prove that $\{\mathrm{x}_n\}$ is a Cauchy sequence. For the simplicity, denote:

$$\begin{aligned} P &= \{x_i \in \{x_n\} : x_i = y_i\}, \\ Q &= \{x_i \in \{x_n\} : x_i \neq y_i\}. \end{aligned}$$

It is obvious that if $x_n \in Q$, then both x_{n-1} and x_{n+1} belong to the set P. We distinguish three possibilities as it follows:

Case 1. If $x_n, x_{n+1} \in P$, having in view the definition of the set P, we have $y_n = x_n$ and $y_{n+1} = x_{n+1}$. We obtain

$$\begin{aligned} d_j(x_n, x_{n+1}) &= d_j(y_n, y_{n+1}) \leq H_j(Tx_{n-1}, Tx_n) + \theta^n \leq \\ &\leq \theta \cdot d_{r(j)}(x_{n-1}, x_n) + L \cdot \underbrace{D_{r(j)}(x_n, Tx_{n-1})}_{=0} + \theta^n \leq \\ &\leq \theta \cdot d_j(x_{n-1}, x_n) + \theta^n, \end{aligned}$$

because $y_n \in Tx_{n-1}$.

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$.
In this case we have $y_n = x_n$ but $y_{n+1} \neq x_{n+1}$. We conclude from here:

$$\begin{aligned} d_j(x_n, x_{n+1}) &\leq d_j(x_n, x_{n+1}) + d_j(x_{n+1}, y_{n+1}) = d_j(x_n, y_{n+1}) = d_j(y_n, y_{n+1}) \leq \\ &\leq H_j(Tx_{n-1}, Tx_n) + \theta^n \leq \\ &\leq \theta \cdot d_{r(j)}(x_{n-1}, x_n) + L \cdot D_{r(j)}(x_n, Tx_{n-1}) + \theta^n \leq \\ &\leq \theta \cdot d_j(x_{n-1}, x_n) + \theta^n, \quad \forall j \in J. \end{aligned}$$

Case 3. If $x_n \in Q$, $x_{n+1} \in P$, then $y_n \neq x_n$, $y_{n+1} = x_{n+1}$, $y_{n-1} = x_{n-1}$ and $y_n \in Tx_{n-1}$. After simple computations, we get

$$\begin{aligned} d_j(x_n, x_{n+1}) &= d_j(x_n, y_{n+1}) \leq d_j(x_n, y_n) + d_j(y_n, y_{n+1}) \leq \\ &\leq d_j(x_n, y_n) + H_j(Tx_{n-1}, Tx_n) + \theta^n \leq \\ &\leq d_j(x_n, y_n) + \theta \cdot d_{r(j)}(x_{n-1}, x_n) + L \cdot D_{r(j)}(x_n, Tx_{n-1}) + \theta^n, \forall j \in J. \end{aligned}$$

Since $\theta < 1$, we have

$$\begin{aligned} d_j(x_n, x_{n+1}) &\leq d_j(x_n, y_n) + d_j(x_{n-1}, x_n) + L \cdot D_{r(j)}(x_n, Tx_{n-1}) + \theta^n = \\ &= d_j(x_{n-1}, y_n) + L \cdot D_{r(j)}(x_n, Tx_{n-1}) + \theta^n \leq \\ &\leq d_j(x_{n-1}, y_n) + L \cdot D_j(x_n, Tx_{n-1}) + \theta^n \leq \\ &\leq d_j(x_{n-1}, y_n) + L \cdot d_j(x_n, y_n) + \theta^n = \\ &= d_j(x_{n-1}, y_n) + L \cdot d_j(x_{n-1}, y_n) - L \cdot d_j(x_{n-1}, x_n) + \theta^n \leq \\ &\leq (1+L) d_j(y_{n-1}, y_n) + \theta^n \leq \\ &\leq (1+L) H_j(Tx_{n-2}, Tx_{n-1}) + (1+L)\theta^{n-1} + \theta^n \leq \\ &\leq (1+L)\theta \cdot d_{r(j)}(x_{n-2}, x_{n-1}) + (1+L) L D_{r(j)}(x_{n-1}, Tx_{n-2}) + \\ &+ (1+L)\theta^{n-1} + \theta^n \leq \\ &\leq (1+L)\theta \cdot d_j(x_{n-2}, x_{n-1}) + (1+L)\theta^{n-1} + \theta^n, \quad \forall j \in J. \end{aligned}$$

Having in view the condition $h = \theta(1+L) < 1$, we obtain

$$d_j(x_n, x_{n+1}) < h \cdot d_j(x_{n-2}, x_{n-1}) + h \cdot \theta^{n-2} + \theta^n, \quad \forall j \in J. \tag{2.228}$$

By combining all three cases, we get

$$d_j(x_n, x_{n+1}) \leq \alpha d_j(x_{n-1}, x_n) + \alpha^n, \quad \forall j \in J, \tag{2.229}$$

or the other possibility:

$$d_j(x_n, x_{n+1}) \leq \alpha d_j(x_{n-2}, x_{n-1}) + \alpha^{n-1} + \alpha^n, \quad \forall j \in J, \tag{2.230}$$

where

$$\alpha = \max\{\theta, h\} = h. \tag{2.231}$$

We inductively obtain:

$$d_j(x_n, x_{n+1}) \leq h^{\frac{n-1}{2}} \cdot \omega + h^{\frac{n}{2}}, \quad \forall j \in J, n \in \mathbb{N}, \tag{2.232}$$

where

$$\omega = \max\{d_j(x_0, x_1), d_j(x_1, x_2)\}. \tag{2.233}$$

By taking $n > m$, we have

$$\begin{aligned} d_j(x_n, x_m) &\leq d_j(x_n, x_{n-1}) + d_j(x_{n-1}, x_{n-2}) + \cdots + d_j(x_{m-1}, x_m) \leq \\ &\leq (h^{\frac{n-1}{2}} + h^{\frac{n-2}{2}} + \ldots h^{\frac{m-1}{2}})\omega + \\ &\quad + \alpha^{\frac{n}{2}} \cdot n + \alpha^{\frac{n-1}{2}} \cdot (n-1) + \cdots + \alpha^{\frac{m}{2}} \cdot m. \end{aligned}$$

These relations show us that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$.
As the subset K is assumed to be sequentially τ-complete, there exists z in K such that

$$z = \lim_{n\to\infty} x_n. \tag{2.234}$$

From the construction of the sequence $\{x_n\}_{n\in\mathbb{N}}$, we conclude: there is a subsequence $\{\mathrm{x}_p\}$ such that

$$y_p = x_p \in Tx_{p-1}. \tag{2.235}$$

In what follows, we propose to prove that $z \in Tz$. In fact, by (i), $x_p \in Tx_{p-1}$. Since $x_p \to z$ as $p \to \infty$, we have

$$D_j(z, Tx_{p-1}) \to 0, \text{ as } q \to \infty. \tag{2.236}$$

It is easy to see that

$$\begin{aligned} D_j(z, Tz) &\leq d_j(z, x_p) + d_j(x_p, Tz) \leq d_j(z, x_p) + H_j(Tx_{p-1}, Tz) \leq \\ &\leq d_j(z, x_p) + \theta \cdot d_{r(j)}(x_{p-1}, z) + L \cdot D_{r(j)}(z, Tx_{p-1}) \overset{mon.}{\leq} \\ &\leq d_j(z, x_p) + \theta \cdot d_j(x_{p-1}, z) + L \cdot D_j(z, Tx_{p-1}), \quad \forall j \in J. \end{aligned}$$

Now, if we let $q \to \infty$, implies that $D_j(z, Tz) = 0$. Thus, we get $z \in Tz$. □

Note that, by Theorem 2.2.196 we obtain a fixed point theorem for multivalued non-self almost contractions stated in [**1**] as a particular case by letting $r(j) = j$, $\theta = \delta, d_j = d, j \in J$.

Corollar 2.2.197. *Assume that $X, J, \mathcal{D}, r, \tau$ and K are as in Definition 2.2.193 and suppose that $T : K \to \mathcal{CB}(X)$ is a multivalued contraction, that is,*

$$H_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y), \quad \forall x, y \in K, \tag{2.237}$$

with $\theta \in (0, 1)$. Further, we assume that the monotonicity property (2.221) *is valid. If T satisfies Rothe's type condition, that is, $x \in \partial K \Longrightarrow Tx \subset K$, then T has a fixed point in K, which means: there exists $z \in K$ such that $z \in Tz$.*

Remark 2.2.198. *Corollary 2.2.197 represents a particular case of Theorem 2.2.196 by taking $L = 0$, therefore we skip over the proof.*

THEOREM 2.2.199. *Let X be a uniform space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudo-metrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $K \subset X$. Let r be a function from J to J. Assume the monotonicity property* (2.220) *satisfied. The operator $T : K \to CB(X)$ satisfies the following contractive condition: for every $x, y \in K$,*

$$\begin{aligned} H_j(Tx,Ty) &\leq \alpha \cdot d_{r(j)}(x,y) + \beta \cdot \max\{D_{r(j)}(x,Tx), D_{r(j)}(y,Ty)\} + \\ &+ \gamma[D_{r(j)}(x,Ty) + D_{r(j)}(y,Tx)], \quad \forall j \in J, \end{aligned} \tag{2.238}$$

with $\alpha, \beta, \gamma \geq 0$ such that $p = \left(\frac{1+\alpha+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) < 1$.
If T satisfies Rothe's boundary condition, that is, $x \in \partial K \Longrightarrow Tx \subset K$, then T has a fixed point in K.

Proof: As in the proof of Theorem 2.2.196, consider two sequences $\{\mathrm{x}_n\}$ and $\{\mathrm{y}_n\}$ by constructing them step by step as it follows:
Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, let $x_1 = y_1$. If $y_1 \notin K$, then there exists $x_1 \in \partial K$ such that

$$d_j(x_0,x_1) + d_j(x_1,y_1) = d_j(x_0,y_1), \quad \forall j \in J. \tag{2.239}$$

Take $y_2 \in Tx_1$ and, according to Lemma 2.2.192, we have:

$$d_j(x_1,y_2) \leq H_j(Tx_0,Tx_1) + (1-\beta-\gamma)\varepsilon, \quad \forall j \in J, \tag{2.240}$$

where $\varepsilon = p$. Again, if $y_2 \in K$, let $x_2 = y_2$. If $y_2 \notin K$, then there exists $x_2 \in \partial K$ such that

$$d_j(x_1,x_2) + d_j(x_2,y_2) = d_j(x_1,y_2), \quad \forall j \in J. \tag{2.241}$$

Continuing in this way, we introduce two sequences $\{\mathrm{x}_n\}$ and $\{\mathrm{y}_n\}$ such that

(i) $y_{n+1} \in Tx_n$,
(ii) $d_j(y_n,y_{n+1}) \leq H_j(Tx_{n-1},Tx_n) + (1-\beta-\gamma)\varepsilon^n$,
(iii) $x_{n+1} = y_{n+1}$ if $y_{n+1} \in K$,
(iv) if $y_{n+1} \notin K$, then x_{n+1} will satisfy condition

$$d_j(x_n,x_{n+1}) + d_j(x_{n+1},y_{n+1}) = d_j(x_n,y_{n+1}), \quad \forall j \in J. \tag{2.242}$$

Our next goal is to prove that $\{\mathrm{x}_n\}$ is a Cauchy sequence. Denote:

$$P = \{x_i \in \{x_n\} : x_i = y_i\},$$

$$Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

We distinguish three cases as it follows:

Case 1. If $x_n, x_{n+1} \in P$, having in view (2.238), we get:

$$\begin{aligned}
d_j(x_n, x_{n+1}) &\leq H_j(Tx_{n-1}, Tx_n) + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq \alpha \cdot d_{r(j)}(x_{n-1}, x_n) + \beta \cdot \max\{D_{r(j)}(x_{n-1}, Tx_{n-1}), D_{r(j)}(x_n, Tx_n)\} + \\
&+ \gamma[D_{r(j)}(x_{n-1}, Tx_n) + D_{r(j)}(x_n, Tx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq \alpha \cdot d_{r(j)}(x_{n-1}, x_n) + \beta \cdot \max\left\{d_{r(j)}(x_{n-1}, x_n), d_{r(j)}(x_n, x_{n+1})\right\} + \\
&+ \gamma d_{r(j)}(x_{n-1}, x_{n+1}) + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq \max\Big\{\frac{(\alpha+\beta+\gamma)d_{r(j)}(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma}, \\
&, \quad \frac{(\alpha+\gamma)d_{r(j)}(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma}\Big\} \leq \\
&\leq \max\left\{\frac{(\alpha+\beta+\gamma)}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta-\gamma}\right\} d_{r(j)}(x_{n-1}, x_n) + \varepsilon^n = \\
&= k \cdot d_{r(j)}(x_{n-1}, x_n) + \varepsilon^n, \quad \forall j \in J.
\end{aligned}$$

where we denote $k := \frac{(\alpha+\beta+\gamma)}{1-\gamma}$.

Case 2. If $x_n \in P$, $x_{n+1} \in Q$, from (2.238), we have:

$$\begin{aligned}
d_j(x_n, x_{n+1}) &\leq d_j(x_n, x_{n+1}) + d_j(x_{n+1}, x_{n+1}) = d_j(x_n, x_{n+1}) + d_j(x_{n+1}, y_{n+1}) = \\
&= d_j(x_n, y_{n+1}) = d_j(y_n, y_{n+1}) \leq \\
&\leq H_j(Tx_{n-1}, Tx_n) + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq \alpha \cdot d_{r(j)}(x_{n-1}, x_n) + \beta \cdot \max\{D_{r(j)}(x_{n-1}, Tx_{n-1}), D_{r(j)}(x_n, Tx_n)\} + \\
&+ \gamma[D_{r(j)}(x_{n-1}, Tx_n) + D_{r(j)}(x_n, Tx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq \alpha \cdot d_{r(j)}(x_{n-1}, x_n) + \beta \cdot \max\{d_{r(j)}(x_{n-1}, x_n), d_{r(j)}(x_n, y_{n+1})\} + \\
&+ \gamma d_{r(j)}(x_{n-1}, y_{n+1}) + (1-\beta-\gamma)\varepsilon^n, \quad \forall j \in J.
\end{aligned}$$

We observe that: if in the term of the coefficient β, the maximum is $d_{r(j)}(x_{n-1}, x_n)$, and if the monotonicity condition holds, then we can write:

$$d_j(x_n, y_{n+1}) \leq \frac{(\alpha+\beta+\gamma)d_j(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma}, \quad \forall j \in J.$$

On the other hand, if in the term of the coefficient β, the maximum is $d_{r(j)}(x_n, y_{n+1})$, and if the monotonicity condition holds, then we get:

$$d_j(x_n, y_{n+1}) \leq \frac{(\alpha+\gamma)d_j(x_{n-1}, x_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma}, \quad \forall j \in J.$$

After analyzing these two cases, we conclude:

$$d_j(x_n, x_{n+1}) \leq k \cdot d_{r(j)}(x_{n-1}, x_n) + \varepsilon^n, \quad \forall j \in J. \tag{2.243}$$

Case 3. If $x_n \in Q$, $x_{n+1} \in P$, then $y_n \neq x_n$, $y_{n+1} = x_{n+1}$, $y_{n-1} = x_{n-1}$ and $y_n \in Tx_{n-1}$. After using (2.238), by applying case 2, we get:

$$\begin{aligned}
d_j(x_n, x_{n+1}) &= d_j(x_n, y_{n+1}) \leq d_j(x_n, y_n) + d_j(y_n, \underbrace{x_{n+1}}_{=y_{n+1}}) \leq \\
&\leq d_j(x_n, y_n) + H_j(Tx_{n-1}, Tx_n) + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq d_j(x_{n-1}, y_n) + \alpha \cdot d_j(x_{n-1}, x_n) + \beta \cdot \max\{D_{r(j)}(x_{n-1}, Tx_{n-1}), D_{r(j)}(x_n, Tx_n)\} + \\
&+ \gamma[D_{r(j)}(x_{n-1}, Tx_n) + D_{r(j)}(x_n, Tx_{n-1})] + (1-\beta-\gamma)\varepsilon^n \leq \\
&\leq d_j(x_{n-1}, y_n) + \alpha \cdot d_j(x_{n-1}, x_n) + \beta \cdot \max\{d_{r(j)}(x_{n-1}, y_n), d_{r(j)}(x_n, x_{n+1})\} + \\
&+ \gamma[d_{r(j)}(x_{n-1}, x_{n+1}) + d_{r(j)}(x_n, y_n)] + (1-\beta-\gamma)\varepsilon^n, \quad \forall j \in J.
\end{aligned}$$

According to the triangle inequality, and the definition of sets P, Q, we can write:

$$\begin{aligned}
d_j(x_{n-1}, x_{n+1}) + d_j(x_n, y_n) &\leq d_j(x_{n-1}, x_n) + d_j(x_n, x_{n+1}) + d_j(x_n, y_n) = \\
&= d_j(x_{n-1}, y_n) + d_j(x_n, x_{n+1}), \quad \forall j \in J.
\end{aligned}$$

Therefore

$$\begin{aligned}
d_j(x_n, x_{n+1}) &\leq \max\Big\{\frac{(1+\alpha+\beta+\gamma)d_{r(j)}(x_{n-1}, y_n) + (1-\beta-\gamma)\varepsilon^n}{1-\gamma}, \\
&\quad \frac{(1+\alpha+\gamma)d_{r(j)}(x_{n-1}, y_n) + (1-\beta-\gamma)\varepsilon^n}{1-\beta-\gamma}\Big\} \leq \\
&\leq \max\left\{\frac{(1+\alpha+\beta+\gamma)}{1-\gamma}, \frac{(1+\alpha+\gamma)}{1-\beta-\gamma}\right\} d_{r(j)}(x_{n-1}, y_n) + \varepsilon^n \leq \\
&\leq \frac{1+\alpha+\gamma}{1-\beta-\gamma} \cdot d_{r(j)}(x_{n-1}, y_n) + \varepsilon^n \leq \\
&\leq p \cdot d_j(x_{n-2}, x_{n-1}) + \frac{(1+\alpha+\gamma)\varepsilon^{n-1}}{1-\beta-\gamma} + \varepsilon^n, \quad \forall j \in J.
\end{aligned}$$

After that step, it results by induction with respect to n, that:

$$d_j(x_{2n}, x_{2n+1}) \leq p^n \left(\omega + \frac{3n}{1-\beta-\gamma}\right), \quad \forall j \in J,$$

and also results the following inequality:

$$d_j(x_{2n+1}, x_{2n+2}) \leq p^{\frac{2n+1}{2}} \left(\omega + \frac{3n+1}{1-\beta-\gamma}\right), \quad \forall j \in J,$$

where we denoted

$$\omega = \max\{d_j(x_0, x_1), d_j(x_1, x_2)\}.$$

The last two inequalities let us to conclude that for any $m > n$,

$$d_j(x_m, x_n) \leq \sum_{i=n}^{m-1} d_j(x_i, x_{i+1}) \leq \omega \sum_{i=n}^{m-1} p^{\frac{i}{2}} + \frac{1}{1-\beta-\gamma} \sum_{i=n}^{m-1} p^{\frac{i}{2}}(3i+1), \forall j \in J,$$

which means that $\{x_n\}_{n\in\mathbb{N}}$ is d_j-Cauchy for each $j \in J$.

As the subset K is assumed to be sequentially τ-complete, $\{x_n\}_{n\in\mathbb{N}}$ is convergent with the limit z. We can always choose a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ such that $x_{n_k} = y_{n_k}$. From that, it follows:

$$\begin{aligned}
D_j(x_{n_k}, Tz) &\leq H_j(Tx_{n_k-1}, Tz) \leq \\
&\leq \alpha d_{r(j)}(x_{n_k-1}, z) + \beta \max\{D_{r(j)}(x_{n_k-1}, Tx_{n_k-1}), D_{r(j)}(z, Tz)\} + \\
&+ \gamma[D_{r(j)}(x_{n_k-1}, Tz) + D_{r(j)}(z, Tx_{n_k-1})] \leq \\
&\leq \alpha d_{r(j)}(x_{n_k-1}, z) + \beta \max\{d_{r(j)}(x_{n_k-1}, x_{n_k}), D_{r(j)}(z, Tz)\} + \\
&+ \gamma[D_{r(j)}(x_{n_k-1}, Tz) + d_{r(j)}(z, x_{n_k})], \quad \forall j \in J.
\end{aligned}$$

Now, if we let $k \to \infty$, we obtain $D_j(z, Tz) \leq (\beta + \gamma) D_j(z, Tz)$, i.e., $z \in Tz$. Thus, T has the fixed point z. $\square$

Remark 2.2.200. *By Corollary 2.2.197, in the particular case* $r(j) = j, d_j = d, \forall j \in J$, *where d is a regular metric on the complete convex metric space* X, *we obtain a fixed point theorem for multivalued non-self contractions stated by Assad and Kirk in* [**4**]. *By working in uniform spaces, the results of this section extend some theorems from* [**4**], [**23**], [**47**], [**69**], [**93**] *and* [**107**].

CHAPTER 3

NON-SELF SINGLE VALUED ALMOST LOCAL CONTRACTIONS

1. Preliminaries

In this chapter, the notion of non-self single valued almost local contractions in real vector spaces is considered. In this framework some new fixed point results are given. Let X be a real vector space and denote by J a family of indices. Let r be a function from J to J. Let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on X. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, closed subset $K \subset X$. Let $T : K \to X$ a non-self single valued almost local contraction, that is a mapping satisfying (1.19).
If $x \in K$ is verifying $Tx \notin K$, then we can always choose an $y \in \partial K$ such that $y = (1 - \lambda)x + \lambda \cdot Tx$, for some $0 < \lambda < 1$, which actually means that

$$d_j(x, Tx) = d_j(x, y) + d_j(y, Tx), y \in \partial K. \tag{3.244}$$

Denote by Y the set of points y satisfying condition (3.244). It is possible that the set Y to contain more than one element.

Remark 3.1.201. *In fact, the existence of $y \in \partial K$ is guaranteed by the extension of Proposition 2.2.191 to the case of pseudometrics $d_j, \forall j \in J$ instead of the metrics d, namely:*

Proposition 3.1.202. *Let X be a real vector space, $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, closed subset $K \subset X$. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that*

$$d_j(x, y) = d_j(x, z) + d_j(z, y), \quad \forall j \in J. \tag{3.245}$$

In order to establish new fixed point results, we will use the following concept from [**28**]:

Definition 3.1.203. [**28**] *Let X be a Banach space and let K a nonempty closed subset of X, $T : K \to X$ a non-self mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by* (3.244)*. If, for any such elements x, we have*

$$d(y, Ty) \leq d(x, Tx), \tag{3.246}$$

for at least one of the corresponding $y \in Y$*, then we say that* T *has property* (M).

In the sequel, we leave the usual metric space in favor of a pseudometric space, adding the new definition of property (M).

Definition 3.1.204. *Let* X *be a real vector space,* $\mathcal{D} = (d_j)_{j \in J}$ *be a family of pseudometrics on* X*, where* J *is a set of indices. Let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$*. Consider a nonempty, closed subset* $K \subset X$ *and let* $T : K \to X$ *be a non-self mapping. For every* $j \in J$*, let* $x_j \in K$ *with* $Tx_j \notin K$ *and let* $y_j \in \partial K$ *be the corresponding elements given by*

$$d_j(x, Tx) = d_j(x, y) + d_j(y, Tx), \quad y_j \in \partial K, \forall j \in J. \tag{3.247}$$

If, for the aforementioned elements x_j*, we have*

$$d_j(y, Ty) \leq d_j(x, Tx), \tag{3.248}$$

for at least one of the corresponding $y_j \in Y$*, then we say that* T *has property* (M).

Example 3.1.205. *Consider* $X = [1,2] \cup \{4\}$ *with the usual norm, let* $K = \{1, 2, 4\} \subset X$ *and take the non-self mapping* $T : K \to X$*, defined by*

$$T(x) = \begin{cases} 0 & \text{if } x \in \{1,2\} \\ \frac{1}{3} & \text{if } x = 4 \end{cases}$$

The only value $x \in K$ *with* $Tx \notin K$ *is represented by* $x = 4$ *and its corresponding set is* $Y = \{2\}$*, having in view that* (3.244) *need to be fulfilled. We have:*

$$\begin{aligned} d(y, Ty) &= d(2, 0) = |2 - 0| = 2, \\ d(x, Tx) &= d\left(4, \frac{1}{3}\right) = \left|4 - \frac{1}{3}\right| = \frac{11}{3}. \end{aligned}$$

Obviously, (3.246) *holds, therefore* T *has property (M).*

Very recently, Rus, I.A. and Şerban, M.A published their work (see [**108**]), regarding the non-self operators.

Definition 3.1.206. [**108**] *Let* (X, d) *be a metric space,* $Y \in \mathcal{C}(X)$ *and* $f : Y \to X$ *be a continuous non-self operator. The maximal displacement functional corresponding to* f *represents the functional* $E_f : \mathcal{P}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ *defined by*

$$E_f(A) := \sup\{d(x, f(x)) | \quad x \in A\}, \tag{3.249}$$

where $\mathcal{P}(X)$ *and* $\mathcal{C}(X)$ *was defined at the beginning of chapter 2.*
Then

(i) $A, B \in P(Y), A \subset B$ *implies* $E_f(A) \leq E_f(B)$*,*
(ii) $E_f(A) = E_f\left(\overline{A}\right)$*, for all* $A \in P(Y)$.

Definition 3.1.207. [**108**] *Let (X, d) be a metric space, $Y \in \mathcal{C}(X)$. An operator $f : Y \to X$ is called α-graphic contraction if there exists $0 \le \alpha < 1$ such that $x \in Y, f(x) \in Y$ imply*

$$d(f^2(x), f(x)) \le \alpha \cdot d(x, f(x)). \tag{3.250}$$

2. Main results

The next theorem states and proves the existence of a fixed point for non-self single valued ALC in uniform spaces.

Theorem 3.2.208. *Let X be a real vector space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$ and let r be a function from J to J. Consider a nonempty, closed subset $K \subset X$. Let $T : K \to X$ a non-self single valued almost local contraction, that is, a mapping for which there exist the constants $\theta \in (0, 1)$ and $L \ge 0$ such that*

$$d_j(Tx, Ty) \le \theta \cdot d_{r(j)}(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in K, \forall j \in J. \tag{3.251}$$

Assume that

$$\lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(K) = 0, \quad \forall j \in J$$

and also assume that the monotonicity property (1.26) for the pseudometrics is valid. If T has property (M) and fulfills Rothe's boundary condition, then T has a fixed point in K.

Proof: If $T(K) \subset K$, then T is, in fact, a self mapping on the closed set K and the conclusion follows by Theorem 1.1.33 for $X = K$. It is natural to consider the case $T(K) \not\subset K$. Let $x_0 \in \partial K$. Using (2.222), we know that $Tx_0 \in K$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the following way:
Let $x_1 = Tx_0$. If $Tx_1 \in K$, set $x_2 = Tx_1$. If $Tx_1 \notin K$, we may select an element $x_2 \in \partial K$, such that

$$x_2 = (1 - \lambda)x_1 + \lambda Tx_1, \text{ for some } 0 < \lambda < 1.$$

The terms of the sequence $\{x_n\}$ defined this way are satisfying one of the following properties:

(i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in K$,
(ii) $x_n = (1 - \lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial K$, $(0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

By introducing the following notations, we will simplify our proof:

$$P = \{x_k \in \{x_n\} : x_k = Tx_{k-1}\},$$

$$Q = \{x_k \in \{x_n\} : x_k \ne Tx_{k-1}\}.$$

It is obvious that $\{x_n\} \subset K$ and also, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, according to (2.222), we cannot have two consecutive terms of $\{x_n\}$

in the set Q (but it is possible to have two consecutive terms of $\{x_n\}$ in the set P).
The strategy is to prove that $\{x_n\}$ is a τ -Cauchy sequence.
To this end, we must discuss three different cases:
Case I. $x_n, x_{n+1} \in P$,
Case II. $x_n \in P$, $x_{n+1} \in Q$,
Case III. $x_n \in Q$, $x_{n+1} \in P$.

Case I. $x_n, x_{n+1} \in P$. Having in view the definition of the set P, we have $x_n = Tx_{n-1}$ and $x_{n+1} = Tx_n$. By (3.251) we get

$$d_j(x_{n+1}, x_n) = d_j(Tx_n, Tx_{n-1}) = d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_n, x_{n-1}) + L \cdot \underbrace{d_{r(j)}(x_n, Tx_{n-1})}_{=0},$$

for all $j \in J$. Therefore

$$d_j(x_{n+1}, x_n) \leq \theta \cdot d_{r(j)}(x_n, x_{n-1}), \quad \forall j \in J. \tag{3.252}$$

Case II. $x_n \in P$, $x_{n+1} \in Q$.
In this case we have $x_n = Tx_{n-1}$ but $x_{n+1} \neq Tx_n$. By applying Proposition 3.1.202 for $x := x_n \in K$, $Tx_n \notin K$, we can choose $y := x_{n+1} \in \partial K$ such that

$$d_j(x_n, x_{n+1}) + d_j(x_{n+1}, Tx_n) = d_j(x_n, Tx_n), \quad \forall j \in J.$$

That means

$$d_j(x_n, x_{n+1}) \leq d_j(x_n, Tx_n) = d_j(Tx_{n-1}, Tx_n), \quad \forall j \in J,$$

and by using (3.251), we get

$$d_j(x_n, x_{n+1}) \leq d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_n, x_{n-1}) + L \cdot \underbrace{d_{r(j)}(x_n, Tx_{n-1})}_{=0} = \theta \cdot d_{r(j)}(x_n, x_{n-1}),$$

which yields again inequality (3.252).

Case III. $x_n \in Q$, $x_{n+1} \in P$.
In this case, we have $x_{n+1} = Tx_n$ and $x_{n-1} \in P$. The mapping T has property (M), this means:

$$d_j(x_n, x_{n+1}) = d_j(x_n, Tx_n) \leq d_j(x_{n-1}, Tx_{n-1}), \quad \forall j \in J. \tag{3.253}$$

Taking into account that $x_{n-1} \in P$, we have $x_{n-1} = Tx_{n-2}$ and by (3.251) we get

$$d_j(Tx_{n-2}, Tx_{n-1}) \leq \theta \cdot d_{r(j)}(x_{n-2}, x_{n-1}) + L \cdot d_{r(j)}(x_{n-1}, Tx_{n-2}) = \theta \cdot d_{r(j)}(x_{n-2}, x_{n-1}),$$

for every $j \in J$. From that, by combining the last inequality with (3.253), we obtain

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_{r(j)}(x_{n-2}, x_{n-1}). \tag{3.254}$$

At this point, after analyzing all three cases, and using (3.252), (3.254), and the monotonicity property for the pseudometrics, it follows that the sequence $\{x_n\}$ verifies the inequality:

$$\begin{aligned} d_j(x_n, x_{n+1}) &\le \theta \cdot max\{d_{r(j)}(x_{n-2}, x_{n-1}), d_{r(j)}(x_{n-1}, x_n)\} \le \\ &\le \theta \cdot max\{d_j(x_{n-2}, x_{n-1}), d_j(x_{n-1}, x_n)\}, \end{aligned} \tag{3.255}$$

for all $n \ge 2$, for each $j \in J$. Now, by induction with respect to $n \ge 2$ we obtain from (3.255):

$$d_j(x_n, x_{n+1}) \le \theta^{[n/2]} \cdot max\{d_j(x_0, x_1), d_j(x_1, x_2)\}, \quad \forall j \in J,$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.
Moreover, for $m > n > N$,

$$d_j(x_n, x_m) \le \sum_{i=N}^{\infty} d_j(x_i, x_{i-1}) \le 2 \cdot \frac{\theta^{[N/2]}}{1-\theta} \cdot max\{d_j(x_0, x_1), d_j(x_1, x_2)\}, \forall j \in J.$$

The last inequality means that $\{x_n\}$ is Cauchy sequence with respect to the topology τ. Note that $\{x_n\} \subset K$ and K is closed, thus $\{x_n\}$ converges to some point in K. Denote

$$x^* = \lim_{n\to\infty} x_n, \tag{3.256}$$

and let $\{x_{n_k}\} \subset P$, be an infinite subsequence of $\{x_n\}$ denoted for simplicity also by $\{x_n\}$. It is clear that such a subsequence always exists.
Using the triangle inequality and the definition of P, we get:

$$d_j(x^*, Tx^*) \le d_j(x^*, x_{n+1}) + d_j(x_{n+1}, Tx^*) = d_j(x_{n+1}, x^*) + d_j(Tx_n, Tx^*),$$

for all $j \in J$. Using (3.251), we obtain

$$d_j(Tx_n, Tx^*) \le \theta \cdot d_{r(j)}(x_n, x^*) + L \cdot d_{r(j)}(x^*, Tx_n), \forall j \in J,$$

and hence

$$d_j(x^*, Tx^*) \le d_j(x^*, Tx_n) + d_j(Tx_n, Tx^*) \le d_j(x_{n+1}, x^*) + \theta \cdot d_{r(j)}(x_n, x^*) + L \cdot d_{r(j)}(x^*, Tx_n), \tag{3.257}$$

for all $n \ge 0$ and for every $j \in J$. Letting $n \to \infty$ in (3.257), we get the final conclusion for our proof, i.e.,

$$d_j(x^*, Tx^*) = 0, \quad \forall j \in J,$$

which shows that x^* is a fixed point of T. □

Remark 3.2.209. *Theorem 3.2.208 is a generalisation of result established in* [**28**] *in the particular case of a Banach space X, obtained for a non-self almost contraction.*

The following theorem assures the uniqueness of the fixed point for a non-self almost local contraction.

THEOREM 3.2.210. *Let X be a real vector space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Let r be a function from J to J. Consider a nonempty, closed subset $K \subset X$ and let $T : K \to X$ be a non-self mapping. Let $T : K \to X$ be a non-self almost local contraction with constants $\theta_1 \in (0,1)$ and $L_1 \geq 0$. Assume that an additional condition holds:*

$$(3.258) \qquad (U) \qquad \lim_{n\to\infty} (\theta_1 + L_1)^n diam_{r^n(j)}(z, K) = 0, \forall z \in K, \forall j \in J.$$

If T has property (M) and satisfies Rothe's boundary condition, then T has a unique fixed point in K.

REMARK 3.2.211. *The proof is quite similar to the case of single valued self almost local contractions (see Theorem 1.1.36), therefore we skip over the proof. In fact, Theorem 3.2.210 is a generalisation of the uniqueness theorem* [**28**] *for non-self almost contractions in Banach spaces.*

Starting from the work of Rus-Şerban [**108**], involving the α-graphic contractions, our main aim is to extend that notion to the more general case of non-self single valued almost local contractions in the framework of a real vector space.

DEFINITION 3.2.212. *Let X be a real vector space, $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics on X, where J is a set of indices. Let τ be the weak topology on X defined by the family $\mathcal{D}$ and let r be a function from J to J. Consider a nonempty, closed subset $K \subset X$. An operator $T : K \to X$ is called α-graphic local contraction if there exists $0 \leq \alpha < 1$ such that $x \in K, f(x) \in K$ imply*

$$(3.259) \qquad d_j(T^2x, Tx) \leq \alpha d_{r(j)}(x, Tx), \quad \forall j \in J.$$

In the sequel, we study several types of non-self almost local contractions, in order to establish if they are α-graphic local contractions.

THEOREM 3.2.213. *Assume that $X, J, \mathcal{D}, r, \tau$ and K are as in Definition 3.2.212 and assume the monotonicity property* (1.26) *is valid for the pseudometrics. Let $T : K \to X$ be a non-self Ćirić-Reich-Rus operator, that is, a mapping for which there exist the constants $\delta, L \in \mathbb{R}_+$ with $\delta + 2L < 1$ such that*

$$(3.260) \qquad d_j(Tx, Ty) \leq \delta \cdot d_{r(j)}(x, y) + L \cdot [d_{r(j)}(x, Tx) + d_{r(j)}(y, Ty)], \forall j \in J,$$

for all $x, y \in K$. Then T is a non-self α-graphic local contraction with $\alpha = \frac{\delta+L}{1-L}$.

Proof: Let $x \in K$ such that $Tx \in K$. The subset K is nonempty and closed, therefore such an $x \in K$ always exists. If we replace $y := Tx$ in the last inequality, after applying the monotonicity property (1.26) for the pseudometrics, we can write:

$$\begin{aligned} d_j(Tx, T^2x) &\leq \delta \cdot d_{r(j)}(x, Tx) + L \cdot [d_{r(j)}(x, Tx) + d_{r(j)}(Tx, T^2x)] \leq \\ &\leq \delta \cdot d_{r(j)}(x, Tx) + L \cdot [d_{r(j)}(x, Tx) + d_j(Tx, T^2x)], \end{aligned}$$

which implies:

$$d_j(Tx,T^2x) \leq \frac{\delta + L}{1-L} \cdot d_{r(j)}(x,Tx), \quad \forall j \in J.$$

□

THEOREM 3.2.214. *Assume that $X, J, \mathcal{D}, r, \tau$ and K are as in Definition 3.2.212 and assume the monotonicity property* (1.26) *is valid for the pseudometrics. Let $T : K \to X$ be a non-self almost local contraction, that is, a mapping for which there exist the constants $\theta \in (0,1)$ and $L \geq 0$ such that*

$$d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in K, \forall j \in J.$$

Then T is a non-self α-graphic local contraction with $\alpha = \theta$.

Proof: Let $x \in K$ such that $Tx \in K$. We use the definition of an almost local contraction, that is,

$$d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in K, \forall j \in J,$$

with $\theta \in (0,1)$ and $L \geq 0$. Again, we apply the same substitution, $y := Tx$ and we obtain:

$$d_j(Tx,T^2x) \leq \theta \cdot d_{r(j)}(x,Tx) + L \cdot d_{r(j)}(Tx,Tx), \forall x,y \in K, \forall j \in J.$$

From that, we conclude:

$$d_j(Tx,T^2x) \leq \theta \cdot d_{r(j)}(x,Tx), \forall x,y \in K, \forall j \in J,$$

which shows that T is a non-self α-graphic local contraction with $\alpha = \theta$. □

REMARK 3.2.215. *A non-self generalized almost local contraction is also a non-self α-graphic local contraction with constant $\alpha = \theta$. In the definition of a generalized ALC, that is, a mapping satisfying*

$$\begin{aligned} d_j(Tx,Ty) \quad &\leq \quad \theta \cdot d_{r(j)}(x,y) + \\ &\quad +L \cdot min\{d_{r(j)}(x,Tx), d_{r(j)}(y,Ty), d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\}, \end{aligned}$$

for all $x,y \in K$ and for every $j \in J$, if we repeat the substitution $y := Tx$, we obtain exactly the same inequality as in the case of non-self almost local contraction:

$$d_j(Tx,T^2x) \leq \theta \cdot d_{r(j)}(x,Tx), \forall x,y \in K, \forall j \in J.$$

Thus, T is a non-self α-graphic local contraction with constant $\alpha = \theta$.

DEFINITION 3.2.216. *Assume that $X, J, \mathcal{D}, r, \tau$ and K are as in Definition 3.2.212, the operator $T : K \to X$ is called non-self generalized Berinde type almost local contraction with respect ($\mathcal{D}, r$) if there exist a constant $\theta \in (0,1)$ and a function $b : K \to [0,\infty)$ such that*

$$d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + b(y) \cdot d_{r(j)}(y,Tx), \forall x,y \in K, \forall j \in J. \tag{3.261}$$

THEOREM 3.2.217. *A non-self generalized Berinde type almost local contraction is a non-self α-graphic local contraction with constant $\alpha = \theta$.*

Proof: Let $x \in K$ such that $Tx \in K$. After applying in (3.261) the substitution $y := Tx$, we get:

$$d_j(Tx, T^2x) \leq \theta \cdot d_{r(j)}(x, Tx) + b(y) \cdot d_{r(j)}(Tx, Tx), \forall x, y \in K, \forall j \in J.$$

From that, we can write:

$$d_j(Tx, T^2x) \leq \theta \cdot d_{r(j)}(x, Tx), \forall x, y \in K, \forall j \in J,$$

which shows that T is a non-self α-graphic local contraction with $\alpha = \theta$. □

THEOREM 3.2.218. *Assume that $X, J, \mathcal{D}, r, \tau$ and K are as in Definition 3.2.212 and assume the monotonicity property* (1.26) *is valid for the pseudometrics. Let $T : K \to X$ be a non-self Chatterjea type almost local contraction, that is, a mapping $T : K \to X$ for which there exists a constant $0 \leq c < \frac{1}{2}$ such that*

$$d_j(Tx, Ty) \leq c \cdot [d_{r(j)}(x, Ty) + d_{r(j)}(y, Tx)], \forall x, y \in K, \forall j \in J. \tag{3.262}$$

Then T is a non-self α-graphic local contraction with $\alpha = \frac{c}{1-c}$.

Proof: Let $x \in K$ such that $Tx \in K$. We use (3.262), and after applying the substitution $y := Tx$, by using the triangle inequality, and the monotonicity property, we get:

$$\begin{aligned} d_j(Tx, T^2x) &\leq c \cdot [d_{r(j)}(x, T^2x) + d_{r(j)}(Tx, Tx)] \leq \\ &\leq c \cdot [d_{r(j)}(x, Tx) + d_{r(j)}(Tx, T^2x)] \overset{mon.}{\leq} \\ &\leq c \cdot [d_{r(j)}(x, Tx) + d_j(Tx, T^2x)], \quad \forall j \in J. \end{aligned}$$

From that, we conclude:

$$d_j(Tx, T^2x) \leq \frac{c}{1-c} d_{r(j)}(x, Tx), \quad \forall j \in J,$$

which means that T is a non-self α-graphic local contraction with $\alpha = \frac{c}{1-c}$. □

EXAMPLE 3.2.219. *Let X be the set of reals with the usual metric, $K = [0, 1]$ and let $T : K \to X$ be defined by $Tx = -\frac{1}{10}$, if $x = \frac{9}{10}$, and $Tx = \frac{x}{x+1}$, if $x \neq \frac{9}{10}$. We choose the identity function $r(j) = j$.*
T verifies condition (3.246)*, because T has property (M), T is discontinuous at $\frac{9}{10}$, the unique fixed point of T is* 0*, and T is continuous at* 0*. T has property (M). If $x = \frac{9}{10} \in K$, $Tx = -\frac{1}{10} \notin K$, then using the condition* (3.246) *we have $\left|\frac{y}{y+1}\right| \leq 1$. This is valid for both $y \in \{0, 1\}$, so $y \in \partial K$. However, T is not satisfying the ALC condition. Taking for example $x \neq \frac{9}{10}$ and $y = \frac{x}{x+1}$ in* (1.19) *to get, for any $x > 0$,*

$$d_j(Tx, Ty) = d_j\left(\frac{x}{x+1}\right) = \left|\frac{x^2}{(x+1)(2x+1)}\right|; \quad d_j(x, y) = \left|\frac{x^2}{(x+1)}\right|; \quad d_j(y, Tx) = 0.$$

By replacing these distances in (1.19), *we get the equivalent form:* $\frac{1}{2x+1} \leq \theta < 1$, $x > 0$. *If we take now* $x \to 0$ *in the last double inequality, we obtain a contradiction:*

$$1 \leq \theta < 1.$$

REMARK 3.2.220. *Theorems 3.2.213, 3.2.214, 3.2.217, 3.2.218 represent generalizations of results established in* [**108**] *regarding the* α*-graphic contractions in metric spaces.*

CHAPTER 4

APPLICATION OF ALMOST LOCAL CONTRACTIONS IN DYNAMIC PROGRAMMING

1. Preliminaries

Dynamic programming represents the foundation stone in economic analysis. Economic problems that require a succession of decisions are determined by the fact that a decision that is adopted in a certain period has both an immediate and a long-term economic effect, influencing the other stages. The optimization of sequential processes is obtained by the methods of a relatively recently established mathematical theory which is called dynamic programming. One of the pioneers of this theory is Richard Bellman, and his fundamental work is Dynamic Programming published in 1957 [**11**]. Dynamic programming has a wide field of application in operational research (production organization, equipment renewal, inventory management), as well as in other fields (reverse connection processes, cosmic navigation, etc.).

A certain sequence of decisions is a policy and the one we are interested in is the optimal policy, for example the one that leads to a minimum total cost of the process.

We distinguish two main types of sequential processes:

a) deterministic dynamic programming, when at each stage the process is completely controlled by the decision we take;
b) stochastic dynamic programming (or Markov decision processes), when the evolution of the process takes place under the double influence of decisions and chance.

Optimal policy is that sequence of decisions that optimizes the process as a whole, being a deterministic process. In the case of a stochastic process, the notion of optimal strategy is used appropriately. Dynamic processes can be continuous or discrete. An example of a discrete process is the following: an enterprise must draw up its annual supply plan for a particular material; 12 periods (months) are considered and for each period the quantity to be supplied is established, so that for the whole year a minimum total cost results. Discrete dynamic processes can have a limited horizon (in the example above 12 periods) or unlimited.

The purpose of this chapter is to study dynamic programming (shortly DP) problems setting as reduced form models. Starting from the work of Martin-da-Rocha and Vailakis (2010) (see [**76**]), Rincón-Zapatero, Rodriguez-Palmero (2003) (see [**94**]) and also Matkowski-Nowak (see [**78**]) it is our aim to extend their results and to obtain new ones. We intend to illustrate the use of local almost contractions in order to solve the Bellman equation in both the deterministic and stochastic case.

Throughout the first part of this chapter we recall some important concepts and results from Matkowski-Nowak [**78**] and Rincón-Zapatero [**94**], as follows:

X will denote a topological space such that

$$X = \bigcup_{j=1}^{\infty} K_j, \tag{4.263}$$

with $\{K_j\}$ an increasing sequence (in the sense of inclusion) of compact subsets of X. Denote by $C(X)$ the set of all real-valued, continuous functions over X. In the sequel, $\mathbf{0}$ denotes the function φ such that $\varphi(x) = 0$ for all $x \in X$. We will introduce a countable family of pseudometrics, as follows:

$$d_j(f, g) := \max_{x \in K_j} |f(x) - g(x)|, \quad j \in \mathbb{N}, \forall f, g \in C(X). \tag{4.264}$$

Definition 4.1.221. [**94**] *The set $A \subseteq C(X)$ is said to be bounded if there exist a sequence $\{m_j\}, m_j < \infty$, such that $d_j(\phi, \mathbf{0}) \leq m_j$, for all $\phi \in A$ and all $j \in \mathbb{N}$.*

Note that: if the set A contains an unbounded function ϕ, then the sequence $\{m_j\}$ also need to be unbounded.

- We introduce a seminorm on $F(X)$, as follows from the work of Matkowski [**78**]. Let $F(X)$ be a vector space of functions $\varphi : X \to \mathbb{R}$.
  The aforementioned seminorm on $F(X)$ is defined by
  $$||\varphi||_j := \sup_{x \in K_j} |\varphi(x)|, \quad \varphi \in F(X).$$
  We assume that the set of all operators $\varphi \in F(X)$ endowed with the seminorm $\| \cdot \|_j$ is a Banach space.
- If $c > 1$ and $m = \{m_j\}$ is an increasing unbounded sequence of positive real numbers, denote by $F_m(X)$ the set of all operators $\varphi \in F(X)$ such that
  $$\sum_{j=1}^{\infty} \frac{\|\varphi\|_j}{m_j c^j} < \infty. \tag{4.265}$$
  Observe that $F_m(X)$ is a vector space.
- The operator $\| \cdot \| : F_m(X) \to \mathbb{R}$ defined by
  $$\|\varphi\| := \sum_{j=1}^{\infty} \frac{\|\varphi\|_j}{m_j c^j} \tag{4.266}$$
  is a complete norm on $F_m(X)$, which means that $(F_m(X), \| \cdot \|)$ is a Banach space.
- Denote by

$$F_{mb}(X) := \{\varphi \in F(X) : \|\varphi\|_j \leq m_j, \text{ for all } j \in \mathbb{N}\}, \tag{4.267}$$

which is a closed subset of $F_m(X)$.

DEFINITION 4.1.222. [**78**] *Take $k \in \{0,1\}$. An operator $T : F_m(X) \to F(X)$ is called k-local contraction relative to the subset $A \subseteq F_m(X)$, if there exists a coefficient $\beta \in [0,1)$ such that*

$$\|T\varphi - T\psi\|_j \leq \beta\|\varphi - \psi\|_{j+k} \quad \forall\varphi,\psi \in A, j \in \mathbb{N}.$$

REMARK 4.1.223. *It is obvious that a 0-local contraction is also a 1-local contraction.*

2. Main results

a) Deterministic dynamic programming

First, we present some concepts and notations.

DEFINITION 4.2.224. [**94**] *The dynamic optimization problem consists in solving the following maximization problem:*

$$\begin{array}{rcll} v^*(x_0) & = & \max\limits_{\{x_{j+1}\}_{j=0}^{\infty}} & \sum\limits_{j=0}^{\infty} \beta^j U(x_j, x_{j+1}) \\ x_{j+1} & \in & \Gamma(x_j) & \qquad (j = 0, 1, 2, ...) \\ x_0 & \in & X \quad \text{fixed}, & \end{array} \tag{4.268}$$

where:
- X is a subset of $\mathbb{R}^l$,
- $\Gamma : X \to \mathcal{P}(X)$ is the technological correspondence giving the set of admissible actions from any $x \in X$, where $\mathcal{P}(X)$ denote the set of all nonempty subsets of X,
- $U : Graph(\Gamma) \to \mathbb{R}$ represents the return function, where
$Graph(\Gamma) = \{(x,y) \in X \times X : y \in \Gamma(x)\}$.
- $\beta \in (0,1)$ is the discounting factor,
- $v^* : X \to \overline{\mathbb{R}}$ v^* is the value function which describes the best possible value of the objective, as a function of the state x
- $v^*(x_0)$ is the optimal value as a function of the initial condition x_0.

At this point, consider the space $Z = X \times X \times \cdots = X^{\mathbb{N}}$ (the set of all sequences with terms from X). Then we define the operator $\Pi : X \to Z$ by

$$\Pi(x_0) = \Big\{\tilde{x} = (x_j) = (x_0, x_1, \cdots) \in Z \quad | \quad x_{j+1} \in \Gamma(x_j), j = 0, 1, \cdots\Big\}, \quad x_0 \in X.$$

For every $\tilde{x} \in \Pi(x_0)$, let

$$S(\tilde{x}) = \sum_{j=0}^{\infty} \beta^j U(x_j, x_{j+1})$$

be the total discounted returns. The following assumptions are frequently used in dynamic programming:

(DP1) Γ is nonempty, continuous and compact valued.
(DP2) $U : Graph(\Gamma) \to \mathbb{R}$ is continuous.

Next, we state the maximum theorem of Berge, which will be used in future demonstrations or argumentations:

THEOREM 4.2.225. [**32**] *Let Θ and X be two metric spaces, $\Gamma : \Theta \rightrightarrows X$ a compact-valued correspondence, and $f : X \times \Theta \to \mathbb{R}$ a continuous function. Define*

$$f^*(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta)$$

and

$$\Gamma^*(\theta) = \arg\max_{x \in \Gamma(\theta)} f(x, \theta),$$

where we denote

$$\arg\max_{x \in \Gamma(\theta)} f(x, \theta) = \{x | x \in \Gamma(\theta) \text{ for all } b \in \Gamma(\theta) : f(x, b) \leq f(x, \theta)\}.$$

If Γ is continuous at $\theta \in \Theta$, then $f^* : \Theta \to \mathbb{R}$ is continuous at θ and $\Gamma^* : \Theta \rightrightarrows X$ is compact-valued and upper hemicontinuous at θ.

DEFINITION 4.2.226. [**94**] *Let V be the set of functions from X to $[-\infty, \infty)$. The Bellman operator $\mathcal{B}$ on V is defined by*

$$\mathcal{B}f(x) = \max_{y \in \Gamma(x)} \Big(U(x, y) + \beta f(y)\Big), \qquad x \in X, f \in V. \tag{4.269}$$

REMARK 4.2.227. *The hypotheses (DP1) and (DP2) enable us to apply Berge's Theorem of the maximum, therefore the Bellman operator is well defined on the space of continuous functions on X.*

The Bellman equation, according to Rincón-Zapatero [**94**], is:

$$\mathcal{B}f = f. \tag{4.270}$$

A solution of the Bellman equation and the value function v^* are closely connected. Stokey, Lucas and Prescott [**112**] have proved that:

- A fixed point of the Bellman operator is the value function of the maximization problem (4.268) and conversely:
- the value function is a solution for the Bellman equation, in case of an upper semicontinuous and finite value function.

The Bellman operator possesses various properties, according to [**67**], [**94**], [**112**], such as:

(1) $\mathcal{B}$ is monotone increasing;
(2) for every $\alpha > 0$ we have $\mathcal{B}(f + \alpha) = \mathcal{B}f + \alpha\beta$;

(3) If the foolowing conditions hold:
a) the technological correspondence Γ is monotone increasing, that is,
$x_1 < x_2$ implies $\Gamma(x_1) \subseteq \Gamma(x_2)$,
b) for $y \in X$ fixed, U is strictly monotone increasing, that is,
$x_1 < x_2$ implies $U(x_1, y) < U(x_2, y)$,
then $\mathcal{B}f$ is also strictly monotone increasing: $\mathcal{B}f(x_1) < \mathcal{B}f(x_2)$ if $x_1 < x_2$.

In the sequel, we consider the metric $d_{\mathbb{R}}$, the usual metric on the real axis. By using this metric, we obtain a family of semidistances (seminorms) $\{d_j\}$ on $C(X)$, defined as

$$d_j(f, g) = \max_{x \in K_j} |f(x) - g(x)| = \|f - g\|_{K_j} \qquad (d_j(f, \mathbf{0}) = \|f\|_{K_j}) \tag{4.271}$$

Consider the metric d:

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}.$$

The following theorem was proved by Rincón-Zapatero and Rodriguez-Palmero in [**94**]:

Theorem 4.2.228. [**94**] *Let $\mathcal{B}$ be a Bellman operator satisfying (DP1) and (DP2) so that there exists a countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X with $X = \bigcup_j K_j$ satisfying $\Gamma(K_j) \subseteq K_j$ for all $j \in \mathbb{N}$.*
Then the following conditions hold:
(a) The Bellman equation has a unique solution $\hat{f}$ on $C(X)$. Furthermore, $\hat{f}$ satisfies

$$\|\hat{f}\|_{K_j} \leq \frac{\|\Psi\|_{K_j}}{1 - \beta}, \quad \Psi \in C(X), j \in \mathbb{N}.$$

(b) The value function v^* is continuous and coincides with the fixed point $\hat{f}$.
(c) For any $f \in C(X)$, $\mathcal{B}^n f \to v^*$ as $n \to \infty$.

Next, we extend the concept of k-local contractions to the more general case of k-almost local contractions (breafly k-ALC). To this end, we apply the basic idea of Matkowski-Nowak (see [**78**]).

Definition 4.2.229. *Let X be a nonempty set. Denote by $\{K_j\}$ a strictly increasing sequence (in the sense of inclusion) of subsets of X and assume that*

$$X = \bigcup_{j=1}^{\infty} K_j.$$

If $c > 1$ and $m = \{m_j\}$ is an increasing unbounded sequence of positive real numbers, denote by $F_m(X)$ the set of all operators $\varphi \in F(X)$ such that

$$\sum_{j=1}^{\infty} \frac{\|\varphi\|_j}{m_j c^j} < \infty \tag{4.272}$$

The operator $\|\cdot\| : F_m(X) \to \mathbb{R}$ defined by

$$\|\varphi\| := \sum_{j=1}^{\infty} \frac{\|\varphi\|_j}{m_j c^j} \tag{4.273}$$

is a complete norm on $F_m(X)$, *which means that* $(F_m(X), \|\cdot\|)$ *is a Banach space. Denote by*

$$F_{mb}(X) := \{\varphi \in F(X) : \|\varphi\|_j \leq m_j, \text{ for all } j \in \mathbb{N}\}, \tag{4.274}$$

which is a closed subset of $F_m(X)$. *Let* $A \subseteq F_m(X)$. *The operator* $T : F_m(X) \to F(X)$ *is called* k*-almost local contraction, with* $k \in \{0,1\}$, *relative to the subset* A *if there exist the constants* $\beta \in [0,1)$ *and* $L \geq 0$ *such that*

$$\|T\varphi - T\psi\|_j \leq \beta\|\varphi - \psi\|_{j+k} + L\|\psi - T\varphi\|_{j+k} \quad \forall, \varphi, \psi \in A, j \in \mathbb{N}. \tag{4.275}$$

In the following, our main goal is to find the conditions of existence for the fixed points on $F_m(X)$ (defined in (4.265)) for a $0 - ALC$, although the operator T need not be a contraction on the metric generated by the family of seminorms. Proposition 4.2.230 below proves that a $0 - ALC$ is an almost contraction relative to a subset of $F_m(X)$.

Proposition 4.2.230. *Assume all conditions from Definition 4.2.229 hold.*
a) Let $T : F_m(X) \to F(X)$ *be a* $0 - ALC$ *relative to* $A = F_m(X)$ *with the constants* $\beta, L \in [0,1)$. *Then for each* $\varphi, \psi \in A$ *we have:*

$$\|T\varphi - T\psi\| \leq \beta\|\varphi - \psi\| + L\|\psi - T\varphi\|, \quad \forall, \varphi, \psi \in A. \tag{4.276}$$

If $T\mathbf{0} \in F_m(X)$, *then* T *maps* $F_m(X)$ *into itself. Furthermore,* T *has a fixed point* $\varphi^* \in F_m(X)$.
If, in addition to the hypothesis, we suppose that

$$\|T\varphi - T\psi\|_j \leq \beta\|\varphi - \psi\|_j + L\|\varphi - T\varphi\|_j \quad \forall, \varphi, \psi \in A, j \in \mathbb{N}, \tag{4.277}$$

then the fixed point is unique.
b) If we add the conditions

$$\begin{aligned} \|T\mathbf{0}\|_j &\leq (1 - \beta - L) \cdot m_j, \quad \forall j \in \mathbb{N} \\ \beta + L &< 1, \end{aligned} \tag{4.278}$$

then T *maps* $F_{mb}(X)$ *into itself, i.e.,* $T : F_{mb}(X) \to F_{mb}(X)$.

Proof: a) Let $\varphi, \psi \in A$. Apply (4.266) and (4.276), and we obtain:

$$\begin{aligned} \|T\varphi - T\psi\| &= \sum_{j=1}^{\infty} \frac{\|T\varphi - T\psi\|_j}{m_j c^j} \leq \sum_{j=1}^{\infty} \frac{\beta\|\varphi - \psi\|_j + L\|\psi - T\varphi\|_j}{m_j c^j} = \\ &= \sum_{j=1}^{\infty} \frac{\beta\|\varphi - \psi\|_j}{m_j c^j} + \sum_{j=1}^{\infty} \frac{L\|\psi - T\varphi\|_j}{m_j c^j} = \\ &= \beta\|\varphi - \psi\| + L\|\psi - T\varphi\|, \quad \varphi, \psi \in A. \end{aligned}$$

Observe that, for all $\varphi \in F_m(X)$, we can write:

$$\begin{aligned} \|T\varphi\| &= \|T\varphi - T\mathbf{0} + T\mathbf{0}\| \leq \|T\varphi - T\mathbf{0}\| + \|T\mathbf{0}\| \leq \\ &\leq \beta\|\varphi - \mathbf{0}\| + L\|\mathbf{0} - T\varphi\| + \|T\mathbf{0}\| = \\ &= \beta\|\varphi\| + L\|T\varphi\| + \|T\mathbf{0}\|. \end{aligned}$$

After rearranging the terms, we obtain:

$$(1-L)\|T\varphi\| \leq \beta\|\varphi\| + \|T\mathbf{0}\|.$$

From that, by dividing with $1 - L > 0$, we get

$$\|T\varphi\| \leq \frac{\beta}{1-L}\big\|\varphi\big\| + \frac{1}{1-L}\big\|T\mathbf{0}\big\|, \quad \varphi \in A.$$

Having in view that $\varphi \in F_m(X)$ and also $T\mathbf{0} \in F_m(X)$, it results that T maps $F_m(X)$ into itself, because the seminorm is finite in $F_m(X)$, which yields:

$$\|T\varphi\| < \infty.$$

Observe that, according to (4.276), the operator T is an almost contraction. The existence of a fixed point is guaranteed by the existence of the fixed points for almost contractions. Thus, T admits a fixed point $\varphi^* \in F_m(X)$.
Moreover, since (4.277) is valid, it follows:

$$\begin{aligned} \|T\varphi - T\psi\| &= \sum_{j=1}^{\infty} \frac{\|T\varphi - T\psi\|_j}{m_j c^j} \leq \sum_{j=1}^{\infty} \frac{\beta\|\varphi - \psi\|_j + L\|\varphi - T\varphi\|_j}{m_j c^j} = \\ &= \sum_{j=1}^{\infty} \frac{\beta\|\varphi - \psi\|_j}{m_j c^j} + \sum_{j=1}^{\infty} \frac{L\|\varphi - T\varphi\|_j}{m_j c^j} = \\ &= \beta\|\varphi - \psi\| + L\|\varphi - T\varphi\|, \quad \varphi, \psi \in A. \end{aligned}$$

In fact, we found the uniqueness condition (0.11) for the fixed point of an almost contraction. Therefore, the fixed point is unique.

b) For the desired conclusion, suppose that for every $j \in \mathbb{N}$ and every $\varphi \in F_{mb}(X)$, conditions (4.278) are fulfilled. Then

$$\begin{aligned} \|T\varphi\|_j &\leq \|T\varphi - T\mathbf{0}\|_j + \|T\mathbf{0}\|_j \leq \beta\|\varphi - \mathbf{0}\|_j + L\|\mathbf{0} - T\varphi\|_j + \|T\mathbf{0}\|_j = \\ &= \beta \underbrace{\|\varphi\|_j}_{\leq m_j} + L\|T\varphi\|_j + \|T\mathbf{0}\|_j, \quad \varphi \in A. \end{aligned}$$

Once again, after rearranging the terms in a proper order, we can write:

$$(1-L) \cdot \|T\varphi\|_j \leq \beta \cdot m_j + (1 - \beta - L) \cdot m_j.$$

It is easy to see:

$$(1-L) \cdot \|T\varphi\|_j \leq (1-L) \cdot m_j, \quad \varphi, \psi \in A.$$

Note that as $L \in [0, 1)$, we have from the last inequality:

$$\|T\varphi\|_j \leq m_j,$$

which means that $T : F_{mb}(X) \to F_{mb}(X)$. □

REMARK 4.2.231. *Condition $\beta + L < 1$ is compulsory in the hypothesis of Proposition 4.2.230, otherwise* $\underbrace{\|T\mathbf{0}\|_j}_{\geq 0} \leq \underbrace{(1-\beta-L)}_{<0} m_j$ *would lead to a contradiction.*

PROPOSITION 4.2.232. *Assume all conditions from Definition 4.2.229 are valid. Let $T : F_m(X) \to F(X)$ be a $1 - ALC$ relative to $A = F_m(X)$. Denote*

$$\zeta := \beta c \cdot \sup\Big\{\frac{m_{j+1}}{m_j}, j \in \mathbb{N}\Big\}, \quad \xi := Lc \cdot \sup\Big\{\frac{m_{j+1}}{m_j}, j \in \mathbb{N}\Big\} \tag{4.279}$$

and suppose that there exists $c > 1$ such that $\zeta < 1$. Then $T : A \to A$ is an almost contraction, having the contraction coefficients $0 < \zeta < 1$, $0 < \xi < 1$ and with the fixed point $\varphi^* \in A$. If, in addition to the hypothesis, we suppose that

$$\|T\varphi - T\psi\|_j \leq \beta\|\varphi - \psi\|_{j+1} + L\|\varphi - T\varphi\|_{j+1} \quad \forall, \varphi, \psi \in A, j \in \mathbb{N}, \tag{4.280}$$

then the fixed point is unique.

Proof: Select $\varphi, \psi \in A$. Then we have:

$$\begin{aligned}
\|T\varphi - T\psi\| &= \sum_{j=1}^{\infty} c^{-j}\frac{\|T\varphi - T\psi\|_j}{m_j} \overbrace{\leq}^{(4.275)} \sum_{j=1}^{\infty} \beta c^{-j}\frac{\|\varphi - \psi\|_{j+1}}{m_j} + \sum_{j=1}^{\infty} Lc^{-j}\frac{\|\psi - T\varphi\|_{j+1}}{m_j} = \\
&= \sum_{j=1}^{\infty}\Big(\beta c\frac{m_{j+1}}{m_j}\Big)c^{-j-1}\frac{\|\varphi - \psi\|_{j+1}}{m_{j+1}} + \sum_{j=1}^{\infty}\Big(Lc\frac{m_{j+1}}{m_j}\Big)c^{-j-1}\frac{\|\psi - T\varphi\|_{j+1}}{m_{j+1}} \leq \\
&\leq \zeta\sum_{j=1}^{\infty}\frac{\|\varphi - \psi\|_{j+1}}{m_{j+1}c^{j+1}} + \xi\sum_{j=1}^{\infty}\frac{\|\psi - T\varphi\|_{j+1}}{m_{j+1}c^{j+1}} = \zeta\|\varphi - \psi\| + \xi\|\psi - T\varphi\|,
\end{aligned}$$

for every $\varphi, \psi \in A$. Hence, T is an almost contraction with the fixed point $\varphi^* \in A$. The uniqueness condition (4.280) lead us to the uniqueness condition of an almost contraction. The proof is very similar to that used in Proposition 4.2.230:

$$\begin{aligned}
\|T\varphi - T\psi\| &= \sum_{j=1}^{\infty} c^{-j}\frac{\|T\varphi - T\psi\|_j}{m_j} \overbrace{\leq}^{(4.275)} \sum_{j=1}^{\infty} \beta c^{-j}\frac{\|\varphi - \psi\|_{j+1}}{m_j} + \sum_{j=1}^{\infty} Lc^{-j}\frac{\|\varphi - T\varphi\|_{j+1}}{m_j} = \\
&= \sum_{j=1}^{\infty}\Big(\beta c\frac{m_{j+1}}{m_j}\Big)c^{-j-1}\frac{\|\varphi - \psi\|_{j+1}}{m_{j+1}} + \sum_{j=1}^{\infty}\Big(Lc\frac{m_{j+1}}{m_j}\Big)c^{-j-1}\frac{\|\varphi - T\varphi\|_{j+1}}{m_{j+1}} \leq \\
&\leq \zeta\sum_{j=1}^{\infty}\frac{\|\varphi - \psi\|_{j+1}}{m_{j+1}c^{j+1}} + \xi\sum_{j=1}^{\infty}\frac{\|\varphi - T\varphi\|_{j+1}}{m_{j+1}c^{j+1}} = \zeta\|\varphi - \psi\| + \xi\|\varphi - T\varphi\|,
\end{aligned}$$

for every $\varphi, \psi \in A$.
This way, the uniqueness of the fixed point for an 1-almost local contraction is proved.
□

Starting from Theorem 4.2.228, we present some other properties of the Bellman operator which are very important.

THEOREM 4.2.233. *Let $\mathcal{B}$ be the Bellman operator satisfying (DP1) and (DP2) such that there exists a countable increasing sequence $\{K_j\}$ of nonempty , compact subsets of X with* (4.263) *fulfilled and $\Gamma(K_j) \subseteq K_j$ for each $j \in \mathbb{N}$.*
Then

(a) $\|\mathcal{B}f - f\|_{K_j} \leq \|f - \mathcal{B}g\|_{K_j} + \beta \cdot \|f - g\|_{K_j}$, for all $f, g \in C(X)$;
(b) $\|\mathcal{B}f - g\|_{K_j} \leq \|\mathcal{B}g - g\|_{K_j} + \beta \cdot \|f - g\|_{K_j}$, for all $f, g \in C(X)$;
(c) $\|\mathcal{B}f - f\|_{K_j} \leq \|\mathcal{B}g - g\|_{K_j} + (\beta + 1) \cdot \|f - g\|_{K_j}$, for all $f, g \in C(X)$.

Proof: (a) Use the well known inequality $x - y \leq |x - y|, \forall x, y \in \mathbb{R}$. If we replace the real numbers $f(x)$ and $f(y)$ instead of x, y, we obtain: $f(x) \leq f(y) + |f(x) - f(y)|$, for every $x, y \in X, \forall f, g \in C(X)$. By using the definition of the Bellman operator, we get:

$$\begin{aligned}
\mathcal{B}f(x) &= \max_{y \in \Gamma(x)} \Big(U(x, y) + \beta f(y) \Big) \leq \\
&\leq \max_{y \in \Gamma(x)} \Big(U(x, y) + \beta g(y) + \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)| \Big) = \\
&= \max_{y \in \Gamma(x)} \mathcal{B}g(x) + \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)|, \forall x \in X, f, g \in C(X).
\end{aligned}$$

We obtain

$$\mathcal{B}f(x) \leq \mathcal{B}g(x) + \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)|, \forall x \in X, f, g \in C(X). \tag{4.281}$$

Changing the roles of f and g and having in view that $\Gamma(K_j) \subseteq K_j$, we deduce

$$\|\mathcal{B}f - \mathcal{B}g\|_{K_j} \leq \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)| \leq \beta \|f - g\|_{K_j}, \forall x \in X, f, g \in C(X). \tag{4.282}$$

The last inequality means that the Bellman operator is a 0-local contraction.
Consequently, by (4.281), we obtain:

$$\mathcal{B}f - f \leq \mathcal{B}g - f + \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)|, \forall x \in X, f, g \in C(X). \tag{4.283}$$

Hence, we can write: $\|\mathcal{B}f - f\|_{K_j} \leq \|f - \mathcal{B}g\|_{K_j} + \beta \cdot \|f - g\|_{K_j}$, for all $f, g \in C(X)$.
(b) The conclusion is immediate from (4.281), since we have:

$$\mathcal{B}f - g \leq \mathcal{B}g - g + \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)|, \forall x \in X, f, g \in C(X),$$

which implies in a similar way as in the beginning of the proof:

$$\|\mathcal{B}f - g\|_{K_j} \leq \|\mathcal{B}g - g\|_{K_j} + \beta \cdot \|f - g\|_{K_j},$$

for all $f, g \in C(X)$.
(c) The inequality (4.281) implies that:

$$\mathcal{B}f(x) \leq \mathcal{B}g(x) + \beta \|f - g\|_{K_j}, \forall x \in X, f, g \in C(X). \tag{4.284}$$

The last inequality implies:

$$\mathcal{B}f(x) \leq \mathcal{B}g(x) + f(x) - f(x) + g(x) - g(x) + \beta \|f - g\|_{K_j}.$$

We obtain:

$$\mathcal{B}f - f \leq \mathcal{B}g - g + \beta\|f-g\| + \|f-g\|.$$

Therefore, we have $\|\mathcal{B}f - f\| \leq \|\mathcal{B}g - g\| + (\beta+1)\|f-g\|$, for all $f, g \in C(X)$. □

REMARK 4.2.234.
- *By introducing a new operator $\Lambda := \mathcal{B} - \mathcal{I}$ ($\mathcal{I}$ being the identity mapping), we can reformulate (c) from Theorem 4.2.233 as follows:*

$$\|\Lambda f\| \leq \|\Lambda g\| + (\beta+1)\|f-g\|, \forall f, g \in C(X). \tag{4.285}$$

- *Theorem 4.2.233 brings some improvements to the study of the Bellman operator in the deterministic case, by providing new properties of that operator. These results will open a window to study them in the context of dynamic programming.*
- *In order to illustrate the strong connection between the almost local contractions studied in Chapter 1 and the k-almost local contractions from Chapter 4, note that Examples 1.2.49, 1.3.98, 1.3.67 and 1.3.127 provide early examples of* 1*-almost local contractions, since they all have in common the function $r : J \to J$, $r(j) = j+1, \forall j \in J$. Also, observe that a* 0*-almost local contraction is, in fact, an almost contraction with many examples in the literature.*

b) Stochastic dynamic programming

The theory of dynamic programming with uncountable state space begins with the essential work of Blackwell (see [**33**]). His results were extended in a large number of directions, with important applications in economy, operations and engineering. In fact, the theory of stochastic optimal growth lies in the framework of dynamic programming. In the beginning of this subsection, we present some preliminary concepts from [**78**].

DEFINITION 4.2.235. [**78**] *Let us consider (X, Σ) a measurable space and Y a separable metric space equipped with the Borel σ-algebra. Let S be the family of nonempty subsets of Y. The mapping $B : X \to S$ is said to be (weakly) measurable if*

$$B^{-1}(D) := \{x \in X : B(x) \cap D \neq \Phi\} \in \Sigma,$$

for every open set $D \subset Y$.

In the sequel, let X be a metric space and let B a set-valued mapping defined on X. Then B is called continuous if $B^{-1}(D)$ is closed for every closed set $D \subset Y$ and open for every open set $D \subset Y$. Obviously, a continuous set-valued mapping B is measurable if Σ is the Borel σ-algebra on X. Note that every measurable mapping B with nonempty compact values $B(x)$ for all $x \in X$ admits a measurable selector, see [**75**].

Let $B : X \to S$ be a measurable compact set-valued mapping, where S represents the family of nonempty subsets of Y. Denote

$$C := \{(x, a) : x \in X, a \in B(x)\}. \tag{4.286}$$

According to Himmelberg's work [**62**], C is a measurable subset of $X \times Y$ equipped with the product σ-algebra.

PROPOSITION 4.2.236. [**78**] *Let us consider the set defined in* (4.286)*. Let* $h : C \to \mathbb{R}$ *be a measurable function such that the correspondence* $a \to h(x, a)$ *is continuous on* $B(x)$*, for every* $x \in X$*. Then*

$$h^*(x) := \max_{a \in B(x)} h(x, a)$$

is measurable and, also, we can find a measurable operator $g^* : X \to Y$ *such that*

$$g^*(x) \in \arg\max_{a \in B(x)} h(x, a),$$

for all $x \in X$, *where we denote*

$$\arg\max_{a \in B(x)} h(x, a) = \{x | x \in B(x) \text{ for all } b \in B(x) : h(x, b) \leq h(x, a)\}.$$

Following [**78**], the *discrete-time Markov decision process* includes the following objects: $X, Y, u, q, \beta, \{A(x)\}_{x \in X}$ such that:

M1: X represents the *state space* endowed with a σ-algebra Σ.

M2: Y is called *the space of actions of the decision maker*, which is actually a separable metric space. For every $x \in X$, the compact subset $A(x) \subset Y$ indicates the set of all actions available in state $x \in X$. The set C is defined according to (4.286).

M3: $u : C \to \mathbb{R}$ represents the (product) measurable *instantaneous return function.*

M4: q indicates a transition probability from C to X, termed the *law of motion among states.*

M5: The coefficient $\beta \in (0, 1)$ represents the *discount factor.*

The sequence $\pi = \{\pi_t\}$ is a *policy*, where π_t is a measurable mapping which realize a connection between an action $a_t \in A(s_t)$ and any admissible history of the process s_t. The set of all policies will be denoted by Π. Our work is focused on non-randomized policies, because they are adequate to study the discounted models. We define the *expected discounted return* over an infinite future as:

$$J(x, \Pi) := E_x^\pi\Big(\sum_{t=1}^{\infty} \beta^{t-1} u(x_t, a_t)\Big), \tag{4.287}$$

for each initial state $x_1 = x$ and any policy $\pi \in \Pi$. In that equality, E_x^π indicates the *expectation operator* regarding to the unique conditional probability measure P_x^π defined by π and the transition probability q stated by the Ionescu Tulcea Theorem (see [**63**]). In the sequel, we assume that the expected returns (4.287) are well-defined.

In the following, we add some regularity assumptions regarding the return and transition probability functions:

(R1): Let X be a metric space and K_j a strictly increasing family of compact sets such that

$$X = \bigcup_{j=1}^{\infty} Int(K_j). \tag{4.288}$$

Denote by $C_c(X)$ the space of all continuous functions on X with compact support. Assume that:

a) the set-valued mapping $x \to A(x)$ is continuous,

b) the return function u is continuous,

c) Denote by $q(dy|x,a)$ represents the law of motion among states from C to X. The correlation

$$(x,a) \to \int_X v(y)q(dy|x,a)$$

is also continuous on the set C, for every $v \in C_c(X)$. The domain of this function is C, defined in (4.286), the codomain is $\mathbb{R}$.

If X is not necessarily a topological space, we use another regularity condition:

(R2): For every $x \in X$ and every measurable set $D \subset X$, the functions $a \to u(x,a)$ and $a \to q(D|x,a)$ are continuous on $A(x)$. The domain of both functions is $A(x)$ and the codomain is $\mathbb{R}$.

Remark 4.2.237. *Having in view the (R2) regularity assumption, we can deduce that q is continuous on C if the space of probability measures on the space X (which is assumed to be σ-compact) is equipped with the vague topology*

The vague topology can be defined by duality with continuous functions having compact support $C_C(X)$ (see [**51**]).

Definition 4.2.238. [**66**] *Let X be a locally compact Hausdorff space. Assume that X is second countable, which means there is a countable base. Then X is a Polish space (there exists a complete separable metrization). Let $\mathfrak{X}$ be a Borel field of X, generated by the (set of open subsets of the) topology of X. Let $\mathfrak{B}$ be ring of all relatively compact elements of $\mathfrak{X}$, the ring of bounded Borel sets. Let $\mathfrak{M}$ be the collection of all Borel measures on X. Let $\mathfrak{F}_C$ be the space of real-valued functions of compact support on X. A sequence of elements $\mu_n \in \mathfrak{M}$ converges to $\mu \in \mathfrak{M}$ if for all $f \in \mathfrak{F}_C$,*

$$\int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx).$$

The topology obtained on $\mathfrak{M}$ is called the vague topology.

If the (R_1) or (R_2) regularity conditions are fulfilled, we define

$$u_j(x) := \max_{a \in A(x)} |u(x,a)|, \quad x \in K_j, \quad r_j := \sup_{x \in K_j} u_j(x). \tag{4.289}$$

Consider the sequences $\{m_j\}$ and $\{K_j\}$ mentioned before. Suppose that (4.263) is valid. In order to obtain our main results, we need some new assumptions, as follows:

(P1): We have the identity:

$$q(K_j|x,a) = 1 \text{ for each } j \in \mathbb{N}, x \in K_j, a \in A(x), \tag{4.290}$$

(P2): Suppose that there exists $c > 1$ such that

$$\zeta := \beta c \cdot \sup_{j\in\mathbb{N}} \frac{m_{j+1}}{m_j} < 1. \tag{4.291}$$

In the sequel, we make two notations:

1) If X is a metric space, the sets K_j are compact and (4.288) holds, then let us denote by $C_m(X)$ (and $C_{mb}(X)$) the set $F_m(X)$ (and $F_{mb}(X)$, respectively);

2) If (X, Σ) is a measurable space, $\{K_j\}$ an increasing sequence of measurable sets with condition (4.263) fulfilled. In this case, $F_m(X)$ turns into $M_m(X)$, which represents the set of all measurable functions on X, bounded on every set K_j and $F_{mb}(X)$ will be denoted by $M_{mb}(X)$, respectively.

Furthermore, there exists a function $g \in M_m(X)$ such that

$$|u_j(x)| \leq g(x), \forall j \in \mathbb{N}, x \in K_j. \tag{4.292}$$

If X is a metric space, then $g \in C_m(X)$. Moreover, if $x \in K_j, a \in A(x)$, $j \in \mathbb{N}$, one obtain $q(K_{j+1}|x,a) = 1$.

Remark 4.2.239. *In fact,* (4.291) *leads to the conclusion:*

$$\sum_{j=1}^{\infty} (\beta c)^j m_j < \infty$$

The basis for the theory of discounted Markov decision processes is represented by the Bellman functional equation. Its form appears in various papers. The Bellman functional equation stated by Matkowski and Nowak [**78**], is:

$$Lv(x,a) := u(x,a) + \beta \int_X v(y) q(dy|x,a), \quad (x,a) \in C. \tag{4.293}$$

for the integrable function $v : X \to \mathbb{R}$.

According to [**78**], the Bellman equation can be put in the form:

$$v^*(x) = \max_{a\in A(x)} Lv^*(x,a), \quad x \in X, \tag{4.294}$$

while the Bellman equation, introduced by Rincón-Zapatero [**94**], is

$$\mathcal{B}f = f. \tag{4.295}$$

Example 4.2.240. [**30**] *(Discounted DP Problems)*

Let X the set of "states" and U be the set of "controls". For every $x \in X$, let $U(x) \subset U$ be a nonempty subset of controls that are feasible at state x. We refer to a function $\mu : X \to U$ with $\mu(x) \in U(x)$, for every $x \in X$, as a "policy". Let us denote by $\mathcal{M}$ the *set of all policies. Let $R(X)$ the set of functions $J : X \to \mathbb{R}$. Let $H : X \times U \times R(X) \to \mathbb{R}$ be a given mapping. Consistent with the DP context, we consider the mapping T defined by*

$$(TJ)(x) = \inf_{u \in U(x)} H(x, u, J), \quad \forall x \in X.$$

Suppose that $(TJ)(x) > -\infty$ for every $x \in X$, which means that T maps $R(X)$ into $R(X)$. For every policy $\mu \in \mathcal{M}$, consider the mapping $T_\mu : R(X) \to R(X)$ defined by

$$(T_\mu J)(x) = H\Big(x, \mu(x), J\Big), \quad \forall x \in X.$$

The purpose is to find a function $J^* \in R(X)$ such that

$$J^*(x) = \inf_{u \in U(x)} H(x, u, J^*), \quad \forall x \in X,$$

which means that we want to find a fixed point of T. Also, we intend to find a policy μ^* such that $T_{\mu^*} J^* = TJ^*$. Note that the mapping T can be defined by

$$(TJ)(x) = \inf_{\mu \in \mathcal{M}} (T_\mu J)(x), \quad \forall x \in X, J \in R(X).$$

Consider an α-discounted total cost DP problem. We have

$$H(x, u, J) = E\{g(x, u, w) + \alpha J(f(x, u, w))\},$$

where $\alpha \in (0, 1)$, the function g is uniformly bounded and represents cost per stage, w is random with distribution, depending on (x, u).
The equation $J = TJ$, i.e.,

$$J(x) = \inf_{u \in U(x)} H(x, u, J) = \inf_{u \in U(x)} E\{g(x, u, w) + \alpha J(f(x, u, w))\}$$

is actually Bellman's equation with the unique solution J^*. The mapping H could be used in different forms, such as:

$$H(x, u, J) = \min\Big[V(x), E\{g(x, u, w) + \alpha J(f(x, u, w))\}\Big],$$

or

$$H(x, u, J) = E\Big\{g(x, u, w) + \alpha \min[V(f(x, u, w)), J(f(x, u, w))]\Big\},$$

where V is a known function with the property $V(x) \geq J^*(x)$ for all $x \in X$. If the solution J^* is not affected by using different variants V for the mapping H, we will apply policy iteration algorithms for a more favorably computation of the value x.

EXAMPLE 4.2.241. [**30**] *(Discounted Semi-Markov Problems)*
With x, y, u as in the previous example, take the mapping

$$H(x, u, J) = G(x, u) + \sum_{y=1}^{n} m_{xy}(u) J(y),$$

where the function G represents cost per stage, U represents the set of controls and $m_{xy}(u)$ are non-negative numbers with

$$\sum_{y=1}^{n} m_{xy} < 1, \forall x \in X, u \in U(x).$$

The equation $J = TJ$ is Bellman's equation for a continuous-time semi-Markov decision problem.

EXAMPLE 4.2.242. [**30**] *(Minimax Problems)*
Consider a minimax version of Example 4.2.240, where an antagonistic player chooses v from a set $V(x, u)$. Consider the mapping

$$H(x, u, J) = \sup_{v \in V(x,u)} \Big[g(x, u, v) + \alpha J(f(x, u, v))\Big].$$

Again, the equation $J = TJ$ is Bellman's equation for an infinite horizon minimax DP problem. A generalization is represented by the mapping

$$H(x, u, J) = \sup_{v \in V(x,u)} E\Big\{g(x, u, v, w) + \alpha J(f(x, u, v, w))\Big\},$$

where w is random with given distribution, and the expected value is with respect to that distribution. This form could be found in zero-sum sequential games.

Our main goal is to state and prove an existence theorem of a solution to the Bellman equation in the space $C_m(X)$, when X is a metric space, or in $M_m(X)$ for a more general case.

THEOREM 4.2.243. *Suppose condition (P1) holds.*
a) If condition (R1) is valid, then there exist an increasing unbounded sequence $m = \{m_j\}$ and a unique mapping $v^* \in C_m(X)$, which satisfies the Bellman equation (4.294)*.*
b) If condition (R2) holds and $r_j < \infty$, for every $j \in \mathbb{N}$ is valid, then there exist an increasing unbounded sequence $m = \{m_j\}$ and a unique mapping $v^* \in M_m(X)$, which satisfies the Bellman equation (4.294)*.*

Proof: a) If we make assumption (R1), then Berge's theorem 4.2.225 assures the continuity of every function u_i on the compact set K_j. Thus, $r_j < \infty, \forall j \in \mathbb{N}$. That means that we can select an increasing unbounded sequence $m = \{m_j\}$ such that $m_j \geq r_j$. Let $C_{mb}(X)$ be the closed subset of the Banach space $C_m(X)$. We introduce the operator T on $C_{mb}(X)$, as follows:

$$\begin{aligned} Tv(x) \;&:=\; \max_{a \in A(x)} \Big((1-\beta)u(x,a) + \beta \int_X v(y) \cdot q(dy|x,a) + \\ &+\; L \int_X (v - Tv)(y) \cdot q(dy|x,a)\Big), \end{aligned} \tag{4.296}$$

for $v \in C_{mb}(X), x \in X$ and some $\beta, L \in (0,1)$ such that $\frac{\beta+L}{1-L} \in (0,1)$. Again, by the Berge's maximum theorem, Tv is continuous on each set K_j. According to (4.288), the mapping Tv is continuous on the set X. It follows that the mapping $T : C_{mb}(X) \to X$

is a non-self mapping. In the light of these considerations, we can easily conclude that T is a Ćirić-Reich-Rus type almost contraction. By using our assumption on q (the identity (4.290) from condition (P1)), we get:

$$\begin{aligned} \|Tv - Tw\|_j &\leq \beta \cdot \|v - w\|_j + L \cdot \Big(\|v - Tv\|_j - \|w - Tw\|_j\Big) \leq \\ &\leq \beta \cdot \|v - w\|_j + L \cdot \Big(\|v - Tv\|_j + \|w - Tw\|_j\Big), \end{aligned}$$

for each $j \in \mathbb{N}$ and $\forall v, w \in C_{mb}(X)$. Remind that a Ćirić-Reich-Rus type contraction is an almost contraction in certain conditions (Lemma 1.3.2. from [**83**]). By applying Proposition 4.2.230, the mapping T have a unique fixed point $w^* \in C_{mb}(X)$ such that $Tw^* = w^*$. Take $v^* = \frac{w^*}{1-\beta} \in C_m(X)$, which is a solution for the Bellman equation (4.294).

b) In this case, the proof is very similar to that of case (a), if we use Proposition14.09.6 and Proposition 4.2.230. Obviously, in that case the solution of the Bellman equation is $v^* \in M_m(X)$. □

Theorem 4.2.244. *Under the conditions of Theorem 4.2.243, but in the space* $M_m(X)$*, there exists an increasing unbounded sequence* $m = \{m_j\}$ *and a unique mapping* $v^* \in M_m(X)$*, which satisfies the Bellman equation* (4.294)*.*

Proof: If we make assumption (R2) and $r_j < \infty$ for every $j \in \mathbb{N}$, the proof is very similar to that of Theorem 4.2.243, by applying Theorem 4.2.230. Obviously, the fixed point is $v^* \in M_m(X)$. □

Theorem 4.2.245. *Suppose condition (P2) is satisfied. If condition (R1) is valid, then the Bellman equation* (4.295) *has the unique solution* $v^* \in C_m(X)$*.*

Proof: If we make assumptions (P2) and (R1), then the operator T from (4.296) is well-defined, with some $\beta, L \in (0,1)$ such that $\frac{\beta+L}{1-L} \in (0,1)$, for any $v \in C_m(X)$, hence $\|v\| < \infty$. Denote

$$\rho := \|g\|, \quad u^*(x) := \max_{a \in A(x)} |u(x,a)|,$$

where g is given by (4.292).

The operator T is defined on the closed ball $B_\rho := \{v \in C_m(X) : \|v\| \leq \rho\} \subset C_m(X)$. That means: $u^* \in B_\rho$. By applying the Berge's maximum theorem, the operator Tv is continuous, for every $v \in B_\rho$. For finalizing the proof, we need another notation:

$$\mu(x) := \max_{a \in A(x)} \Big| \int_X v(y) \cdot q(dy|x,a) \Big|, \quad x \in X. \tag{4.297}$$

Obviously, the mapping μ is continuous. By condition (P2), we conclude that $\|\mu\|_j \leq \|v\|_{j+1}$, for each $j \in \mathbb{N}$ and for $x \in K_j$. By using that $\zeta < 1$ and $\|v\| \leq \rho$, we obtain:

$$\begin{aligned}\|\mu\| &= \sum_{j=1}^{\infty}\frac{\|\mu\|_j}{m_j c^j} \le \sum_{j=1}^{\infty}\frac{\|v\|_{j+1}}{m_j c^j} = \sum_{j=1}^{\infty}\frac{\|v\|_{j+1}}{m_{j+1}c^{j+1}}\cdot\frac{m_{j+1}c^{j+1}}{m_j c^j} \le \\ &\le \|v\|\cdot\frac{1}{\beta}\cdot\frac{\beta c\cdot m_{j+1}}{m_j} = \frac{\|v\|\zeta}{\beta} \le \frac{\rho}{\beta}.\end{aligned}$$

Again, the operator T is a non-self mapping, similar to that in Theorem 4.2.244. In the light of the last inequalities regarding μ, we obtain that T is a Ciric-Reich-Rus type 1-ALC.

$$\|Tv - Tw\|_j \le \beta\cdot\|v-w\|_{j+1} + L\cdot\Big(\|v-Tv\|_{j+1} + \|w-Tw\|_{j+1}\Big).$$

By applying Proposition 4.2.232, remind that a Ćirić-Reich-Rus type contraction is an almost contraction. These considerations lead to the conclusion that the mapping T is a 1-almost local contraction with unique fixed point $w^* \in C_m(X)$ such that $Tw^* = w^*$. Take $v^* = \frac{w^*}{1-\beta} \in C_m(X)$, which is a solution for the Bellman equation (4.294). □

Remark 4.2.246. *The main results obtained in the deterministic case are significantly more general that the results presented in the studies of Rincón-Zapatero* [**94**] *and Matkowski and Novak* [**78**]*, because we are working with k-almost local contractions instead of k-local contractions. So, in the light of the examples of almost local contractions by means of which we have illustrated our theoretical results in Chapter 1, we extend effectively the results from* [**78**] *and* [**94**] *to the more general case of k-almost local contractions, thus providing new existence theorems for Bellman equation and some new properties of the Bellman operator.*

On the other hand, the main results obtained in our thesis for the stochastic case provide another kind of novelty in dynamic programming by connecting the classical results with some of the classes of almost local contractions studied in the first chapter of the thesis.

CONCLUSIONS

In this part, we summarise the main results of the thesis. Then, we give a list of some possible further directions of research, in order to develop the new concepts, methods and results presented in the present thesis.

As mentioned in the beginning of this thesis, the theory of fixed points and Banach's contraction principle represent the foundation stone of nonlinear analysis with a huge development in the last years. The theory of Picard operators has various applications in nonlinear analysis. Most of the results contained in the present thesis were presented at the 17^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, Romania (2015), at the 18^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, Romania (2016), at the 19^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, Romania, (2017), and also at the Conference of Mathematics and Informatics with applications, organized by Babeş-Bolyai University, Cluj Napoca, Romania, (2016).

We started this thesis with a detailed review of the most important results involving contractions, fixed points and the two types of contractions: almost contractions and local contractions. The first chapter reiterated elementary results on both contractions, that represent the starting point in the subsequent chapters.

In Chapter 2 we introduce a new type of mappings with the use of the aforementioned two type of contractions, namely: the almost local contractions. We have obtained new existence and uniqueness theorems for this new mappings, then we extend the obtained results in order to obtain new classes of almost local contractions, such as:

- generalized almost local contractions,
- Ćirić type strong almost local contractions,
- quasi almost local contractions,
- Ćirić-Reich-Rus type almost local contractions,
- Chatterjea type almost local contractions,
- generalized Berinde type almost local contractions,
- almost local φ-contractions,
- B-almost local contractions, etc.

We have included a large collection of examples for several types of ALC-s for the purpose of a better understanding and in order to underline the applicability of certain theorems, lemmas. This chapter also contains a detailed survey about approximate fixed points with quantitative and qualitative results for mappings in pseudometric spaces. These results are contained in [**118**], [**119**], [**120**].

In Chapter 3 we continue our work with the study of the multivalued almost local contractions, both self and non-self types.The chapter is mainly containing the results from 19^{th} Symposium of Symbolic and Numeric Algorithms for Scientific Computing SYNASC Timişoara, where I presented my research results "Multivalued Self Almost Local Contractions".

In Chapter 4 we use the same ideas to study the non-self single valued almost local contractions, providing illustrative examples, as well. According to Chapters 2-4, we provided existence and uniqueness theorems for different type of mappings, using various types of contractions, as listed below:

$$d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in A.$$

$$d_j(Tx,Ty) \leq \theta_u \cdot d_{r(j)}(x,y) + L_u \cdot d_{r(j)}(x,Tx), \forall x,y \in A,$$

$$\begin{aligned} d_j(Tx,Ty) \quad &\leq \quad \theta \cdot d_{r(j)}(x,y) + \\ &+L \cdot min\{d_{r(j)}(x,Tx), d_{r(j)}(y,Ty), d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\}. \end{aligned}$$

$$d(Tx,Ty) \leq \alpha \cdot M_1(x,y) + L \cdot d(y,Tx), \text{ for all x,y} \in X,$$

where

$$M_1(x,y) = max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\Big\}.$$

$$d_j(Tf,Tg) \leq \theta \cdot M_{r(j)}(f,g), \text{ for all f,g} \in A,$$

where

$$M_{r(j)}(f,g) = \max\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\}.$$

$$d_j(Tf,Tg) \leq \delta \cdot d_{r(j)}(f,g) + L \cdot [d_{r(j)}(f,Tf) + d_{r(j)}(g,Tg)],$$

$$d_j(Tf,Tg) \leq c \cdot [d_{r(j)}(f,Tg) + d_{r(j)}(g,Tf)], \forall f,g \in A, \forall j \in J.$$

$$d_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + b(y) \cdot d_{r(j)}(y,Tx), \forall x,y \in A, \forall j \in J.$$

$$d_j(Tx,Ty) \leq \varphi(d_{r(j)}(x,y)) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in A, \forall j \in J.$$

$$d_j(Tx,Ty) \leq \Upsilon(d_{r(j)}(x,y)) + L_1 \cdot d_{r(j)}(x,Tx)$$

$$d_j(Tx,Ty) \leq \varphi(d_{r(j)}(x,y)), \forall x,y \in A, \forall j \in J.$$

$$H_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot D_{r(j)}(y,Tx), \forall x,y \in S, \forall j \in J.$$

$$\begin{aligned} H_j(Tx,Ty) \leq \alpha(d_{r(j)}(x,y))d_{r(j)}(x,y) + L \cdot \min\Big\{&d_{r(j)}(x,Tx), \\ d_{r(j)}(y,Ty), d_{r(j)}(x,Ty), d_{r(j)}(y,Tx)\Big\}&, \forall x,y \in S, \forall j \in J. \end{aligned}$$

$$H_j(Tx,Ty) \leq \theta \cdot d_{r(j)}(x,y) + L \cdot D_{r(j)}(y,Tx), \forall x,y \in K, \forall j \in J.$$

$$\begin{aligned} H_j(Tx,Ty) &\leq \alpha \cdot d_{r(j)}(x,y) + \beta \cdot \max\{D_{r(j)}(x,Tx), D_{r(j)}(y,Ty)\} + \\ &+ \gamma[D_{r(j)}(x,Ty) + D_{r(j)}(y,Tx)], \quad \forall j \in J, \end{aligned}$$

with $\alpha, \beta, \gamma \geq 0$ such that $p = \left(\frac{1+\alpha+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) < 1$.

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)).$$

$$d_j(T^2x, Tx) \leq \alpha d_{r(j)}(x,Tx), \forall j \in J.$$

$$\|T\varphi - T\psi\|_j \leq \beta\|\varphi - \psi\|_{j+k} \quad \forall, \varphi, \psi \in A, j \in \mathbb{N}.$$

$$\|T\varphi - T\psi\|_j \leq \theta_j\|\varphi - \psi\|_{j+k} + L_j\|\psi - T\varphi\|_{j+k} \quad \forall, \varphi, \psi \in A, j \in \mathbb{N}.$$

In Chapter 4 we extend the results obtained in the previous chapters in order to apply the almost local contractions in dynamic programming. The goal of this chapter is to solve the Bellman equation, and also to study the existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming. Building on the research paper by the pair of authors: Martins-da-Rocha and Vailakis (2010) and respectively, Rincón-Zapatero, Rodriguez-Palmero (2003), we attempt to improve their models with unbounded returns. Applications of recursive methods appear in every area of economics: public finance, growth theory etc. We intend to present these methods and applications in a systematic way.

Finally, in the part with the conclusions, we offer an outlook on possible directions for future research. There are some possible extensions to the almost local contractions. We think that our results have applicability in solving dynamic programming problems. The obtained results naturally generate new questions, open problems, which could be studied, eventually, with the same methods or new techniques.
The present thesis can be extended in the following directions:

(1) extending the almost local contractions with constant coefficient of contractions to that of variable coefficients of contraction, as follows:

Definition 4.2.247. *Let X be a uniform space and denote by J a family of indices. Let r be a function from J to J. Let $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics defined on X. Let τ be the weak topology on X defined by the family $\mathcal{D}$. Consider a nonempty, τ-bounded, sequentially τ-complete subset $A \subset X$. Let the operator $T : A \to A$ and assume that the subset A is T-invariant.*
The operator $T : A \to A$ is called almost local contraction with regard to ($\mathcal{D}$,r) if, for every $j \in J$, there exist the constants $\theta_j \in (0,1)$ and $L_j \geq 0$ such that

$$d_j(Tx,Ty) \leq \theta_j \cdot d_{r(j)}(x,y) + L_j \cdot d_{r(j)}(y,Tx), \forall x,y \in A.$$

It would be interesting to examine the conditions for existence and uniqueness of the fixed points for that new type of contraction.

(2) analyzing the fixed points of almost local contractions using other types of contractions, such as:
- Bianchini, in [**48**]: there exist $0 \leq h < 1$ such that
$$d(Tx, Ty) \leq h \cdot \max\{d(x, Tx), d(y, Ty)\}, \forall x, y \in X, ;$$
- Reich in [**91**]: There exist nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that, for each $x, y \in X$,
$$d(f(x), f(y)) < ad(x, f(x)) + bd(y, f(y)) + cd(x, y).$$
- Reich in [**88**]: There exist monotonically decreasing functions $a, b, c : (0, \infty) \to [0, 1]$ satisfying $a(t) + b(t) + c(t) < 1$ such that, for each $x, y \in X, x \neq y$,
$$d(f(x), f(y)) < a(d(x, y))d(x, f(x)) + b(d(x, y))d(y, f(y)) + c(d(x, y))d(x, y).$$
- Hardy and Rogers in [**60**]: There exist nonnegative constants a, b, c, d, e, satisfying $a + b + c + d + e < 1$ such that, for each $x, y \in X$,
$$d(f(x), f(y)) < a \cdot d(x, y) + b \cdot d(x, f(x)) + c \cdot d(y, f(y)) + +d \cdot d(x, f(y)) + e \cdot d(y, f(x)).$$
- Guseman in [**57**]: There exists a number a, $0 < a < 1$, such that for each $x \in X$ there exists an integer $p(x)$ and we have
$$d(f^{p(x)}(x), f^{p(x)}(y)) \leq a \cdot d(x, y), \forall y \in X.$$
- Generalized Z-contraction in [**2**]: Let (X, d) be a complete metric space, let $T : X \to X$ be a mapping and let $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a simulation function, that is, a function satisfying the following conditions:
  (a) $\eta(t, s) < s - t$, for all $s, t > 0$
  (b) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that
$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \text{ implies } \limsup_{n\to\infty} \eta(t_n, s_n) < 0.$$
  Then T is called a *generalized Z-contraction* with respect to η if the following condition is satisfied:
$$\eta(d(Tx, Ty), r_T(x, y)) \geq 0, \quad \forall x, y \in X,$$
  where
$$r_T(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{d(x, y)}\right\}, \text{ if } x \neq y \text{ and } r_T(x, y) = 0 \text{ if } x = y$$
  and Z denote the set of all simulation functions.

(3) estimating fixed points of almost local contractions using another iteration methods, such as: Krasnoselskij, Mann, Ishikawa iterations etc.

(4) Study the existence and uniqueness for the solution of the Bellman equation in the case of k-local contractions.

(5) Extend the k-almost local contractions in order to study new classes of k-almost local contractions, analysing the existence and uniqueness of the fixed points and providing error estimates.

Bibliography

[1] Alghamdi, M.A., Berinde, V., Shahzad, N., *Fixed points of multivalued non-self almost contractions*, J. Appl. Math., (2013), Article ID 621614, 1-6.

[2] Alsubaie, R., Alqahtani, B., Karapınar, E., Roldán López de Hierro, A.F., *Extended Simulation Function via Rational Expressions*, Mathematics, (2020), 8, 710, 1-29.

[3] Angelov, V.G., *Fixed Points in Uniform Spaces and Applications*, Cluj University Press (2009).

[4] Assad, N.A., Kirk, W.A., *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43** (1972), 553–562.

[5] Azam, A., Fisher, B., Khan, M., *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim., **32** (2011), no. 3, 243–253.

[6] Babu, G. V. R., Sandhya, M.L., Kameswari, M. V. R., *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math., **24** (2008), no. 1., 08–12.

[7] Bakhtin, I.A., *The contraction principle in quasimetric spaces*, (Russian), Func. An., Ulianowsk, Gos. Fed. Ins., **30** (1989), 26–37.

[8] Caccioppoli, R., *Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale*, Rend. Accad. Lincei, **11** (1930), 794–799.

[9] Banach S., *Sur les opérations dans les ensembles abstraits et leur applications aux equations intégrales*, Fund. Math., **3** (1922), 133–181.

[10] Barroso, C.S., *The approximate fixed point property in Hausdorff topological vector spaces and applications*, Discrete Cont Dyn S, **25** (2009), no. 2, 467–479.

[11] Bellman, R., *Dynamic programming*, Princeton University Press (1957).

[12] Berinde, V., *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory (Preprint), "'Babeş-Bolyai"' University of Cluj- Napoca, **3** (1993), 3-9.

[13] Berinde, V., *Sequences of operators and fixed points in quasimetric spaces*, Stud. Univ. "Babeş-Bolyai", Math., **16** (1996), no. 4, 23-27.

[14] Berinde, V., *Generalized Contractions and Applications* (in Romanian), Editura Cub Press 22, Baia Mare, (1997).

[15] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), no. 1, 7–22.

[16] Berinde, V., *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theor-RO, 4 (2003), no. 2, 131–147.

[17] Berinde, V., *Approximating fixed points of weak φ-contractions using Picard iteration*, Fixed Point Theory, **4** (2003), no. 2, 131-142.

[18] Berinde, V., *On the convergence of the Ishikawa iteration in the class of quasi contractive operators*, Acta Math. Univ. Comenianae, **73** (2004), no.1, 119–126.

[19] Berinde V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum **9** (2004), no. 1, 43–53.

[20] Berinde, V., Berinde, M., *On Zamfirescu's fixed point theorem*, Carpathian J. Math., **50** (2005), no. 5-6, 443-453.

[21] Berinde, V., *A convergence theorem for some mean value fixed point iteration procedures in the class of quasi contractive operators*, Demonstr. Math., **38** (2005), 177–184.

[22] Berinde, M., *Approximate fixed point theorems*, Studia Univ. "Babeş-Bolyai" Mathematica, **LI** (2006), no. 1, 11-25.

[23] Berinde, V., Berinde, M., *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl., **326** (2007), 772–782.

[24] Berinde, V., *Iterative Approximation Of Fixed Points*, Springer Verlag, Berlin, Heidelberg, New York, (2007).

[25] Berinde, V., Păcurar, M., *Fixed points and continuity of almost contractions*, Fixed Point Theory, **9** (2008), no. 1, 23–24.

[26] Berinde, V., *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), no. 2, 10-19.

[27] Berinde, V., *Some remarks on a fixed point theorem for Ćirić- type almost contractions*, Carpathian J. Math., **25** (2009), no. 2, 157-161.

[28] Berinde, V., Păcurar, M. *Fixed points theorems for nonself single-valued almost contractions*, Fixed Point Theory, **14** (2013), no. 2, 301–312.

[29] Berinde, V., Păcurar, M., *Iterative Approximation of Fixed Points of Single-valued Almost Contractions*, Fixed Point Theory and Graph Theory, (2016), 29-97.

[30] Bertsekas, D.P., *Weighted sup-norm contractions in dynamic programming: a review and some new applications*, Dept. of Electr. Engineering and Comp. Science, M.I.T., Cambridge, Mass., 02139. Research supported by NSF Grant ECCS-0801549 and by the Air Force Grant FA9550-10-1-0412.

[31] Bertsekas, D.P., *Dynamic programming and optimal control*, 4th ed., **II** (2012), Athena Scientific, Belmont, MA.

[32] Berge, C., *Topological spaces*, MacMillan, (1963), New York.

[33] Blackwell, D., *Discounted dynamic programming*, Ann. Math. Statist., **36** (1965), 226-235.

[34] Boriceanu, M., *Dynamical aspects in the theory of multivalued operators*, Ph.D. Thesis, Cluj-Napoca, 2009.

[35] Boriceanu, M., Petruşel, A., Rus, I.A., *Fixed point theorems for some multivalued generalized contractions in b-metric spaces*, International J. Math. Statistics, **6** (2010), S10, 65–76.

[36] Browder, F.E., Petryshyn, W.V., *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72** (1966), 571–575.

[37] Bourbaki, N., *Topologie générale*, Herman, Paris, (1961).

[38] Brown, T.A., Comfort, W.W., *New method for expansion and contractive maps in uniform spaces*, Proc. Amer. Math. Soc., **11** (1960), 483–486.

[39] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc., **10** (1968), 71–76.

[40] Chatterjea, S.K., *Fixed-point theorems*, C.R. Acad. Bulgare Sci., **25** (1972), 727–730.

[41] Bianchini, R.M.T., *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Bollettino U.M.I., **4** (1972), no. 5, 103-108.

[42] Reich, S., *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **5** (1972), 26–42.

[43] Ćirić, Lj.B., *On contraction type mappings*, Math. Balkanica, **1** (1971), 52–57.

[44] Ćirić, Lj.B., *Quasi-contractions in Banach spaces*, Publications de l'Institut Mathématique, **35** (1977), 41–48.

[45] Ćirić, Lj.B., *A generalization of Banach's Contraction Principle*, Proc. Am. Math. Soc., **45** (1974), 267–273.

[46] Czerwik, S., *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti. Sem. Mat. Univ. Modena, **46** (1998), 263-276.

[47] Daffer, P.Z., Kaneko, H., *Fixed points of generalized contractive multivalued mappings*, J. Math. Anal. Appl., **192** (1995), 655–666.

[48] Bianchini, R.M.T., *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Bolletino U.M.I., **4** (1972), no. 5, 103–106.

[49] De Blasi, F.S., Myjak, J., *Sur la porosité des contractions sans point fixe*, C.R. Acad. Sci. Paris, **308** (1989), 51-54.

[50] Colojoară, I., *Sur un théoreme de point fixe dans les espaces uniformes complets*, Com. Acad. R. P. Română, **11** (1961), 281-283.

[51] Dieudonné, J., *The vague topology*, Treatise on analysis, II, Academic Press, **13.4** (1970).

[52] Feng, Y., Liu, S., *Fixed point theorems for multivalued contractive mappings and multivalued Caristi mappings*, J. Math. Anal. Appl., **317** (2006), 103-112.

[53] Gheorghiu, N., *The contractions theorem in uniform spaces*, (Romanian), Stud. Cerc. Mat., **19** (1967), 119-122.

[54] Gheorghiu, N., Rotaru, E., *Fixed point theorems in uniform spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat., **18** (1972), 311-314.

[55] Granas, A., Dugundji, J., *Fixed Point Theory*, Springer-Verlag, New York, (2003).

[56] Groze, S., Goldner, G., *Successive approximations in uniform spaces*, Proceedings of the colloquium on approximation and optimization, Univ. Cluj-Napoca, (1985), 105–110.

[57] Guseman, L.F. Jr., *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc., **26** (1970), 615–618.

[58] Hopf, H., *Freie Uberdeckungen und freie Abbildungen*, Fund. Math., **28** (1937), 33–57.

[59] Harder, A.M., Hicks, T.L., *Stability results for fixed point iteration procedures*, Math. Japon., **33** (1988), no. 5, 693–706.

[60] Hardy, G.E., Rogers, T.D, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201–206.

[61] Hicks, T.L., Rhoades, B.E., *A Banach type fixed point theorem*, Math. Japonica, **24** (1979), no. 3, 327–330.

[62] Himmelberg, C.J., *Measurable relations*, Fund. Math., **87** (1975), 63-72.

[63] Ionescu Tulcea, C.T., *Mesures dans les espaces produits*, Atti Accad. Naz. Lincei Rend., **7** (1949), 208–211.

[64] Ivanov, A.A., *Fixed points of metric space mappings* (in Russian), Isledovaniia po topologii. II, Akademia Nauk, Moskva, (1976), 5-102.

[65] Jovanović, M., Kadelburg, Z, Radenović, S., *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. 2010;2010:978121. doi: 10.1155/2010/978121.

[66] Kallenberg, O., *Random measures*, Akademie Verlag & Acad. Press (1986), Chapt. 15

[67] Kamihigashi, T., *Existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming*, (2011), RIEB Discussion Paper DP2011, 23.

[68] Kaneko, H., *A general principle for fixed points of contractive multivalued mappings*, Math. Japon., **31** (1986), 407–411.

[69] Kaneko, H., *Generalized contractive multivalued mappings and their fixed points*, Math. Japon., **33** (1988), 57–64.

[70] Kannan, R., *Some results on fixed points*, II, Amer. Math. Monthly. **76** (1969), 405-408.

[71] Kelley, J.E., *General topology*, D. van Nostrand Co., Inc. Princeton, New Jersey, (1953).

[72] Kirk, W.A., Sims, B. (eds.), *Handbook of metric fixed point theory*, Kluwer Acad. Publishers, Dordrecht, (2001).

[73] Klee, V.R., Jr., *Stability of the fixed point property*, Colloq. Math., **8** (1961), 43-46.

[74] Klim, D., Wardowski, D., *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334** (2007), 132–139.

[75] Kuratowski, K., Ryll-Nardzewski, C., *A general theorem on selectors*, Bull. Acad. Pol. Sci. (Ser. Mathematique), **13** (1965), 397-403.

[76] Martins-da-Rocha, V.F., Vailakis, Y., *Existence and uniqueness of a fixed point for local contractions*, Econometrica, **78** (2010), no. 3, 1127–1141.

[77] Matouskowa, E., Reich, S., Zaslawski, A.J., *Genericity in nonexpansive mapping theory*, Proceed. of the 1st international school "Advanced courses of mathematical analysis 1", Cádiz, Spain (A. Aizpuru-Tomás ed.), (2002), 81-98.

[78] Matkowski, J., Nowak, A.S., *On Discounting Dynamic Programming with Unbounded Returns*, MPRA Paper, (2008), no. 12215, 1–14.

[79] Mizoguchi, N., Takahashi, W., *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177–188.

[80] Nadler, S.B., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475-488.

[81] Neveu, J., *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco, (1965).

[82] Osilike, M.O., *Stability of the Ishikawa Iteration method for quasi-contractive maps*, Indian J. Pure Appl. Math., **28** (1997), no. 9, 1251–1265.

[83] Păcurar, M., *Iterative Methods for Fixed Point Approximation*, Cluj-Napoca: Editura Risoprint, (2009).

[84] Păcurar, M., *A fixed point result for φ-contractions on b-metric spaces without the boundedness assumption*, Fasc. Math., **43** (2010), 127–137.

[85] Rao, K.P.R., Swamy, P.R., Prasad, J.R., *A common fixed point theorem in complex valued b-metric spaces*, Bulletin of Mathematics and Statistics Research, **1** (2013), no. 1, 1–8.

[86] Ramasamy, C.T., Ganesa Moorthy, C., *Compact nonlinear mappings in metric spaces*, Proc. International Seminar on Recent Trends in Topology & its Applications, St. Joseph's College, Irinjalakuda, 2009.

[87] Reich, S., Zaslawski, A.J., *Well-posedness of fixed point problems*, Far East J. Math. Sci., Special Volume, **III** (2001), 393–401.

[88] Reich, S., *Kannan's fixed point theorem*, Boll. Un. Mat. Ital., **4** (1971), 1-11.

[89] Reich, S., *A fixed point theorem for locally contractive multivalued functions*, Rev. Roumaine Math. Pures Appl., **17** (1972), 569–572.

[90] Reich, S., *Some problems and results in fixed point theory*, Contemp. Math., **21** (1983), 179–187.

[91] Reich, S., *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124.

[92] Rhoades, B.E., *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc., **226** (1977), 257-290.

[93] Rhoades, B.E., *Fixed points and continuity for multivalued mappings*, Internat. J. Math. Math. Sci., **15** (1992), no. 1, 15–30.

[94] Rincón-Zapatero, J. P., Rodrigues-Palmero, C., *Existence and uniqueness of solutions to the Bellman equation in the unbounded case*, Econometrica, **71** (2003), 1519-1555.

[95] Rincón-Zapatero, J. P., Rodrigues-Palmero, C., *Recursive utility with unbounded aggregators*, Economic Theory, **33** (2007), 381-391.

[96] Rudin, W., *Functional Analysis*, New York: McGraw-Hill, (1991).

[97] Rus, I.A., *Metrical fixed point theorems*, Univ. of Cluj-Napoca, (1979).

[98] Rus, I.A., *Principles and Applications of the Fixed Point Theory*, (in Romanian) Editura Dacia, Cluj-Napoca, (1979).

[99] Taskovic, M., *Osnove teorije fiksne tacke*, (Fundamental Elements of Fixed Point Theory), Matematicka Biblioteka **50** (1986), Beograd.

[100] Rus, I.A., *Generalized Contractions*, Seminar on Fixed Point Theory **3** (1983), 1–130.

[101] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, (2001).

[102] Rus, I.A., *Picard operators and applications*, Babeş-Bolyai Univ., (1996).

[103] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, (2008).

[104] Rus, I.A., *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), no. 2, 541-559.

[105] Rus, I.A., *Picard operators and well-posedness of fixed point problems*, Studia Univ. Babeş-Bolyai, Math., **52** (2007), no. 3, 147-156.

[106] Rus, I.A., *Basic problems of the metric fixed point theory revisited* (II), Stud. Univ. Babeş-Bolyai, **36** (1991), 81–99.

[107] Rus, I.A., Petruşel, A., Sîntămărian, A., *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52** (2003), 1947-1959.

[108] Rus, I.A., Şerban, M.A., *Some fixed point theorems for non-self generalized contraction*, Miskolc Math Notes, **17** (2017), no. 2, 1021-1031.

[109] Schweizer, B., Sklar, A., Thorp, E., *The metrization of statistical metric spaces*, Pacific J. Math., **10** (1960), 673 -675.

[110] Singh, S.L., Bhatnagar, C., Mishra, S.N., *Stability of iterative procedures for multivalued maps in metric spaces*, Demonstratio Math., **37** (2005), 905-916.

[111] Singh, S.L., Czerwik, S., Król, K., Singh, A., *Coincidences and fixed points of hybrid contractions*, Tamsui Oxf. J. Math. Sci. 2008; 24: 401-416.

[112] Stokey, N. L., Lucas, R.E, Jr., Prescott, E.C., *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press, (1989).

[113] Suzuki, T., *Fixed Point Theorems For Berinde Mappings*, Bull. Kyushu Inst. Tech., Pure Appl. Math., **58** (2011), 13-19.

[114] Suzuki, T., *Generalized metric spaces do not have the compatible topology.*, Abstr. Appl. Anal. 2014;2014:458098. [Google Scholar]

[115] Suzuki, T., *Basic inequality on a b-metric space and its applications*, J Inequal Appl. 2017 (1): 256.

[116] Tijs, S., Torre, A., Brânzei, R., *Approximate Fixed Point Theorems*, Libertas Mathematica, **23** (2003), 35-39.

[117] van der Walt, T., *Fixed and almost fixed points*, Amsterdam, (1963).

[118] **Zákány, M., *Fixed Point Theorems For Local Almost Contractions*, Miskolc Math. Notes, 18 (2017), no. 1, 499-506**.

[119] **Zákány, M., *On the continuity of almost local contractions*, Creat. Math. Inform., 26 (2017), no. 2, 241-246**.

[120] **Zákány, M., *New classes of local almost contractions*, Acta Univ. Sapientiae, Mathematica, 10 (2018), no. 2, 378-394.**

[121] **Zákány, M., *Fixed points of (B)-almost local contractions in pseudo-metric spaces* (submitted)**

[122] **Zákány, M., *Fixed points of multivalued self almost local contractions in pseudo-metric spaces* (submitted)**

[123] **Zákány, M., Fixed point theorems for local almost contractions, SYNASC 2015- IAFP Workshop, 2015.**

[124] **Zákány, M., The continuity of almost local contractions, SYNASC 2016- IAFP Workshop, 2016.**

[125] **Zákány, M., Multivalued self almost local contractions, SYNASC 2017- IAFP Workshop, 2017.**

[126] **Zákány, M., Almost local contractions in b-pseudometric spaces, SYNASC 2019- IAFP Workshop, 2019.**

[127] Zamfirescu, T., *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292-298.

[128] Weil, A., *Sur les espaces à structure uniforme et sur la topologie generale*, Hermann&C-ie, Ed. Paris, (1938).

[129] Wilson, W. A., *On semimetric spaces*, Am. J. Math., **53** (1931), no. 2, 361-373.

[130] Willard, S., *General topology*, Addison-Wesley Publishing Company USA, (1970), 16-173.