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Ph.D. Thesis

(ABSTRACT)

ON THE APPROXIMATION ORDER AND VORONOVSKAJA'S TYPE THEOREM FOR CERTAIN LINEAR POSITIVE OPERATORS

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Introduction

The title of doctoral thesis is part of the field approximation theory, which covers a great deal of mathematical territory. In the present context, the focus is primarily on the approximation of real-valued continuous functions by simpler class of functions, for instance the class of algebraic polynomials. Such issues have attracted the attention of thousands mathematicians in the last two centuries.

A fundamental result in development of functions approximation theory is known as *First Weierstrass approximation theorem*, given by K. Weierstrass [**170**] in 1885, namely: *For any function $f \in C[a,b]$ and $\varepsilon > 0$, there exists an algebraic polynomial $P(x)$ with real coefficients, such that $|f(x) - P(x)| < \varepsilon$, for any $x \in [a,b]$.* First proof of Weierstrass approximation theorem was long and complicated, provoked many famous mathematicians to find simpler and more instructive proofs.

In 1905 E. Borel [**9**] proposed determination of an interpolation process, that allows finding polynomials $P(x)$, which converge uniformly to the function $f \in C[a,b]$.

In 1912 S.N. Bernstein [**6**] was able to give an outstanding solution for the problem proposed by E. Borel. On this occasion the (now) very well-known Bernstein polynomials were constructed

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \text{ for any } f \in C[0,1],\ x \in [0,1],\ n \in \mathbb{N}.$$

The Bernstein operators prove simple and constructive the Weierstrass approximation theorem for the case $C[0,1]$. With the help of the function $t : [0,1] \to [a,b]$, $t(x) = (1-x)a + xb$, the result extends to $C[a,b]$. These operators are, very probably, the most studied linear positive operators and were generalized and modified in a great number of variants. The advantages of the Bernstein operators consist in their simplicity, and on their sharp properties of approximation.

The next result provides one of the most simple and at the same time one of the most powerful criterion for establishing the convergence of a linear positive operator towards the identity operator: *For any* $n \in \mathbb{N}$, *let* $L_n : C[a,b] \to C[a,b]$ *be a sequence of linear positive operators. If the following three conditions hold*

i) $L_n(e_0; x) = 1 + u_n(x)$,

ii) $L_n(e_1; x) = x + v_n(x)$,

iii) $L_n(e_2; x) = x^2 + w_n(x)$,

such that $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} w_n(x) = 0$ *uniformly on* $[a,b]$, *then for any* $f \in C[a,b]$ *and* $x \in [a,b]$, $\lim_{n\to\infty} L_n(f; x) = f(x)$ *uniformly on* $[a,b]$. It was discovered independently and proved by three mathematicians in three consecutive years: T. Popoviciu [**123**] in 1951, H. Bohman [**8**] in 1952 and P.P. Korovkin [**75**] in 1953. This classical result of uniform approximation by linear positive operators is known mostly under the name of Bohman-Korovkin theorem, because T. Popoviciu's contribution remained unknown for a long time. The power of this qualitative result impressed many mathematicians and hence, during the last sixty years a considerable amount of research extended this theorem in different directions.

Having a sequence of operators which approximate a given function arises the question of evaluation of the committed error. This is given by the approximation order, which depends on the smoothness properties of functions. In estimates of the approximation degree a convenient tool for measuring the smoothness of functions is represented by the modulus of continuity

$$\omega(f,\delta) := \sup\{|f(x)-f(y)| \,:\, x,y\in I,\ |x-y|\le\delta\}, \quad \delta\ge 0.$$

We mention that estimates of the approximation order involving modulus of continuity or some modifications of it, are possible in a larger context when the domain of functions is a metric space, for instance, we refer to the papers of H.H. Gonska [**47**], [**50**]. However, in the classical context of functions defined on an interval, the results are stronger. The simplest method of estimating the approximation order, with the aid of the modulus of continuity was established first in 1968 by O. Shisha and B. Mond [**136**]. This quantitative result is given by the following

$$|L_n(f;x)-f(x)| \le |f(x)||L_n(1;x)-1|+(L_n(1;x)+1)\cdot\omega\left(f, \sqrt{L_n((t-x)^2;x)}\right).$$

Now, let the function f be given. In order to approximate the function f by linear positive operators we should take into account the following two problems. One of them refers to compliance of the conditions for which the function approximation process by linear positive operators is realizable. This is a qualitative problem. The another one refers to getting the best evaluation of the committed error in the function approximation process, which means that we have to do with a quantitative problem.
In order to understand as well as possible the meaning of this doctoral thesis we divided it into five chapters.

In the **first chapter** we present preliminaries and auxiliary results that will be used throughout this thesis. Concretely, we give some basic definitions and certain elementary properties concerning linear positive operators, which play an important role in approximation theory and this fact is proved by the vast literature on this topic. Many great mathematicians investigated such of operators, in order to use in the theory of uniform approximation of functions. In the first section, we recall the most studied linear positive operators and taking into account the idea used by C. Mortici [**103**] we introduced in [**26**], [**117**] two classes of linear positive operators depending on a certain analytic function φ. The monomials e_k, $k = 0, 1, 2$ play an important role in uniform approximation by linear positive operators and they are often called Korovkin test-functions. For a linear positive operator L, the following quantities represent indispensable tools. The calculation of the test functions, namely $L(e_k; x)$, where $e_k(x) = x^k$, for $k \in \mathbb{N}_0$, the moments of order k, namely $L\left((e_1 - x)^k; x\right)$, for $k \in \mathbb{N}_0$, $(e_1 - x)^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^i e_1^{k-i}$ and also the absolute moments of odd order k, namely $L\left(|e_1 - x|^k; x\right)$, for $k \in \mathbb{N}$. A further way to derive information on all moments of linear operators is given in [**55**], by: *For a linear operator L and $k \in \mathbb{N}_0$, it follows*

$$L\left((e_1 - x)^k; x\right) = L(e_k; x) - \sum_{i=0}^{k-1} \binom{k}{i} x^{k-i} L\left((e_1 - x)^i; x\right).$$

Knowing the computation of the first three test functions by a linear positive operator the convergence towards the identity operator can be proved using the well-known criterion established by T. Popoviciu [**123**], H. Bohman [**8**] and P.P. Korovkin [**75**]. The next question is: how can we evaluate the committed error in the function approximation process? A convenient tool is the modulus of continuity, which was introduced by H. Lebesgue in 1910 and it appeared also in 1911 in the Ph.D. thesis of D. Jackson [**69**], although in essence the concept was known earlier. Another important tool for evaluating of the committed error is the modulus of smoothness of second order. The first estimates involving the second order modulus of smoothness were established by H. Esser [**42**] in 1976, and improved later in 1984 by H.H. Gonska [**49**]. Estimates using combinations of first and second order modulus of smoothness are more refined then estimates using only the modulus of continuity. Such combinations decompose the error of approximation in three components, corresponding to three specific features of the functions that affect the error: amplitude, deviation from the linear functions and deviation from the polynomials of second degree. Roughly speaking, these moduli measure the deviation from the test functions of the algebraic Chebyshev system. We recall a general quantitative result involving moduli of smoothness of first and second order, such estimates were first established by H.H. Gonska [**49**], then later refined by R. Păltănea as far as the constants are concerned. Păltănea's result (see [**127**]) reads as follows: *Let $I=[a,b]$, $I'\subset I$ and $L: C(I)\to C(I')$ be a linear positive operator, then for any $x\in I'$ and each $0<\delta\le\frac{1}{2}\cdot length(I)$, the following holds*

$$\begin{aligned}|L(f;x)-f(x)|\le&|L(e_0;x)-1|\cdot|f(x)|+\tfrac{1}{\delta}|L(e_1-x;x)|\cdot\omega_1(f,\delta)\\&+\left(L(e_0;x)+\tfrac{1}{2\delta^2}L\left((e_1-x)^2;x\right)\right)\cdot\omega_2(f,\delta).\end{aligned}$$

The condition $\delta\le\frac{1}{2}\cdot\text{length}(I)$ in the above relation can be eliminated for operators which preserve linear functions. In the last paragraph of this chapter we can find some elements concerning univariate and bivariate divided differences, respectively convex functions of higher order. The convex functions of higher order represented a theme of study thoroughly for T. Popoviciu. As witness stand remarkable works *Notes sur les functions convexes d'order supérieur* published since 1936 in various prestigious journals. In 1944 his contributions in this area are collected in monograph [**122**]. The first person who dealt with the topic in question was E. Hopf. He considered in his Ph.D. thesis [**66**] in 1926 the functions with non-negative divided differences without naming them at all. The notion of higher order convexity was introduced by T. Popoviciu in his famous Ph.D. thesis [**119**] in 1934.

In the **second chapter** the main purpose is to do a detailed exposure on the linear positive operators presented in the first chapter, involving qualitative and quantitative results. In order to reach this aim, we proved in each case important results concerning uniform convergence and estimates with first and second order modulus of smoothness, which are direct applications of the properties and formulas recalled in the first chapter. Then, we established general formulas for computation of the test functions for each operator, similarly with that proved first time by S. Karlin and Z. Ziegler [**73**], for Bernstein operators, given by: *If* $j, n \in \mathbb{N}$ *and* $x \in [0,1]$ *then*

$$B_n(e_j; x) = \tfrac{1}{n^j} \sum_{i=1}^{j} S(j,i)(n)_i x^i,$$

where e_j *are test functions,* $S(j,i)$ *Stirling numbers of second kind and* $(n)_i$ *falling factorial denoted by Pochhammer symbol.*

The computation of the moments up to the fourth order for the presented operators represents another idea, which can be followed in whole chapter. These results can be found in various papers written by author, for instance [**90**], [**92**], [**89**], [**94**], [**88**], [**86**] or written jointly with D. Bărbosu, O.T. Pop and P.I. Braica [**26**], [**99**], [**27**], [**96**], [**95**], [**98**], [**97**], [**115**], [**117**]. As you can remark in this chapter, the author's results are many, therefore I wish to stop to the following two important papers [**94**], [**86**], where we got some interesting and nice properties concerning Stancu operators. The computation of the test functions by Stancu operators was done long time ago and can be found in [**144**]. Based on the fact that many properties of Bernstein operators can be transfered to the Stancu operators, different mathematicians studied fast all problems by this standpoint. In [**94**], we revisited the computation of the test functions using another technique, i.e., taking into account only the Vandermonde convolution formula and the following relation

$$t^{[i+j,h]} = t^{[i,h]}(t-ih)^{[j,h]}, \text{ for any } i,j \in \mathbb{N} \text{ and } h \neq 0.$$

Assuming that the parameter α has a fixed non-negative value in each term of the sequence $\left(P_n^{\langle\alpha\rangle}\right)_{n\geq 1}$, in [**144**] D.D. Stancu established an important relationship between two consecutive terms of this sequence, which is useful for proving a monotonicity property of it, in the case of convex or concave functions of first order. Also in [**94**], we revisited and established that the monotonicity property is just an intermediate form of the following relation: *For any* $n \in \mathbb{N}$*, the difference between two consecutive terms of Stancu* *operators is given by*

$$P_{n+1}^{\langle\alpha\rangle}(f;x) - P_n^{\langle\alpha\rangle}(f;x) = -\frac{x(1-x)}{n(n+1)(1+\alpha)} \sum_{k=0}^{n-1} p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right],$$

where $p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) = \binom{n-1}{k} \frac{(x+\alpha)^{[k,-\alpha]}(1-x+\alpha)^{[n-1-k,-\alpha]}}{(1+2\alpha)^{[n-1,-\alpha]}}$.

According to our knowledge and some important papers, for instance [**166**], [**5**], [**143**], [**32**], the proof concerning the monotonicity property for the sequence of Stancu polynomials was not established in totality. After our presentation at ICAM 8 - 2011 it was brought to the author's attention by D. Bărbosu that Popoviciu's Theorem can not be applied to the Stancu operators. The explanation is related to the fact that a relationship regarding the difference between Stancu operator and appropriate function had not been proved. Being motivated by this remark we proved in [**86**] the following:

If the function f *is*

i) convex of first order on $[0,1]$*, then for any* $x \in [0,1]$ *the sequence* $\left(P_n^{\langle\alpha\rangle}\right)_{n\in\mathbb{N}}$ *is decreasing and* $P_n^{\langle\alpha\rangle}(f;x) > f(x)$*.*

ii) concave of first order on $[0,1]$*, then for any* $x \in [0,1]$ *the sequence* $\left(P_n^{\langle\alpha\rangle}\right)_{n\in\mathbb{N}}$ *is increasing and* $P_n^{\langle\alpha\rangle}(f;x) < f(x)$*.*

In [**94**], [**86**] the study on the computation of the test functions by a general formula and on the monotonicity property, for the particular case (special case) of Stancu operators is done in an analogous manner as in the case of Stancu operators. In order to understand better the meaning of qualitative property of the operators, we consider the function $f : [0,1] \to \mathbb{R}$, given by $f(x) = x(x-1)\left(x - \frac{1}{2}\right)$, which we shall approximate by some Bernstein type operators.

The **third chapter** is dedicated to presenting various qualitative and quantitative versions of Voronovskaja's type theorem applied for a large class of linear positive operators. The result proved by E.V. Voronovskaja [**169**] for the Bernstein operators is well-known and can be found in monograph of

R.A. DeVore and G.G. Lorentz [**34**]: *If f is bounded on $[0,1]$, differentiable in a neighborhood of x and has second derivative $f^{(2)}(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left(B_n(f;x) - f(x)\right) = \tfrac{x(1-x)}{2} f^{(2)}(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.
This result has attracted the attention of many authors in the last 80 years. Inspired by the result of E.V. Voronovskaja, her scientific advisor S.N. Bernstein generalized the above result, showing in [**7**] the asymptotic behavior of Bernstein operators for $f \in C^q[0,1]$, q even as follows: *If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, then uniformly in $x \in [0,1]$,*

$$n^{\frac{q}{2}}\left(B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n\left((e_1-x)^r\right)\tfrac{f^{(r)}(x)}{r!}\right) \to 0, \text{ when } n \to \infty.$$

In 1985, V.S. Videnskij [**168**] established a quantitative estimate in Voronovskaja type theorem given by the following:

$$\left|n\left(B_n(f;x) - f(x)\right) - \tfrac{x(1-x)}{2} f^{(2)}(x)\right| \le x(1-x)\cdot\omega\left(f^{(2)}, \sqrt{\tfrac{2}{n}}\right).$$

Another recent result concerning quantitative version of Voronovskaja's type theorem, involving the least concave majorant of the modulus of continuity was established by H.H. Gonska, P. Piţul and I. Raşa [**56**]. The same result can be found also in Ph.D. thesis of P. Piţul [**106**], given by the following: *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that $Le_i = e_i$, for $i = 0, 1$. If $f \in C^2[0,1]$ and $x \in [0,1]$, then*

$$\left|L(f;x) - f(x) - L\left((e_1-x)^2;x\right)\tfrac{f^{(2)}(x)}{2}\right|$$
$$\le \tfrac{1}{2} L\left((e_1-x)^2;x\right)\cdot\tilde{\omega}\left(f^{(2)}, \tfrac{1}{3}\sqrt{\tfrac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}\right).$$

In this chapter we used a technique developed in some recent papers by O.T. Pop (see [**107**], [**110**], [**108**], [**111**]), in order to get the asymptotic behavior of the presented operators in first chapter, the uniform convergence and the approximation order of the approximated functions. These results can be found in various papers written by author, for instance [**90**], [**92**], [**94**], [**88**] or written jointly with O.T. Pop, D. Bărbosu and P.I. Braica [**99**], [**118**], [**96**], [**98**], [**97**], [**115**], [**117**]. In order to have a correspondence between the second, respectively third chapter and to present the best results of the author, we refer again to the paper [**94**]. The generalization of the asymptotic behavior of Stancu operators and also some quantitative forms of Voronovskaja's type formula are done in [**94**] and are given by: *Let* $f \in C[0,1]$ *be given. If* $n \in \mathbb{N}$, $x \in [0,1]$ *and* f *is* s *times differentiable in a neighborhood of* x, *then*

$$\lim_{n\to\infty} P_n^{\langle\alpha\rangle}(f;x) = f(x),$$

$$\lim_{n\to\infty} n\left(P_n^{\langle\alpha\rangle}(f;x) - f(x)\right) = \tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)} f^{(2)}(x),$$

$$\lim_{n\to\infty} n^2\left(P_n^{\langle\alpha\rangle}(f;x) - f(x) - \tfrac{x(1-x)(1+\alpha n)}{2n(1+\alpha)} f^{(2)}(x)\right)$$
$$= \tfrac{x(1-x)(1-2x)(1+\alpha n)(1+2\alpha n)}{6(1+\alpha)(1+2\alpha)} f^{(3)}(x) + \tfrac{(x(1-x))^2(1+\alpha n)^2}{8(1+\alpha)(1+2\alpha)(1+3\alpha)} f^{(4)}(x).$$

Assume that f *is* s *times differentiable on* $[0,1]$, *then the convergence from the above relations is uniform on* $[0,1]$. *Moreover, we get*

$$\left|P_n^{\langle\alpha\rangle}(f;x) - f(x)\right| \le \left(1 + \tfrac{1+\alpha n}{4(1+\alpha)}\right)\cdot \omega\left(f, \tfrac{1}{\sqrt{n}}\right) \ \textit{and}$$

$$n\left|P_n^{\langle\alpha\rangle}(f;x) - f(x) - \tfrac{x(1-x)(1+\alpha n)}{2n(1+\alpha)} f^{(2)}(x)\right|$$
$$\le \tfrac{1+\alpha n}{8(1+\alpha)}\left(1 + \tfrac{3(1+\alpha n)}{4(1+2\alpha)(1+3\alpha)}\right)\cdot \omega\left(f^{(2)}, \tfrac{1}{\sqrt{n}}\right).$$

Estimates using the least concave majorant of the modulus of continuity can be given, taking into account the result proved in [**56**] and the computation of the moments up to the fourth order proved in [**94**], by: *For the Stancu* *operators, we get*

$$\left| n\left(P_n^{\langle\alpha\rangle}(f;x) - f(x)\right) - \tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)} f^{(2)}(x) \right| \le \tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)} \cdot \tilde{\omega}\left(f^{(2)}, \tfrac{2}{3}\sqrt{\tfrac{1+\alpha n}{n(1+3\alpha)}}\right).$$

The **fourth chapter** represents a continuation of the study on certain linear positive operators done in the second chapter. The main idea is univariate and bivariate approximation of the functions by some linear positive operators presented, as well as establishment of upper bounds estimations for the appropriate remainder terms.
For any $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, the following

$$f(x) = B_n(f;x) + R_n(f;x)$$

is called the Bernstein approximation formula, where R_n is the remainder operator associated to the Bernstein operators B_n. The first representation for the remainder term R_n was proved by O. Aramă [**5**] and is given by the following:
Let $f : [0,1] \to \mathbb{R}$ be given. For any $n \in \mathbb{N}$, the representation of the remainder term of the Bernstein approximation formula is given by

$$R_n(f;x) = -\tfrac{x(1-x)}{n}[\xi_0, \xi_1, \xi_2; f], \ \textit{where } \xi_0, \xi_1, \xi_2 \in [0,1].$$

Another representation for the remainder term associated to the Bernstein approximation formula was proved by D.D. Stancu [**142**], by

$$R_n(f;x) = -\tfrac{x(1-x)}{n}\sum_{k=0}^{n-1} p_{n-1,k}(x)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right],$$

for $x \in [0,1] \setminus \left\{\frac{k}{n} \mid k = 0, \ldots, n\right\}$.
W.J. Gordon [**62**] has introduced the basic notions of the algebraic theory of multivariate functions approximation, a theory which was studied and developed by F.J. Delvos and W. Schempp [**33**]. The method of parametric

extension is a procedure for constructing linear operators on the space of multivariate functions, starting from linear operators defined on spaces of univariate functions, (see [**33**]). Having in mind the above results, in [**91**], [**93**], [**94**], [**85**], [**86**], [**87**] we proved interesting and important results concerning univariate, respectively bivariate approximation formulas by certain linear positive operators. For illustration, we recall the Stancu approximation formula given by

$$f(x) = P_n^{\langle\alpha\rangle}(f;x) + R_n^{\langle\alpha\rangle}(f;x),\ \text{for any } f \in C[0,1],\ x \in [0,1],\ n \in \mathbb{N}.$$

The study on the remainder term associated to the Stancu operators was done in [**144**], in terms of divided differences of first, respectively second order of the function f. Three years later, in [**149**] D.D. Stancu established an expression of this remainder term by using only divided differences of second order, given by the following

$$R_n^{\langle\alpha\rangle}(f;x) = -\sum_{k=0}^{n-1} \tfrac{(x+k\alpha)(1-x+\overline{n-1-k}\alpha)}{n(1+\overline{n-1}\alpha)} p_{n-1,k}^{\langle\alpha\rangle}(x)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right].$$

For establishing a representation of a certain remainder term, associated to a linear operator, we can use also the well-known criterion proved by T. Popoviciu [**124**]. In order to get an appropriate form of the remainder term, as an application, in [**86**] we applied the Popoviciu's Theorem to the Stancu operators

$$R_n^{\langle\alpha\rangle}(f;x) = -\tfrac{x(1-x)(1+\alpha n)}{n(1+\alpha)}[\xi_0, \xi_1, \xi_2; f],\ \text{where } \xi_0, \xi_1, \xi_2 \in [0,1].$$

The idea of revision appears when we look at result proved with Popoviciu's Theorem and the above relation. The evaluation of the remainder term was revisited in [**94**] by the author. *The representation of the remainder term associated to Stancu operators is given by*

$$R_n^{\langle\alpha\rangle}(f;x) = -\tfrac{x(1-x)(1+\alpha n)}{n(1+\alpha)} \sum_{k=0}^{n-1} p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right],$$

where $p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) = \binom{n-1}{k} \frac{(x+\alpha)^{[k,-\alpha]}(1-x+\alpha)^{[n-1-k,-\alpha]}}{(1+2\alpha)^{[n-1,-\alpha]}}$.
The upper bound estimation for the remainder term is given also in [**94**], by: *Suppose that* $f \in C^2[0,1]$ *and divided differences of the second order of* f *are all bounded on* $[0,1]$, *then we get*

$$\left| R_n^{\langle\alpha\rangle}(f;x) \right| \leq \tfrac{x(1-x)(1+\alpha n)}{2n(1+\alpha)} M_2[f] \leq \tfrac{1+\alpha n}{8n(1+\alpha)} M_2[f],$$

where $M_2[f] := \max\limits_{0 \leq x \leq 1} |f^{(2)}(x)|$. In [**87**] we revisited also the form of remainder term associated to the bivariate approximation formula of Stancu operators, using bivariate divided differences. The revision is motivated by two ideas. One of them is contained in [**94**], where the revisited form of the reminder term associated to the univariate approximation formula of Stancu operators is established. The another one, is based on the fact that Stancu operators are not projectors and the decomposition formula of the identity operator for determining the form of the bivariate remainder term can not be applied. Concerning the second idea, the reader is invited to see the paper [**23**], where a detailed and complete exposure for the case of Bernstein operators was given.

In the **last chapter** we focus our attention on the approximation of linear functionals. The equality $I(f) = Q(f)+R(f)$, where $Q(f) = \sum_{k=0}^{n} A_k \lambda_k(f)$ is called numerical integration formula for the function f or quadrature formula. Parameters A_k, $k = \overline{0,n}$, $n \in \mathbb{N}$ are called weights or coefficients of the formula, and $R(f)$ is its remainder term.
Practical necessity of the quadrature formulas is owed especially the manner of definition of integral by process of crossing to the limit. There exist situations in which the integral function admits no primitive, therefore the well-known Leibniz-Newton formula can not be applied. There exist situations in which the integrated functions admit primitives, but *the cost* for getting primitives is too large to be calculated. The study on quadrature formulas can be done only for univariate functions and the extension to the

bivariate functions implies another formulas, which are called cubature formulas. We refer to the study on quadrature, respectively cubature formulas applied to the well-known linear positive Bernstein operator.

Considering and integrating on $[0,1]$ the Bernstein approximation formula, from the above chapter, we get the Bernstein quadrature formula [**158**], [**15**], [**25**], given by

$$\int_0^1 f(x)dx = \tfrac{1}{n+1}\sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) + R_n[f],\ \text{for } n\in\mathbb{N},$$

where $|R_n[f]| \le \frac{1}{12n}M_2[f]$.

The next aim is to construct the composite Bernstein quadrature formula [**20**]. In order to get this, the interval $[0,1]$ will be divided in n equally spaced subintervals $\left[\frac{j-1}{n}, \frac{j}{n}\right]$, for any $j=\overline{1,n}$. On each such type of interval, the Bernstein quadrature formula will be applied. Next, adding the mentioned quadrature formulas, the desired composite Bernstein quadrature formula on $[0,1]$ will be got.

For any $f\in C^2[0,1]$, the following composite Bernstein quadrature formula holds

$$\int_0^1 f(x)dx = \tfrac{1}{n(n+1)}\sum_{j=1}^{n}\sum_{k=0}^{n} f\left(\tfrac{jn-n+k}{n^2}\right) + R_n[f],\ \textit{for } n\in\mathbb{N},$$

where $|R_n[f]| \le \frac{1}{12n}M_2[f]$. Considering and integrating on $[0,1]\times[0,1]$ the bivariate Bernstein approximation formula, we get the Bernstein cubature formula [**158**], which was revisited by D. Bărbosu and O.T. Pop [**23**]. In [**19**], we constructed the composite Bernstein cubature formula. To reach this, the bidimensional interval $[0,1]\times[0,1]$ will be divided in mn equally spaced subintervals $\left[\frac{i-1}{m}, \frac{i}{m}\right]\times\left[\frac{j-1}{n}, \frac{j}{n}\right]$, for any $i=\overline{1,m}$, $j=\overline{1,n}$. On each such type of interval, the Bernstein cubature formula will be applied. Next, adding the mentioned cubature formulas, the desired composite Benstein cubature formula on $[0,1]\times[0,1]$ will be got. The above mentioned results can be found in [**25**], [**20**], [**19**], papers written jointly with D. Bărbosu and O.T. Pop.

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CHAPTER 1

Preliminaries and auxiliary results

In this chapter we present preliminaries and auxiliary results that will be used throughout this thesis. Concretely, we give some basic definitions and certain elementary properties concerning linear positive operators, which play an important role in approximation theory and this fact is proved by the vast literature on this topic. For more information on this topic the reader is invited to see [**4**], [**30**], [**3**], [**157**], [**14**] and [**17**].

1. Linear positive operators

1.1. Definitions and properties

1.2. Examples

2. Uniform approximation by linear positive operators

3. Modulus of continuity

4. Different types of moduli of smoothness

5. Evaluation of the approximation order

6. Divided differences and convex functions of higher order

CHAPTER 2

Selected results for certain linear positive operators

The main purpose of this chapter is to do a detailed exposure on the linear positive operators presented in the first chapter, involving qualitative and quantitative results. In order to reach this aim, we proved in each case important results concerning uniform convergence and estimates with first and second order modulus of smoothness, which are direct applications of the properties and formulas recalled in the first chapter. Then, we established general formulas for computation of the test functions for each operator, similarly with that proved first time by S. Karlin and Z. Ziegler [**73**], for Bernstein operators. The computation of the moments up to the fourth order for the presented operators represents another idea, which can be followed in whole chapter. These results can be found in various papers written by author, for instance [**90**], [**92**], [**89**], [**94**], [**88**], [**86**] or written jointly with D. Bărbosu, O.T. Pop and P.I. Braica [**26**], [**99**], [**27**], [**96**], [**95**], [**98**], [**97**], [**115**], [**117**].

1. Bernstein type operators

1.1. Bernstein operators

These operators are, very probably, the most studied linear positive operators and were generalized and modified in a great number of variants. The advantages of the Bernstein operators consist in their simplicity, and on their sharp properties of approximation. From certain points of view the

Bernstein operators play an extremal position in some classes of operators. In order to get some results concerning computation of the test functions by Bernstein operators, we recall first the main result established in [**116**], by O.T. Pop and M. Farcaş.

Proposition 2.1.1. *If $j, n \in \mathbb{N}$ and $x \in [0,1]$ then*

$$B_n(e_j; x) = \frac{1}{n^j} \sum_{i=1}^{j} S(j,i)(n)_i x^i, \tag{2.1}$$

where e_j are test functions, $S(j,i)$ Stirling numbers of second kind and $(n)_i$ falling factorial denoted by Pochhammer symbol.

Remark 2.1.2. During the preparation of our paper [**96**], making detailed researches we discovered that the relation (2.1) had been proved earlier by S. Karlin and Z. Ziegler [**73**]. As a special case, we can find the same relation in [**1**], where the asymptotic expansion of multivariate Bernstein polynomials on a simplex is considered.

The computation of higher order moments is tedious, but rather mechanical work. Taking into account some known results, once we find the moments of low orders, we get the ones of higher order. So, it follows:

Lemma 2.1.3. (**D. Miclăuş** and P.I. Braica, [**96**]) *The Bernstein operators satisfy*

$$B_n\left((e_1 - x)^2; x\right) = \frac{x(1-x)}{n},$$
$$B_n\left((e_1 - x)^3; x\right) = \frac{x(1-x)(1-2x)}{n^2},$$
$$B_n\left((e_1 - x)^4; x\right) = \frac{3(x(1-x))^2}{n^2} + \frac{x(1-x) - 6(x(1-x))^2}{n^3}.$$

Application 2.1.4. In the following we consider the function $f : [0,1] \to \mathbb{R}$, given by $f(x) = x(x-1)\left(x - \frac{1}{2}\right)$, which we shall approximate by Bernstein operator for different choices of the knots number.

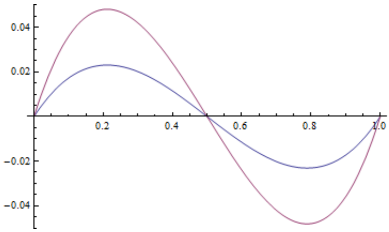


Figure 1. Approximation of f by Bernstein operator, for $m = 5$

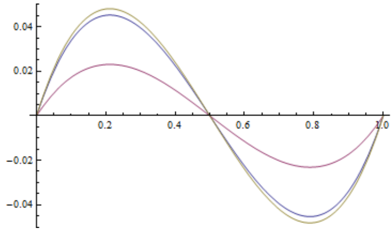


Figure 2. Approximation of f by Bernstein operator, for $m = 5$ and $m = 50$

1.2. Kantorovich operators

1.3. Bernstein-Schurer operators

1.4. Stancu operators

These operators have been introduced and investigated by D.D. Stancu in the memoir [**144**], for any non-negative parameter α, which may depend only on the natural number n. It was studied further in his subsequent papers [**147**], [**148**], [**159**] as well as in several papers published by other authors [**104**], [**40**], [**83**], [**84**], [**32**] and [**163**]. The original operator of Bernstein type is connected with the probability distribution of Markov-Polya and with the Vandermonde convolution formula. In terms of probabilistic interpretation, the Stancu operators can be generated by starting from the Bernoulli probability distribution, so it is possible to arrive to these operators, as D.D. Stancu has shown in [**147**], by using a more general probability distribution, which is connected with the Markov-Polya urn scheme. Another details concerning probabilistic interpretation can be found in [**32**]. In [**94**] we proved a general formula concerning computation of the test functions by Stancu operators, given by:

Theorem 2.1.5. (**D. Miclăuş**, [**94**]) *For any $j, n \in \mathbb{N}$ and $x \in [0,1]$, the following holds*

$$P_n^{\langle\alpha\rangle}(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n)_{j-i}\frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}}. \tag{2.2}$$

Remark 2.1.6. The relation (2.2) was proved also in [**89**], taking into account another technique, i.e., using the properties of Bernstein operators.

The computation of higher order moments is given by the following:

Lemma 2.1.7. (**D. Miclăuş**, [**94**]) *The Stancu operators satisfy*

$$P_n^{\langle\alpha\rangle}\left((e_1-x)^2;x\right) = \frac{x(1-x)(1+\alpha n)}{n(1+\alpha)},$$

$$P_n^{\langle\alpha\rangle}\left((e_1-x)^3;x\right) = \frac{x(1-x)(1-2x)(1+\alpha n)(1+2\alpha n)}{n^2(1+\alpha)(1+2\alpha)},$$

$$P_n^{\langle\alpha\rangle}\left((e_1-x)^4;x\right) = \frac{(x(1-x))^2\big((3n-18\alpha n)(1+\alpha n)^2-6(1+\alpha n)\big)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}$$

$$+\frac{x(1-x)\big(6\alpha n(1+\alpha n)^2+(1-\alpha)(1+\alpha n)\big)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

Assuming that the parameter α has a fixed non-negative value in each term of the sequence $\left(P_n^{\langle\alpha\rangle}\right)_{n\geq 1}$, in [**144**] D.D. Stancu established an important relationship between two consecutive terms of this sequence, which is useful for proving a monotonicity property of it, in the case of convex or concave functions of first order.

Theorem 2.1.8. [**144**] *For any $n \in \mathbb{N}$, between the polynomials $P_{n+1}^{\langle\alpha\rangle}(f;x)$ and $P_n^{\langle\alpha\rangle}(f;x)$ there exists the following relationship*

$$\begin{aligned} &P_{n+1}^{\langle\alpha\rangle}(f;x) - P_n^{\langle\alpha\rangle}(f;x) \\ &= -\frac{1}{n(n-1)}\sum_{k=0}^{n-1}\frac{(x+k\alpha)(1-x+\overline{n-1-k}\alpha)}{(1+n\alpha)(1+\overline{n-1}\alpha)}p_{n-1,k}^{\langle\alpha\rangle}(x)\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]. \end{aligned} \tag{2.3}$$

The monotonicity property was revisited in [**94**]. We proved that the relation (2.3) is just an intermediate form of the monotonicity property and we established the final form, given by:

Theorem 2.1.9. (**D. Miclăuş**, [**94**]) *For any $n \in \mathbb{N}$, the difference between two consecutive terms of Stancu operators is given by*

(2.4)

$$P_{n+1}^{\langle\alpha\rangle}(f;x) - P_n^{\langle\alpha\rangle}(f;x) = -\frac{x(1-x)}{n(n+1)(1+\alpha)} \sum_{k=0}^{n-1} p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right],$$

where $p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) = \binom{n-1}{k} \frac{(x+\alpha)^{[k,-\alpha]}(1-x+\alpha)^{[n-1-k,-\alpha]}}{(1+2\alpha)^{[n-1,-\alpha]}}$.

In the particular case, when $\alpha = 0$, the relation (2.4) reduces to the appropriate Bernstein formula (see [**5**], [**143**]) given by

$$B_{n+1}(f;x) - B_n(f;x) = -\frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f\right].$$

According to our knowledge and some important papers, for instance [**166**], [**5**], [**143**], [**32**], the proof concerning the monotonicity property for the sequence of Stancu polynomials was not established in totality. After our presentation at ICAM 8 - 2011 it was brought to the author's attention by D. Bărbosu that Popoviciu's Theorem can not be applied to the Stancu operators. The explanation is related to the fact that a relationship regarding the difference between Stancu operators and appropriate function had not been proved. Being motivated by this remark we proved in [**86**] the following:

Theorem 2.1.10. (**D. Miclăuş**, [**86**]) *If the function f is*

i) convex of first order on $[0,1]$, then for any $x \in [0,1]$ the sequence $\left(P_n^{\langle\alpha\rangle}\right)_{n\in\mathbb{N}}$ is decreasing and $P_n^{\langle\alpha\rangle}(f;x) > f(x)$.

ii) concave of first order on $[0,1]$, then for any $x \in [0,1]$ the sequence $\left(P_n^{\langle\alpha\rangle}\right)_{n\in\mathbb{N}}$ is increasing and $P_n^{\langle\alpha\rangle}(f;x) < f(x)$.

Application 2.1.11. In the following we consider the function $f : [0,1] \to \mathbb{R}$, given by $f(x) = x(x-1)\left(x - \frac{1}{2}\right)$, which we shall approximate by Stancu operator for different choices of the knots number and different values of the parameter α.

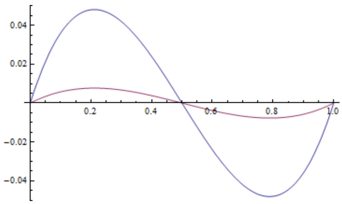


Figure 3. Approximation of f by Stancu operator, for $m = 5$ and $\alpha = 0.5$

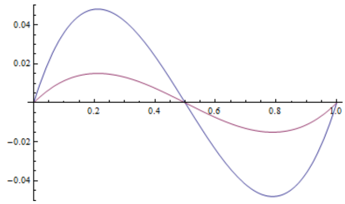


Figure 4. Approximation of f by Stancu operator, for $m = 50$ and $\alpha = 0.5$

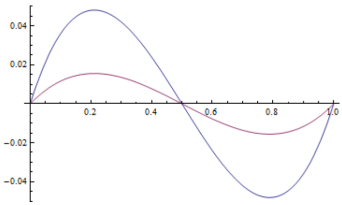


Figure 5. Approximation of f by Stancu operator, for $m = 100$ and $\alpha = 0.5$

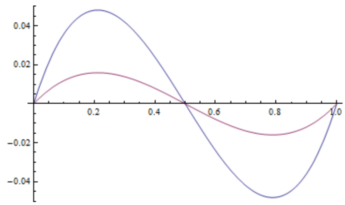


Figure 6. Approximation of f by Stancu operator, for $m = 500$ and $\alpha = 0.5$

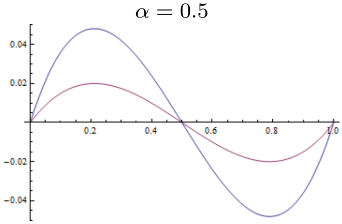


Figure 7. Approximation of f by Stancu operator, for $m = 5$ and $\alpha = 0.05$

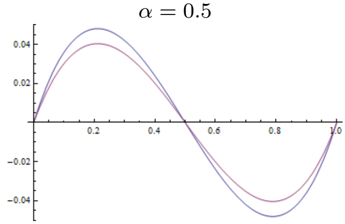


Figure 8. Approximation of f by Stancu operator, for $m = 100$ and $\alpha = 0.05$

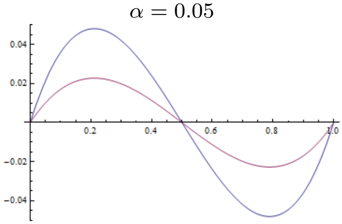


Figure 9. Approximation of f by Stancu operator, for $m = 5$ and $\alpha = 0.005$

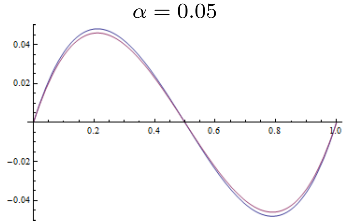


Figure 10. Approximation of f by Stancu operator, for $m = 100$ and $\alpha = 0.005$

1.5. Bernstein-Stancu operators

1.6. Schurer-Stancu operators

1.7. A general class of Bernstein type operators

In 1972 D.D. Stancu [**150**] introduced a general class of Bernstein type operators, where n is a natural number, p is a non-negative integer, while β and γ are non-negative real parameters satisfying the relation $0 \le \beta \le \gamma$. Concerning the parameter α, it may depend only on the natural number n, i.e., $\alpha = \alpha(n) \to 0$ as $n \to \infty$. These operators had been investigated before by quite a number of authors and the most detailed paper was made by H.H. Gonska and J. Meier [**54**]. In what follows, we want to present only some particular cases of this general class, which are contained also in [**54**].

Case 1.

Choice of parameters: $p = 0$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

Explicit representation of $L_{n,p}^{\langle \alpha,\beta,\gamma \rangle}(f;x)$:

$$L_{n,0}^{\langle 0,0,0 \rangle}(f;x) := B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right).$$

Case 2.

Choice of parameters: $p \neq 0$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

Explicit representation of $L_{n,p}^{\langle \alpha,\beta,\gamma \rangle}(f;x)$:

$$L_{n,p}^{\langle 0,0,0 \rangle}(f;x) := \tilde{B}_{n,p}(f;x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\tfrac{k}{n}\right).$$

Case 3.

Choice of parameters: $p = 0$, $\alpha \neq 0$, $\beta = 0$, $\gamma = 0$.

Explicit representation of $L_{n,p}^{\langle \alpha,\beta,\gamma \rangle}(f;x)$:

$$L_{n,0}^{\langle \alpha,0,0 \rangle}(f;x) := P_n^{\langle \alpha \rangle}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} f\left(\tfrac{k}{n}\right).$$

Case 4.

Choice of parameters: $p = 0$, $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$.

Explicit representation of $L_{n,p}^{\langle\alpha,\beta,\gamma\rangle}(f;x)$:

$$L_{n,0}^{\langle 0,\beta,\gamma\rangle}(f;x) := P_n^{(\beta,\gamma)}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\beta}{n+\gamma}\right).$$

Case 5.

Choice of parameters: $p \neq 0$, $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$.

Explicit representation of $L_{n,p}^{\langle\alpha,\beta,\gamma\rangle}(f;x)$:

$$L_{n,p}^{\langle 0,\beta,\gamma\rangle}(f;x) := \tilde{S}_{n,p}^{(\beta,\gamma)}(f;x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\frac{k+\beta}{n+\gamma}\right).$$

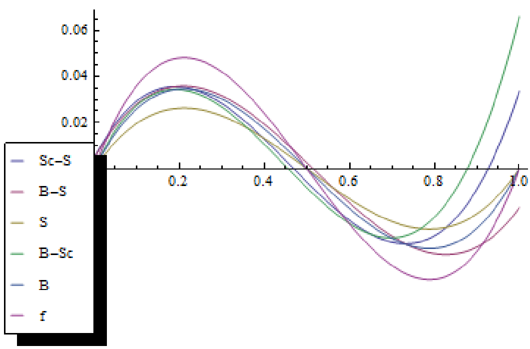
Case 6. (special case)

Choice of parameters: $p = 0$, $\alpha = \frac{1}{n}$, $\beta = 0$, $\gamma = 0$.

Explicit representation of $L_{n,p}^{\langle\alpha,\beta,\gamma\rangle}(f;x)$:

$$L_{n,0}^{\left\langle \frac{1}{n},0,0\right\rangle}(f;x) := P_n^{\left\langle \frac{1}{n}\right\rangle}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{\left[k,-\frac{1}{n}\right]}(1-x)^{\left[n-k,-\frac{1}{n}\right]}}{1^{\left[n,-\frac{1}{n}\right]}} f\left(\frac{k}{n}\right).$$

Application 2.1.12. In the following we consider the function $f : [0,1] \to \mathbb{R}$, given by $f(x) = x(x-1)\left(x - \frac{1}{2}\right)$, which we shall approximate by Bernstein, Bernstein-Schurer, Stancu, Bernstein-Stancu, respectively Schurer-Stancu operator for different choices of the knots number and the same values of the parameters p, α, β.



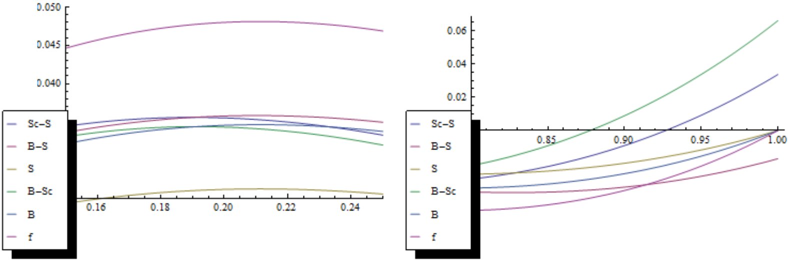


Figure 11. Approximation of f by Bernstein type operators, for $m = 10$, $p = 1$, $\alpha = 0.1$, $\beta = 0.5$

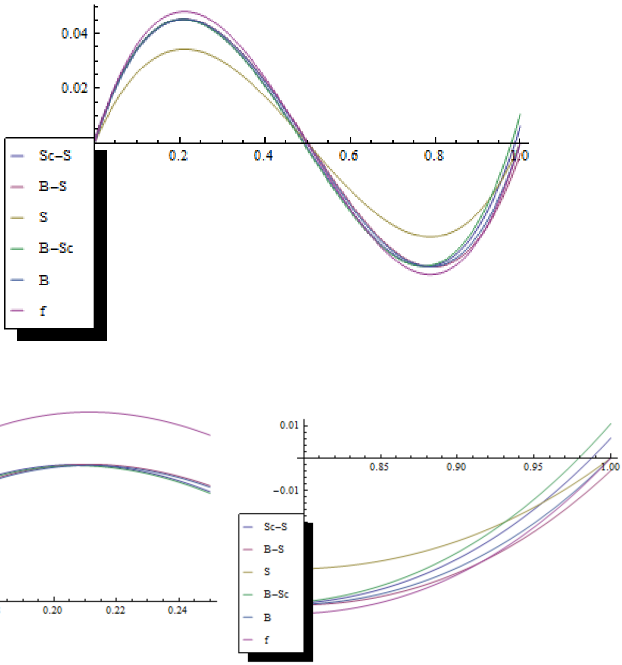


Figure 12. Approximation of f by Bernstein type operators, for $m = 50$, $p = 1$, $\alpha = 0.1$, $\beta = 0.5$

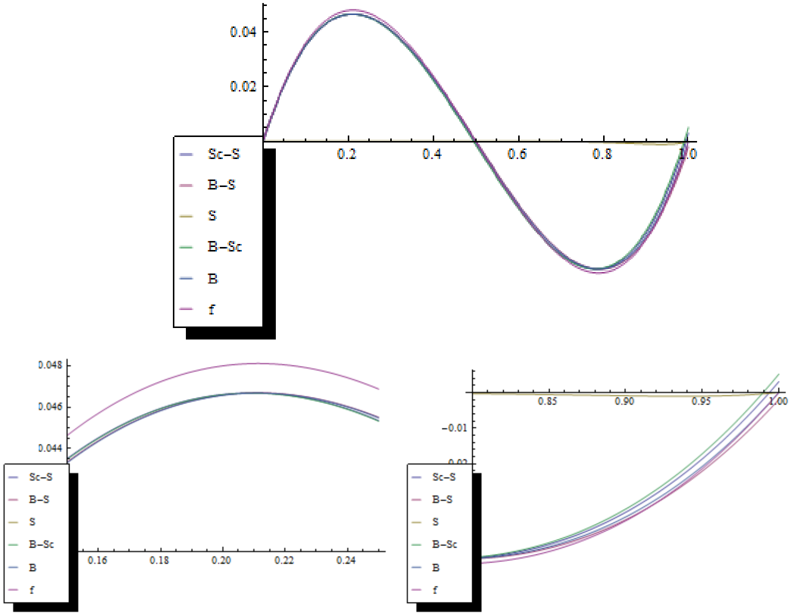


Figure 13. Approximation of f by Bernstein type operators, for $m = 100$, $p = 1$, $\alpha = 0.1$, $\beta = 0.5$

2. Szász-Mirakjan type operators

2.1. Mirakjan-Favard-Szász operators

In order to get certain qualitative and quantitative results for the functions defined on unbounded intervals, in the following we present some linear positive operators defined on the space

$$C_2[0,+\infty) = \left\{ f \in C[0,+\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} \text{ exists and is finite} \right\},$$

which is endowed with the norm $\|f\|_2 := \sup\limits_{x\geq 0} \frac{|f(x)|}{1+x^2}$.

Mirakjan-Favard-Szász operators represent generalization of Bernstein operators to the infinite interval. In this class of the operators defined on unbounded intervals, Mirakjan-Favard-Szász operators play a central role and were intensively studied, for instance some recent papers [**36**], [**130**], [**44**], [**103**], [**80**], [**27**], [**97**].

2.2. Szász-Mirakjan-Kantorovich operators

2.3. Szász-Mirakjan-Schurer operators

3. φ-Szász-Mirakjan type operators

3.1. φ-Szász-Mirakjan operators

Recently, C. Mortici [**103**] defined a new class of linear positive operators depending on a certain function $\varphi : \mathbb{R} \to (0, +\infty)$. The operators contained in this new class generalize the well-known Mirakjan-Favard-Szász operators. In order to understand better the meaning of these operators we give some examples:

Example 2.3.13. (**D. Miclăuş** et al., [**27**]) Let $\varphi(x) = (x+1)^2 e^x$ be given, then for any $k \in \mathbb{N}$ and $x \in [0, +\infty)$, it follows

$$\varphi^{(k)}(x) = \left(x^2 + 2x(k+1) + k^2 + k + 1\right) e^x.$$

In this case, for any $f \in C_2[0, +\infty)$ and $n \in \mathbb{N}$, we get

$$\varphi S_n(f; x) = \frac{e^{-nx}}{(nx+1)^2} \sum_{k=0}^{\infty} \frac{k^2+k+1}{k!} (nx)^k f\left(\frac{k}{n}\right).$$

Example 2.3.14. (**D. Miclăuş** et al., [**27**]) Let $\varphi(x) = \frac{2(x+1)e^x}{x+2}$ be given, then for any $k \in \mathbb{N}$ and $x \in [0, +\infty)$, it follows

$$\varphi^{(k)}(x) = 2e^x \left(\frac{x+1}{x+2} + \sum_{i=1}^{k} \binom{k}{i} \left(\frac{x+1}{x+2}\right)^{(i)} \right)$$

$$= 2e^x \left(\frac{x+1}{x+2} + \sum_{i=1}^{k} \binom{k}{i} \left(1 - \frac{1}{x+2}\right)^{(i)} \right) = 2e^x \left(\frac{x+1}{x+2} + \sum_{i=1}^{k} \binom{k}{i} \frac{(-1)^{i+1} \cdot i!}{(x+2)^{i+1}} \right).$$

In this case, for any $f \in C_2[0,+\infty)$ and $n \in \mathbb{N}$, we get

$$\varphi S_n(f;x) = \frac{(nx+2)e^{-nx}}{2(nx+1)} \sum_{k=0}^{\infty} \frac{1+\sum\limits_{i=1}^{\infty} \binom{k}{i} \frac{(-1)^{i+1} \cdot i!}{2^i}}{k!} (nx)^k f\left(\frac{k}{n}\right).$$

3.2. φ-Szász-Mirakjan-Kantorovich operators

Taking into account the idea used by C. Mortici [**103**] to defining the φ-Szász-Mirakjan operators, we constucted in [**26**] a new class of linear positive operators depending on a certain function $\varphi : \mathbb{R} \to (0,+\infty)$. These operators are called φ-Szász-Mirakjan-Kantorovich operators, because in the case when $\varphi(x) = e^x$, they reduce to the classical Szász-Mirakjan-Kantorovich operators. In order to understand better the meaning of these operators we give the following:

Example 2.3.15. (**D. Miclăuş** et al., [**26**]) Let $\varphi(x) = (x+1)e^x$ be given, then for any $k \in \mathbb{N}$ and $x \in [0,+\infty)$, it follows

$$\varphi^{(k)}(x) = (x+k+1)e^x.$$

In this case, for any $f \in C_2[0,+\infty)$ and $n \in \mathbb{N}$, we get

$$\varphi \tilde{K}_n(f;x) = \frac{ne^{-nx}}{nx+1} \sum_{k=0}^{\infty} \frac{k+1}{k!} (nx)^k \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt.$$

3.3. A general class of φ-Szász-Mirakjan type operators

This section is devoted to defining a new general class of linear positive operators depending on a certain function $\varphi : \mathbb{R} \to (0,+\infty)$. The operators contained in this new general class generalize the well-known Mirakjan-Favard-Szász, respectively φ-Szász-Mirakjan operators.

CHAPTER 3

Voronovskaja's type theorems applied to certain linear positive operators

This chapter is dedicated to presenting various qualitative and quantitative versions of Voronovskaja's type theorem applied for a large class of linear positive operators. We used a technique developed in some recent papers by O.T. Pop (see [**107**], [**110**], [**108**], [**111**]), in order to get the asymptotic behavior of the presented operators in first chapter, the uniform convergence and the order of approximation for the approximated functions. These results can be found in various papers written by author, for instance [**90**], [**92**], [**94**], [**88**] or written jointly with O.T. Pop, D. Bărbosu and P.I. Braica [**99**], [**118**], [**96**], [**98**], [**97**], [**115**], [**117**].

1. Qualitative and quantitative Voronovskaja's type theorems

2. Application to Bernstein type operators

The result proved by E.V. Voronovskaja [**169**] for the Bernstein operators is well-known and can be found in monograph of R.A. DeVore and G.G. Lorentz [**34**].

Theorem 3.2.16. *If the function f is bounded on $[0,1]$, differentiable in a neighborhood of x and has second derivative $f^{(2)}(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left(B_n(f;x) - f(x)\right) = \tfrac{x(1-x)}{2} f^{(2)}(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.

This result has attracted the attention of many authors in the last 80 years.

Bernstein operators

Theorem 3.2.17. (**D. Miclăuş** and P.I. Braica, [**96**]) *Let $f \in C[0,1]$ be given. If $x \in [0,1]$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} B_n(f;x) = f(x), \text{ for } s=0; \tag{3.5}$$

$$\lim_{n\to\infty} n(B_n(f;x) - f(x)) = \tfrac{x(1-x)}{2} f^{(2)}(x), \text{ for } s=0; \tag{3.6}$$

$$\lim_{n\to\infty} n^2 \left(B_n(f;x) - f(x) - \tfrac{x(1-x)}{2n} f^{(2)}(x)\right) \tag{3.7}$$
$$= \tfrac{x(1-x)(1-2x)}{6} f^{(3)}(x) + \tfrac{(x(1-x))^2}{8} f^{(4)}(x), \text{ for } s=4.$$

Assume that f is s times differentiable on $[0,1]$. Then the convergence from (3.5)–(3.7) is uniform on $[0,1]$. Moreover, we get

$$|B_n(f;x) - f(x)| \le \tfrac{5}{4} \cdot \omega\left(f, \tfrac{1}{\sqrt{n}}\right), \text{ for } s=0 \text{ and} \tag{3.8}$$

$$n\left|B_n(f;x) - f(x) - \tfrac{x(1-x)}{2n} f^{(2)}(x)\right| \le \tfrac{7}{32} \cdot \omega\left(f^{(2)}, \tfrac{1}{\sqrt{n}}\right), \text{ for } s=2. \tag{3.9}$$

Stancu operators

Theorem 3.2.18. (**D. Miclăuş**, [**94**]) *Let $f \in C[0,1]$ be given. If $x \in [0,1]$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} P_n^{\langle\alpha\rangle}(f;x) = f(x), \text{ for } s=0; \tag{3.10}$$

$$\lim_{n\to\infty} n\left(P_n^{\langle\alpha\rangle}(f;x) - f(x)\right) = \tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)} f^{(2)}(x), \text{ for } s=2; \tag{3.11}$$

$$\lim_{n\to\infty} n^2\left(P_n^{\langle\alpha\rangle}(f;x) - f(x) - \tfrac{x(1-x)(1+\alpha n)}{2n(1+\alpha)} f^{(2)}(x)\right) \tag{3.12}$$
$$= \tfrac{x(1-x)(1-2x)(1+\alpha n)(1+2\alpha n)}{6(1+\alpha)(1+2\alpha)} f^{(3)}(x) + \tfrac{(x(1-x))^2(1+\alpha n)^2}{8(1+\alpha)(1+2\alpha)(1+3\alpha)} f^{(4)}(x), \text{ for } s=4.$$

Assume that f is s times differentiable on $[0,1]$*, then the convergence from* (3.10)–(3.12) *is uniform on* $[0,1]$. *Moreover, we get*

$$\left|P_n^{\langle\alpha\rangle}(f;x)-f(x)\right| \le \left(1+\tfrac{1+\alpha n}{4(1+\alpha)}\right)\cdot\omega\left(f,\tfrac{1}{\sqrt{n}}\right),\ \textit{for } s=0 \textit{ and}$$

$$n\left|P_n^{\langle\alpha\rangle}(f;x)-f(x)-\tfrac{x(1-x)(1+\alpha n)}{2n(1+\alpha)}f^{(2)}(x)\right| \le \tfrac{1+\alpha n}{8(1+\alpha)}\left(1+\tfrac{3(1+\alpha n)}{4(1+2\alpha)(1+3\alpha)}\right)\cdot\omega\left(f^{(2)},\tfrac{1}{\sqrt{n}}\right),\ \textit{for } s=2.$$

Estimates using the least concave majorant of the modulus of continuity can be given, taking into account the result proved in [**56**], by

Proposition 3.2.19. *For the Stancu operators, we get*

$$\left|n\left(P_n^{\langle\alpha\rangle}(f;x)-f(x)\right)-\tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)}f^{(2)}(x)\right| \le \tfrac{x(1-x)(1+\alpha n)}{2(1+\alpha)}\cdot\tilde{\omega}\left(f^{(2)},\tfrac{2}{3}\sqrt{\tfrac{1+\alpha n}{n(1+3\alpha)}}\right).$$

3. Application to Szász-Mirakjan type operators

Mirakjan-Favard-Szász operators

Szász-Mirakjan-Kantorovich operators

Szász-Mirakjan-Schurer operators

4. Application to φ-Szász-Mirakjan type operators

φSzász-Mirakjan operators

φ-Szász-Mirakjan-Kantorovich operators

φSzász-Mirakjan type operators

CHAPTER 4

Approximation of functions by linear positive operators

The main idea of this chapter is univariate and bivariate approximation of the functions by some linear positive operators presented, as well as establishment of upper bounds estimations for the appropriate remainder terms. In [**91**], [**93**], [**94**], [**85**], [**86**], [**87**] we proved interesting and important results concerning univariate, respectively bivariate approximation formulas by certain linear positive operators.

1. Univariate and bivariate Bernstein approximation formulas

2. Univariate and bivariate Stancu type approximation formulas

Univariate approximation formulas

For any $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, the following

$$f(x) = P_n^{\langle\alpha\rangle}(f;x) + R_n^{\langle\alpha\rangle}(f;x) \tag{4.13}$$

is called Stancu approximation formula, where $R_n^{\langle\alpha\rangle}$ is the remainder operator associated to the Stancu operator $P_n^{\langle\alpha\rangle}$, i.e. $R_n^{\langle\alpha\rangle}$ is the remainder term of the approximation formula (4.13). The study on the remainder term associated to the Stancu operators was done in [**144**], in terms of divided differences of first, respectively second order of the function f. Three years later, in [**149**] D.D. Stancu established an expression of this remainder term by using only divided differences of second order, given by:

Theorem 4.2.20. *Let $f : [0,1] \to \mathbb{R}$ be given. For any $n \in \mathbb{N}$, the remainder term associated to Stancu operators can be represented in the following form*

$$R_n^{\langle\alpha\rangle}(f;x) = -\sum_{k=0}^{n-1} \frac{(x+k\alpha)(1-x+\overline{n-1-k}\alpha)}{n(1+\overline{n-1}\alpha)} p_{n-1,k}^{\langle\alpha\rangle}(x)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right]. \tag{4.14}$$

For establishing a representation of a certain remainder term, associated to a linear operator, we can use also the well-known criterion proved by T. Popoviciu [**124**]. In order to get an appropriate form of the remainder term, as an application, in [**86**] we applied the Popoviciu's Theorem to the Stancu operators.

Application 4.2.21. (**D. Miclăuş**, [**86**]) Applying the Popoviciu's Theorem to the Stancu operators, it follows that linear functional $R_n^{\langle\alpha\rangle}$ on $C[0,1]$, defined by $R_n^{\langle\alpha\rangle}(f;x) = f(x) - P_n^{\langle\alpha\rangle}(f;x)$, satisfies

i) $R_n^{\langle\alpha\rangle}(e_0;x) = R_n^{\langle\alpha\rangle}(e_1;x) = 0$, $R_n^{\langle\alpha\rangle}(e_2;x) = -\frac{x(1-x)(1+\alpha n)}{n(1+\alpha)} \neq 0$;

ii) For any function $f \in C[0,1]$ and convex of first order, $R_n^{\langle\alpha\rangle}(f;x) \neq 0$.

Then, for each function $f \in C[0,1]$, there exist three points $0 \le \xi_0 < \xi_1 < \xi_2 \le 1$, such that

$$R_n^{\langle\alpha\rangle}(f;x) = R_n^{\langle\alpha\rangle}(e_2;x)[\xi_0,\xi_1,\xi_2;f] = -\frac{x(1-x)(1+\alpha n)}{n(1+\alpha)}[\xi_0,\xi_1,\xi_2;f]. \tag{4.15}$$

The idea of revision appears when we look at result proved with Popoviciu's Theorem and the relation (4.14). The evaluation of the remainder term was revisited in [**94**] by the author.

Theorem 4.2.22. (**D. Miclăuş**, [**94**]) *The representation of the remainder term associated to Stancu operators is given by*

$$R_n^{\langle\alpha\rangle}(f;x) = -\frac{x(1-x)(1+\alpha n)}{n(1+\alpha)} \sum_{k=0}^{n-1} p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right], \tag{4.16}$$

where $p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha) = \binom{n-1}{k}\frac{(x+\alpha)^{[k,-\alpha]}(1-x+\alpha)^{[n-1-k,-\alpha]}}{(1+2\alpha)^{[n-1,-\alpha]}}$.

Remark 4.2.23. (**D. Miclăuş**, [**94**]) It should be remarked that the expression (4.14) of the remainder term is an intermediate form of the expression (4.16). In one of the most important paper [**32**], which is a survey of the principal results obtained in the theory of uniform approximation of functions by means of linear positive operators of D.D. Stancu, we find out that both the expressions (4.14), (4.15) for the remainder term are presented. Making detailed researches, we discovered in [**154**] the representation (4.16), which is proved using $\frac{(x+k\alpha)(1-x+\overline{n-1-k}\alpha)}{n(1+\overline{n-1}\alpha)}p_{n-1,k}^{\langle\alpha\rangle}(x) = \frac{x(1-x)(1+\alpha n)}{n(1+\alpha)}p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha)$, which fortifies the fact that (4.14) is just an intermediate form.

Corollary 4.2.24. (**D. Miclăuş**, [**94**]) *Suppose that* $f \in C^2[0,1]$ *and the divided differences of the second order of* f *are all bounded on* $[0,1]$*, then we get*

$$\left|R_n^{\langle\alpha\rangle}(f;x)\right| \leq \frac{x(1-x)(1+\alpha n)}{2n(1+\alpha)}M_2[f] \leq \frac{1+\alpha n}{8n(1+\alpha)}M_2[f]. \tag{4.17}$$

Bivariate approximation formulas

Using the same procedure for constructing linear operators on the space of multivariate functions, as in the case of Bernstein operators we can define the parametric extensions of Stancu operators, by

$$ {}_xP_m^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)f\left(\tfrac{i}{m},y\right), \quad \text{respectively} \tag{4.18}$$

$$ {}_yP_n^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)f\left(x,\tfrac{j}{n}\right). \tag{4.19}$$

Considering the operators (4.18) and (4.19), two kinds of bivariate Stancu operators can be defined. One of them is called GBS operator and is defined by

$$\begin{aligned} U_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) &= {}_xP_m^{\langle\alpha,\beta\rangle}(f;x,y) + {}_yP_n^{\langle\alpha,\beta\rangle}(f;x,y) - P_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\left(f\left(\tfrac{i}{m},y\right)+f\left(x,\tfrac{j}{n}\right)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right). \end{aligned}$$

The another one can be got by the tensorial product of parametric extensions and is given by

$$ {}_xP_m^{\langle\alpha,\beta\rangle}\left({}_yP_n^{\langle\alpha,\beta\rangle};x,y\right) = P_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)f\left(\tfrac{i}{m},\tfrac{j}{n}\right). $$

The operators $P_{m,n}^{\langle\alpha,\beta\rangle}$ defined on the polygonal domain S are called the bivariate Stancu operators and can be found in [**149**], [**32**]. We shall prove:

Theorem 4.2.25. (**D. Miclăuş**, [**85**]) *The remainder term associated to the GBS Stancu operator can be represented under the following form*

$$ R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) $$

$$ = \tfrac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left[\begin{array}{ccc} x, \frac{i}{m}, \frac{i+1}{m} & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{array}\right], $$

where $p_{m-1,i}^{\langle\alpha\rangle}(x+\alpha) = \binom{m-1}{i}\frac{(x+\alpha)^{[i,-\alpha]}(1-x+\alpha)^{[m-1-i,-\alpha]}}{(1+2\alpha)^{[m-1,-\alpha]}}$.

The upper bound estimation for the remainder term is given in:

Corollary 4.2.26. (**D. Miclăuş**, [**85**]) *If the function f has the following properties*

i) $f \in C^{2,2}(S)$,

ii) there exists $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *on* $(0,1)\times(0,1)$,

iii) $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *is bounded on S, then the inequalities hold*

$$ \left|R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)\right| \le \tfrac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{4mn(1+\alpha)(1+\beta)}M_{2,2}[f] \le \tfrac{(1+m\alpha)(1+n\beta)}{64mn(1+\alpha)(1+\beta)}M_{2,2}[f], $$

for any $(x,y)\in S$ and $m,n\in\mathbb{N}$.

Theorem 4.2.27. (**D. Miclăuş**, [**87**]) *The remainder term associated to the Stancu bivariate approximation formula can be represented under the* *following form*

$$R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) = -\tfrac{x(1-x)(1+m\alpha)}{m(1+\alpha)} \sum_{i=0}^{m-1}\sum_{j=0}^{n} p_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y) \left[\begin{matrix} x, \frac{i}{m}, \frac{i+1}{m} \\ \frac{j}{n} \end{matrix} ;f\right]$$

$$-\tfrac{y(1-y)(1+n\beta)}{n(1+\beta)} \sum_{i=0}^{m}\sum_{j=0}^{n-1} p_{m,i}^{\langle\alpha\rangle}(x)p_{n-1,j}^{\langle\beta\rangle}(y+\beta) \left[\begin{matrix} \frac{i}{m} \\ y, \frac{j}{n}, \frac{j+1}{n} \end{matrix} ;f\right]$$

$$+\tfrac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n-1,j}^{\langle\beta\rangle}(y+\beta) \left[\begin{matrix} x, \frac{i}{m}, \frac{i+1}{m} \\ y, \frac{j}{n}, \frac{j+1}{n} \end{matrix} ;f\right].$$

The upper bound estimation for the remainder term is given in:

Corollary 4.2.28. (**D. Miclăuş**, [**87**]) *If the function f has the following properties*

i) $f \in C^{2,2}(S)$,

ii) there exist $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *on* $(0,1)\times(0,1)$,

iii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on S, then the inequalities hold*

$$\left|R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)\right|$$

$$\le \tfrac{x(1-x)(1+\alpha m)}{m(1+\alpha)} M_{2,0}[f] + \tfrac{y(1-y)(1+\beta n)}{n(1+\beta)} M_{0,2}[f] + \tfrac{xy(1-x)(1-y)(1+\alpha m)(1+\beta n)}{mn(1+\alpha)(1+\beta)} M_{2,2}[f]$$

$$\le \tfrac{1+\alpha m}{8m(1+\alpha)} M_{2,0}[f] + \tfrac{1+\beta n}{8n(1+\beta)} M_{0,2}[f] + \tfrac{(1+\alpha m)(1+\beta n)}{64mn(1+\alpha)(1+\beta)} M_{2,2}[f],$$

for any $(x,y)\in S$ and $m,n\in\mathbb{N}$.

3. Univariate and bivariate Mirakjan-Favard-Szász type approximation formulas

CHAPTER 5

Approximation of linear functionals

1. Preliminaries

The focus of this paragraph is to approximate the definite integral of a given function over a finite interval, using numerical informations, as well as global informations. Numerical informations are values of functions at certain points and can be used actually in approximation of the values of the definite integral. In the case when a definite integral is approximated by such a method, the evaluation of error can be done using the global informations, which refer to the affiliation of the integrated function to certain classes of functions. Formula $I(f) = Q(f) + R(f)$, where $Q(f) = \sum\limits_{k=0}^{n} A_k \lambda_k(f)$ is called numerical integration formula for the function f or quadrature formula. Parameters A_k, $k = \overline{0, n}$ are called weights or coefficients of the formula, and $R(f)$ is its remainder term. The study on quadrature formulas can be done only for univariate functions and the extension to the bivariate functions implies another formulas, which are called cubature formulas.
In what follows, we refer to the study on quadrature, respectively cubature formulas applied to the well-known linear positive Bernstein operator.

2. Bernstein quadrature formula

2.1. The composite Bernstein quadrature formula

The aim of this section is to construct the composite Bernstein quadrature formula. In order to get this, the interval $[0, 1]$ will be divided in n

equally spaced subintervals $\left[\frac{j-1}{n}, \frac{j}{n}\right]$, for any $j = \overline{1,n}$. On each such type of interval, the Bernstein quadrature formula will be applied. Next, adding the mentioned quadrature formulas, the desired composite Bernstein quadrature formula on $[0,1]$ will be got.

Theorem 5.2.29. (D. Bărbosu and **D. Miclăuş**, [**20**]) *For any $f \in C^2[0,1]$, the following composite Bernstein quadrature formula holds*

$$\int_0^1 f(x)dx = \frac{1}{n(n+1)} \sum_{j=1}^{n} \sum_{k=0}^{n} f\left(\frac{jn-n+k}{n^2}\right) + R_n[f],$$

where $|R_n[f]| \leq \frac{1}{12n} M_2[f]$.

3. Bernstein cubature formula

3.1. The composite Bernstein cubature formula

The focus of this section is to construct the composite Bernstein cubature formula. For this aim, the bidimensional interval $[0,1] \times [0,1]$ will be divided in mn equally spaced subintervals $\left[\frac{i-1}{m}, \frac{i}{m}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$, $i = \overline{1,m}$, $j = \overline{1,n}$. On each such type of interval, the Bernstein cubature formula will be applied. Next, adding the mentioned cubature formulas, the desired composite Benstein cubature formula on $[0,1] \times [0,1]$ will be got.

Theorem 5.3.30. (D. Bărbosu and **D. Miclăuş**, [**19**]) *For any $f \in C^{2,2}([0,1] \times [0,1])$, the following composite Bernstein cubature formula holds*

$$\int_0^1 \int_0^1 f(x,y)dxdy$$
$$= \frac{1}{mn(m+1)(n+1)} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{h=0}^{m} \sum_{j=0}^{n} f\left(\frac{im-m+h}{m^2}, \frac{jn-n+l}{n^2}\right) + R_{m,n}[f],$$

where $|R_{m,n}[f]| \leq \frac{1}{12m} M_{2,0}[f] + \frac{1}{12n} M_{0,2}[f] + \frac{1}{144(mn)^2} M_{2,2}[f]$.

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The scientific research performed in period 1.10.2008 - present

This doctoral thesis is developed and based on the following published, accepted, submitted and communicated papers:

List of the published papers

1. D. Bărbosu, O.T. Pop and **D. Miclăuş**, *Some quadrature formulas based on linear and positive operators*, J. of Science and Arts, **11** (2009), no. 2, 198–205.

2. D. Bărbosu and **D. Miclăuş**, *On the composite Bernstein type quadrature formula*, Rev. Anal. Numér. Théor. Approx., **39** (2010), no. 1, 3–7.

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3. **D. Miclăuş**, *On the Stancu type bivariate approximation formula*, Studia Scientiarum Mathematicarum Hungarica, (submitted).

4. O.T. Pop, **D. Miclăuş** and D. Bărbosu, *The Voronovskaja type theorem for a general class of Szász-Mirakjan operators*, Miskolc Math. Notes, (submitted).

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