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Summary of the PhD Thesis

TOPOLOGICAL ENDOMORPHISM RINGS

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Introduction

The theory of endomorphism rings of Abelian groups, and, to a wider extent, of modules, is a topic of algebra which is at the intersection of set theory, homological algebra, modules theory, mathematical logic, rings theory and general topology. Studying endomorphism rings is important because it provides the opportunity of using new specific methods in the study of groups and rings. Now it is difficult to establish the border between rings theory and modules theory; actually these theories became only one. We mention that in structure theorems, we have the endomorphism rings of modules used in a great number of cases. Also, the theory of endomorphism rings can be considered a part of Abelian groups theory as well as a part of ring representation theory. In the abstract theory of rings we work with concepts like: maximal ideal, Jacobson radical, primitive ring, etc.. There has to be taken into consideration that the ring theory has certain limits in studying these concepts. But in the case of topological rings of endomorphisms, they can be studied at a deeper level, as besides algebraic methods we also have at our disposal concepts and topological methods.

For every Abelian group one can associate an associative ring with identity, namely, the ring of all its endomorphisms. As a model we have the ring of linear transformations of a vectorial space. A very important particular case is the one of the finite dimensional space. In this case, the endomorphism ring is isomorphic with the ring of all matrices of dimension equal to the dimension of the vectorial space. The first papers in this area were written by Shoda [**50**]. We will illustrate the non-triviality of the endomorphism ring theory in the case of finite Abelian groups through the following example: In the ring theory the following problem has been studied: Let R be a finite ring. Is there a commutative ring S and a natural number n so that R is embedded in $M(n, S)$? The unexpected answer was given by G. Bergman [**14**]. He showed that for every prime number p the ring $\mathrm{End}(Z(p) \oplus Z(p^2))$ does not admit a representation of this kind. The problem of finite rings representation with the help of matrices over commutative rings is difficult and far from being solved. It is unsolved (2010) even for the case of finite local rings [**9**].

Baer [**12**] was the first who studied endomorphism rings as special rings, succeeding to characterize endomorphism rings of finite exponent groups. Different approaches were made by Liebert in [**40**], [**41**], [**42**], obtaining characterizations in the case of separable p-groups.

There are classes of rings whose structure is well-known; thus reaching to the idea of studying the possibility of having endomorphism rings in these classes. The initiator of such a program was Szele [**52**], and since then a great deal of attention has been given to this research ([**47**] and [**48**]). The main objective of the program was to obtain as much information as possible about the structure of groups on which endomorphisms were defined on.

In the process of studying endomorphism rings it is of interest to find the relations between the Abelian group and the ring of endomorphisms. In time, two research directions were emphasized in this theory. One by imposing restrictions on the endomorphism ring and finding characterizations of the Abelian group, and the other one by finding conditions under which an arbitrary ring is actually the endomorphism ring of some Abelian group.

The class of Abelian groups is divided in three subclasses: torsion groups, mixed groups and torsion free groups. The reacher in results is the class of torsion groups. Also, the class of mixed groups contains enough results. The class of torsion free groups is more difficult to study (we mention that until the sixties of the last century not many examples of reduced torsion free not free group were known).

The theory of endomorphism rings is presented in [**27**] chapters XV-XVIII. We mention that the main results refer to the torsion groups. Interesting results regarding endomorphism rings of torsion free groups can be found in the book [**23**]. An important result for the case of torsion free groups, was made by Corner in [**19**]. Studies in this direction can be found in the papers of Corner [**20**], [**21**], Brenner and Butler [**16**], Kishkina [**34**], Krol [**35**]. Budanov in [**18**] describes the elements belonging to the Jacobson radical (from the point of view of their action) for a large class of torsion free groups. Less attention was given to mixed groups. In [**53**], T. Szele proved that there are no mixed groups whose rings of endomorphisms are containing zero divisors. The commutativity of the endomorphism ring was also solved for the case of two large classes of mixed groups by Szele and Szendrei in [**54**]. A generalization of this result was given by Schultz in [**49**].

The main problem in the theory of topological rings is the following: What is the influence of the topologies on a ring R over its algebraic properties and converse? This problem is more concrete when we take in account some constructions in the class of topological rings. For instance, there are studied the relations between the properties of a topological space X and the properties of the rings of continuous functions $C(X)$, the properties of a pro-p-group G and the completed group ring $\mathbb{Z}_p[[G]]$, etc..

The endomorphism ring of an Abelian group carries a natural ring topology - the finite topology. In the case of infinite torsion groups, the finite topology is not discrete.

The results of this thesis are connected with the following two basic problems:

i) Let $\mathcal{P}$ be a topological property. Under which conditions on A the topological ring $(\text{End}(A), \mathcal{T})$ has the property $\mathcal{P}$?

ii) Consider that the topological ring $(\text{End}(A), \mathcal{T})$ has the property $\mathcal{P}$. What is the structure of the group A?

Let A be an Abelian group and $\mathcal{T}$ the finite topology on $\text{End}(A)$. We mention that it can be studied a more general problem: Let A be an Abelian locally compact group and $\text{End}_c(A)$ the ring of continuous endomorphisms endowed with the compact-open topology. What are the connections between the properties of the ring $\text{End}_c(A)$ and the topological group A?

The following general properties are useful:

i) the ring $\text{End}_c(A)$ has a fundamental system of neighborhoods of 0 consisting of sub-semi-groups of the multiplicative semigroup [**56**].

ii) $\text{End}_c(A)$ is a complete ring[**56**].

iii) since $(\mathrm{End}(A), \mathcal{T})$ has a fundamental system of neighborhoods of 0 consisting left ideals (in other words is a left linear ring) it is a left bounded ring and 0 dimensional in the sense of dimension ind. In other words, $(\mathrm{End}(A), \mathcal{T})$ has a base consisting of closed and open subsets.

The first important result was given by L. Fuchs, when he characterized Abelian groups having $(\mathrm{End}(A), \mathcal{T})$ compact([**27**] Proposition 107.4):

If A is an Abelian group then $\mathrm{End}(A)$ is compact if and only if $A = \oplus_{p\in\mathbb{P}} A_p$ where each primary component A_p is a finite direct sum of cocyclic groups.

Other results in this direction were obtained also in [**56**] and [**57**].

The following notion of the general theory of topological rings is an algebraic analogue of the notion of a compact space. A topological ring $(R, \mathcal{T})$ is called *bounded* if for every neighborhood U of 0 there exists a neighborhood V of 0 such that $V.R \subseteq U$ and $R.V \subseteq U$. The class of bounded rings contains compact rings and discrete rings. Any topological ring having a fundamental system of neighborhoods of 0 consisting of ideals is bounded.

Since the notion of a bounded ring generalizes the notion of a compact ring, we studied the following problem: When the ring $(\mathrm{End}(A), \mathcal{T})$ is bounded? Equivalently: Under which conditions on A the ring $\mathrm{End}(A)$ endowed with the finite topology has a fundamental system of neighborhoods of 0 consisting of ideals? We note that if $\mathrm{End}(A)$ is commutative, it is bounded. The characterization of the groups A having the ring $\mathrm{End}(A)$ commutative, is a famous unsolved problem in the theory of Abelian groups.

In the preliminaries of the second chapter are presented a few technical results. In Proposition 2.1.31 we show that if a group is a direct sum of fully invariant subgroups, then the endomorphism ring is topologically isomorphic with the product of the endomorphism rings of those subgroups(Corollary 2.1.32 is for the case of torsion groups).

In the beginning of the second section we deal with locally boundedness for the endomorphism rings. In Lemma 2.2.37 it is proved that if $0 \neq A = \bigoplus_{\mathfrak{m}} B$, $\mathfrak{m}$ is infinite, then $\mathrm{End}(A)$ is not locally bounded. As a consequence we obtain a characterization of divisible torsion groups for which the endomorphism ring is locally bounded. In

Theorem 2.2.40 are characterized divisible groups for which the endomorphism ring is bounded. If we will consider Abelian divisible groups for which the endomorphism ring is locally bounded, then this characterization is given in the Theorem 2.2.42.

We prove in Proposition 2.2.44 that the ring of endomorphisms of an infinite separable reduced p-group is not bounded, and so we can give a characterization of reduced p-groups A ($|A| \leq \omega$) for which $\mathrm{End}(A)$ is bounded. One important result in this thesis gives a characterization of countable torsion groups whose ring of endomorphisms $\mathrm{End}(A)$ is bounded (Theorem 2.2.49). At the end of this section we present some results concerning the cardinalities of the group and basic subgroups of Abelian p-groups and reduced groups, respectively. We give in Theorem 2.2.54 a characterization of bounded subsets of $(\mathrm{End}A, \mathcal{T}_\lambda)$ where A is an elementary or free group.

In the last section of this chapter (Theorem 2.3.57) we give an example of a non-discrete ring of endomorphisms of an Abelian group which admits right bounded topologies but it does not admit bounded non-discrete topologies.

Regular rings were first introduced by John von Neumann [**25**]. The book of K.R. Goodearl [**30**] develops the theory for different types of regular rings. In the theory of Abelian groups the accent is in determination of those groups whose ring of endomorphisms is regular or has similarly properties ([**27**] section 112, [**28**], [**29**]). Brown and McCoy give a new perspective to the research, showing [**17**] that every ring contains an unique maximal ideal whose elements are regular elements in the ring.

Theorem 3.1.61 contains a criterion for m-regularity for a single endomorphism of the ring $\mathrm{End}_c(A)$ for any topological Abelian group A, not necessarily Hausdorff. Theorem 3.1.62 is an important theorem presented in this thesis. It contains a characterization of topological Abelian groups, not necessarily locally compact, for which the endomorphism ring is π-regular, extending in this way the Theorem of Rangaswamy ([**27**], Proposition 112.2) about regular endomorphism rings. The next results gives us the opportunity of constructing an example of a group A whose ring of endomorphisms is not regular but $\mathrm{End}_c(A)$ is. Proposition 3.1.70 shows a group for which $\mathrm{End}(A)$ is regular but $\mathrm{End}_c(A)$ is not (it is also proved that the ring

$\mathrm{End}_c(A)$ of the group is prime). In Theorem 3.1.72 we obtained a characterization of locally compact torsion Abelian groups for which $\mathrm{End}_c(A)$ regular.

In the second section of chapter III we study endomorphism rings of topologically discrete quasi-injective modules. In Theorem 3.2.73, which is an important result of the thesis, we extend a well-known theorem about the regularity of the endomorphism ring of a quasi-injective module modulo its Jacobson radical (Utumi, Wang and Johnson, Osofsky, Faith and Utumi). The result presented in Theorem 3.2.76 gives us a characterization of quasi-injective module for which the endomorphism ring is compact.

In section 3.3 is given an example of a linearly compact Abelian group whose ring of continuous endomorphisms is regular. It is also studied the linearly compact-open topology, a topology which together with the endomorphism ring forms a topological ring.

The category of topological rings is too wide to allow developing a structure theory. We mention two classes of topological rings where we can apply successfully the concepts and methods of the abstract theory of rings:

i) locally compact rings;

ii) Banach algebras and their generalizations.

The Jacobson radical plays an important role in the study of abstract rings. Of course, Jacobson radical cannot be used successfully in studying topological rings unless it is closed. There exist examples of endomorphism rings of modules having the Jacobson radical not closed [**15**]. Recently were constructed examples of endomorphism rings of Abelian groups for which the Jacobson radical is not closed (see, for instance, [**18**]). This example shows us the difficulties that can occur when studying topological endomorphism rings.

In both cases presented before the radical is closed. For the case of Banach algebras, this result is true because the algebra has a neighborhood of 0 consisting of quasi-regular elements. This explains the importance of the Q-ring notion in the category of topological ring. In a topological Q-ring every maximal ideal, right or left, is closed, hence the Jacobson radical is closed. The most general form of this theorem is contained in [**56**]. All this facts turned our attention to the following question:

When the ring End(A) is a Q-ring?

The first theorem of the section 4.1 gives us some equivalent conditions for a topological left bounded ring to be a Q-ring. As a corollary, we state the theorem for the case of endomorphism rings (Corollary 4.1.84).

In the beginning of the second section it is proved that for the case of infinite direct sums, the ring of endomorphisms is non-discrete. Theorem 4.2.87 shows that in the case of torsion Abelian groups, for the ring End(A) to be discrete is necessary and sufficient that the group A to be finite. It is also studied the case when the endomorphism ring has a neighborhood of 0 without non-zero nilpotent elements and topological nilpotent elements, in this case being also necessary and sufficient that the group A to be finite (Theorem 4.2.88). It is proved that if A is a separable group and if the endomorphism ring has a commutative neighborhood then A is finite. In Theorem 4.2.91 we show that if End(A) has a nilpotent 0-neighborhood and A is torsion then the group is a finite direct sum of p-groups.

Because Theorem 4.1.83 gives us no information about Abelian groups whose rings of endomorphisms are Q-rings, the third section gives some sufficient conditions for this fact to be true. Beside some preliminary results, it was obtained a characterization of the torsion divisible groups and separable reduced p-groups in the case when the Jacobson radical of the endomorphism ring is open(Theorems 4.3.95 and 4.3.96).

The character and the pseudo-character are two basic notions in general topology. In section 4.4 we describe this notions for the case of endomorphism rings, through the invariants of the groups, and show the relations that appear between the two notions. The first result gives us information about the direct sum in which an Abelian group is decomposed, when the character of the endomorphism ring is less than a cardinal number. In Proposition 4.4.103 we obtained a relation between the character and the pseudo-character of the endomorphism ring of a direct sum of copies of a non-zero group of cardinality $\leq \omega$. A very interesting group in the theory of groups is Kulikov's group. Theorem 4.4.105 gives us information about the character and the pseudo-character of the endomorphism ring of Kulikov's group.

Theorem 4.4.107 is an important result in this thesis. It characterizes infinite torsion groups for which the pseudo-character is $\leq \tau$ where τ is an infinite cardinal. In Theorem 4.4.110 it is proved that for Baer-Specker's group $A = \mathbb{Z}^{\mathbb{N}}$ the character $\chi(\mathrm{End}(A)) = 2^{\omega}$ and the pseudo-character $\psi(\mathrm{End}(A)) = \omega$.

The rings of quotients theory is an important part of the modern ring theory. The theory of topological rings of quotients is not so well elaborated. There exists different ways of defining the ring of quotients notion, but in the topological case, the situation is more difficult. An important role in the theory of quotients rings is played by the prime rings. It is important to find non-trivial examples of prime rings of endomorphisms. We find a large class of prime rings in section 4.5 constructed by using of topological groups.

In section 4.6 are studied the conditions under which the endomorphism ring of a unitary right module endowed with the finite topology is a minimal topological ring, which means that there is no coarser topology the the finite topology. We prove (Theorem 4.6.114) that the endomorphism ring of a free module over a finite ring is minimal. Actually, we indicate a large class of minimal sub-rings of the endomorphism ring of a free module over a finite ring.

The last section of the thesis deals with admissible topologies on endomorphism rings. A non-discrete ring topology on the endomorphism ring of an Abelian group A is called *admissible* if the group as a left module over $\mathrm{End}(A)$ is a topological module, A being endowed with the discrete topology. It is given a class of admissible topologies on the endomorphism ring of an infinite Abelian group. All that topologies are complete. We construct a topology for the ring of endomorphisms, which for the case of elementary groups is not comparable with the finite topology, hence it is not admissible. We also give an example of a ring topology which defined on the endomorphism ring gives a topological ring, this topology being admissible and different from the topologies contained in the class presented previous. We have introduced the notion of admissible group topology and constructed one using ultra-filters.

The results form Chapter II can be found in the papers [**5**] and [**3**]. Them were presented in the following international conferences: 26-29 may 2005, AAA70 70th Workshop on General Algebra, Institute of Discrete Mathematics and Geometry, TU

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The results from the first 4 sections of Chapter IV are sent for publication to Algebra Colloquium [**6**], those in section 6 are in [**2**] and those in the last section are in [**4**]. Them were presented in the following international conferences: 31 August-6 September 2008 Summer School on Algebra and Ordered Sets, Charles University - Eduard Cech Center - Czech Academy of Sciences, Trest, Czech Republic, 18-21 September 2008, 6th International Conference on Applied Mathematics, North University, Baia Mare, Romania, 21- 25 June 2010 International Conference on Algebras and Lattices, Charles University and Czech Mathematical Society, Prague, Czech Republic, 26-28 August 2010 The Thirteenth International Conference on Applied Mathematics and Computer Science, Technical University, Cluj-Napoca, România.

CHAPTER 1

Preliminaries

1. Definition and conventions

$\mathbb{P}$ stands for the set a all positive prime numbers. ω stands for the first infinite ordinal number or the set af natural numbers. If $n, m \in \omega, n \leq m$, then $[n, m]$ is the set $\{n, n+1, \ldots, m\}$. If $f : X \to Y$ is a mapping and $Z \subseteq X$ a subset, then by $f \restriction_Z$ is denoted the restriction of f on Z. We will denote by $K \subset_f X$ the fact that K is a finite subset of X. The subgroup of a group A generated by a subset X is denoted by $\langle X \rangle$. The order of an element a of a group is denoted by $o(a)$. If A is a direct sum of the subgroups B and C, we will write $A = B \oplus C$. Let X, Y be two subsets of the ring R. Then $X.Y := \{xy : x \in X, y \in Y\}$ and $XY = \{\sum_{i=0}^{n} x_i y_i : x_i \in X, y_i \in Y, n \in \omega\}$. The closure of a subset B of a topological space is denoted by $\overline{B}$. If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ then by $\mathcal{T} \restriction_Y$ is denoted the induced topology. A family $\mathfrak{B}$ of open subsets of a topological space X is called *sub-base* in $x \in X$ if the family consisting of the finite meetings of the elements belonging to $\mathfrak{B}$ forms a fundamental system of neighborhoods of x. A torsion group is called a p-group if the order of every element is some power of p. For an Abelian group A and a cardinal number $\mathfrak{m}$, $\oplus_{\mathfrak{m}} A$ denotes the direct sum of $\mathfrak{m}$ copies of the group A. Every ring is associative and with 1. The endomorphism ring $\mathrm{End}(A)$ of an Abelian group A is considered as a topological ring furnished with the finite topology. If K is a finite subset of A then by $T(K)$ we denote the left ideal $\{\varphi \in \mathrm{End}(A) : \varphi(K) = 0\}$ of $\mathrm{End}(A)$. for simplifying the notations, for $a \in A$ we will write $T(a)$ instead of $T(\{a\})$. The family $\{T(K)\}$, where K runs all finite subsets of an Abelian group A, forms a fundamental system of neighborhoods of 0, of the topological ring $\mathrm{End}(A)$. $A \cong B$ means that the groups A and B are isomorphic and $R_1 \cong_{top} R_2$ means that the topological rings R_1 and R_2 are topologically isomorphic. The modules (left or right) are assumed to be unitary.

We denote by $\mathrm{End}_c(A)$ the ring of all continuous endomorphisms of an Abelian group A. When A is an Abelian compact group (shortly, LCA), the ring $\mathrm{End}_c(A)$ with the compact-open topology is a topological ring. The additive group of the ring R is denoted by $R(+)$ and the center of the ring R by $Z(R)$. Let $\{R_\alpha\}_{\alpha\in\Omega}$ be a family of topological rings and $S_\alpha \subseteq R_\alpha$ an open fixed subring for every $\alpha \in \Omega$. Consider the subring $A \subseteq \prod_{\alpha\in\Omega} R_\alpha$ of the Cartesian product R_α, $A = \{(x_\alpha) \in \prod_{\alpha\in\Omega} R_\alpha \mid x_\alpha \in S_\alpha$ for almost all $\alpha \in \Omega\}$ (almost all means all excepting a finite number). Then the product $\prod_{\alpha\in\Omega} S_\alpha$ of the topological rings $S_\alpha, \alpha \in \Omega$, gives a ring topology on A. This ring is called *local direct product* of R_α over $S_\alpha, \alpha \in \Omega$ and it is denoted by $\prod_{\alpha\in\Omega}(R_\alpha : S_\alpha)$ (see [**15**], p. 46) and ([**56**], p. 211). It is important to note that if all S_α are locally compact and almost all S_α are compact, then the ring $\prod_{\alpha\in\Omega}(R_\alpha : S_\alpha)$ is locally compact. This construction plays an important role in the algebraic theory of numbers.

Recall an important construction in the theory of topological rings:

Let R be an abstract ring. Let S be a subring and assume that on S we have a ring topology. Let $A = \{x \in R|$ for every neighborhood U of 0_S there exists a neighborhood V of 0_S such that $Vx \subset U, xV \subset U\}$. Then A is a ring and the neighborhoods of 0_S gives a ring topology. The subring A is maximal for which the neighborhoods of S gives a ring topology on R.

We will give the definitions of the most important notions that appear in the thesis.

Definition 1.1.1. *An Abelian group A is called* separable *if any finite subset K can be embedded in a direct summand S of the group A which is a direct sum of groups having rank 1(*[**27**]*, section 65).*

Definition 1.1.2. *A subgroup B of an Abelian p-group G is called* basic subgroup *if it is a pure subgroup, if it is decomposable in a direct sum of cyclic groups and if the quotient group G/B is divisible.*

Definition 1.1.3. *A topological ring $(R, \mathcal{T})$ is called* almost bounded *if there exists a ring topology $\mathcal{T}' \leq \mathcal{T}$ such that $(R, \mathcal{T}')$ is bounded.*

Definition 1.1.4. *A topological ring is called* left linear *if it has a fundamental system of neighborhoods of 0 consisting from left ideals.*

Definition 1.1.5. *The topological ring is called* locally bounded *provided it contains a bounded open subset.*

Definition 1.1.6. *An Abelian group A is called* self-small *if $Hom(A, -)$ preserves direct sums of copies of A.*

Definition 1.1.7. *Let $P = \prod_{n=1}^{\infty} \langle e_n \rangle$, where $o(e_n) = \infty$. The torsion free group G is called* slender *if for every homomorphism $\eta : P \to G$ the equality $\eta e_n = 0$ holds for almost all natural numbers n.*

Definition 1.1.8. *An ideal P of a ring R is called* left primitive *if it is the largest ideal contained in a maximal left ideal.*

Definition 1.1.9. *A ring is called* left primitive *if 0 is a left primitive ideal.*

Definition 1.1.10. *The ideal P of a ring R is* prime, *if $P \neq R$ and if $AB \subset P$ implies $A \subset P$ or $B \subset P$, for any ideals A and B of R.*

Definition 1.1.11. *A ring R is called* prime *provided 0 is a prime ideal.*

The following concept is important in the ring theory:

Definition 1.1.12. *A module M is called* injective *if for every exact sequence $0 \to A \overset{\alpha}{\to} B$ and every homomorphism $\beta : A \to M$ there exists a homomorphism $\gamma : B \to M$ such that $\beta = \gamma \circ \alpha$*[**39**].

Definition 1.1.13. *A submodule of a module A_R is called* large *if it has a non-zero intersection with every non-zero submodule of A_R*[**39**].

Definition 1.1.14. *A topological Abelian group A is called linearly compact, if:*

i) the topology on A is linearly, this means that it has a fundamental system of neighborhoods of 0 consisting of subgroups.

ii) if $\mathfrak{F}$ is a filter base consisting of subgroups of the form $x + B$, where B is a closed subgroup, then $\cap \mathfrak{F} \neq \emptyset$ (see [**43**], [**61**]*).*

Definition 1.1.15. The Kulikov's group *is the torsion part of the group $\prod_{n=1}^{\infty} B_n$ $(n = 1, 2, \ldots)$ where B_n is a direct sum of cyclic groups of order p^n.*

Definition 1.1.16. *A group is called* residually finite *provided it is a subgroup of a product of finite groups.*

The set of all cardinal numbers of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for the topological space $(X, \mathcal{T})$, has a smallest element.

Definition 1.1.17. The weight *of the topological space* $(X, \mathcal{T})$ *is the smallest cardinal number from the set of cardinal numbers having the form* $|\mathcal{B}|$*, where* $\mathcal{B}$ *is a base of the topological space and we will denote this number by* $w((X, \mathcal{T}))$ *or* $w(X)$*.*

Definition 1.1.18. The character of a point x *of the topological space* $(X, \mathcal{T})$ *is the smallest number of the form* $|\mathcal{B}(x)|$*, where* $\mathcal{B}(x)$ *is a base for* $(X, \mathcal{T})$ *in* x*. This cardinal number is denoted by* $\chi(x, (X, \mathcal{T}))$ *or* $\chi(x, X)$*.*

Definition 1.1.19. The character of the topological space $(X, \mathcal{T})$ *is defined as the biggest number* $\chi(x, (X, \mathcal{T}))$ *for every* $x \in X$*. This cardinal is denoted by* $\chi((X, \mathcal{T}))$*.*

Definition 1.1.20. The pseudo-character of a point x *of a* T_1 *topological space* X *is the smallest cardinal number of the form* $|\mathcal{U}|$*, where* $\mathcal{U}$ *is a family of open subsets of* X *such that* $\cap\mathcal{U} = \{x\}$*; this cardinal number is denoted by* $\psi(x, X)$*.* The pseudo-character of a T_1 space *is defined as the biggest number* $\psi(x, X)$ *for* $x \in X$*; this cardinal number is denoted by* $\psi(X)$*.* [**7**]

Definition 1.1.21. *An Abelian group* E *is called* S-group *provided every finite subset* $F \subseteq E$ *is contained in a free subgroup* H *of finite rank and* H *is a direct summand of* E *(see* [**11**]*).*

2. Endomorphism rings

On the set of all endomorphisms on an Abelian group we take the addition and multiplication as follows:

$(\alpha + \beta)a = \alpha a + \beta a$ and $(\alpha\beta)a = \alpha(\beta a)$, for every $a \in A$. In this way we obtain an associative ring with identity called *the endomorphism ring* End(A) *of* A.

We will present the endomorphism ring of a direct sum of groups. A matrix $[\alpha_{ji}]$ with elements from End(A) is called *column convergent* if, for every column i, the sum $\sum_j \alpha_{ji}$ exists in the finite topology on End(A).

Let $A = \oplus_{i\in I} A_i$ be a direct sum and $\varepsilon_i (i \in I)$ the corresponding projections, considered as idempotents in $\mathrm{End}(A)$. Any $a \in A$ can be written in the following form $a = \sum_i \varepsilon_i a$, where almost all $\varepsilon_i a$ are zero. For $\alpha \in \mathrm{End}(A)$, we have $\alpha a = \sum_i \alpha\varepsilon_i a = \sum_{i,j} (\varepsilon_j \alpha \varepsilon_i) a$. In this way, with every $\alpha \in \mathrm{End}(A)$ one can associate an $I \times I$-matrix:

$$\phi : \alpha \mapsto [\alpha_{ji}]_{j,i\in I}$$

where $\alpha_{ji} = \varepsilon_j \alpha \varepsilon_i$. If $\beta \in \mathrm{End}(A)$ and if $[\beta_{ji}]$ with $\beta_{ji} = \varepsilon_j \beta \varepsilon_i$ being the corresponding matrix, then the matrices associated with $\alpha - \beta$ and $\alpha\beta$ are exactly the matrices $[\alpha_{ji} - \beta_{ji}]$ and $[\sum_k \alpha_{jk}\beta_{ki}]$, respectively, hence ϕ is a ring homomorphism.

From the definition it results that the zero matrix can appear only from the zero endomorphism. For every i, $\alpha\varepsilon_i a = \sum_j \alpha_{ji} a$ exists for every $a \in A$, hence the matrices $[\alpha_{ji}]$ are column convergent. Conversely, if $[\alpha_{ji}]_{i,j\in I}$ is a column convergent matrix with $\alpha_{ji} \in \varepsilon_j \mathrm{End}(A)\varepsilon_i$, then it results that for an $\alpha \in$ End(A)

$$\alpha a = \sum_{i,j} \alpha_{ji} a.$$

If we identify $Hom(A_i, A_j)$ with a subgroup ε_jEnd(A)ε_i of End(A) then we can obtain the following theorem:

Theorem 1.2.22 ([**27**], Theorem 106.1)**.** *Let $A = \oplus_{i\in I} A_i$ a direct sum decomposition of A. Then* $\mathrm{End}(A)$ *is isomorphic with the ring of all $I \times I$ column convergent matrices.*

$$[\alpha_{ji}]_{j,i\in I}, \alpha_{ji} \in Hom(A_i, A_j).$$

3. The finite topology and the compact-open topology

The rings of endomorphisms admits different topologies. Let, for example, α be an infinite cardinal number and $\mathcal{P}_\alpha$ the family of all subsets of A of cardinality $< \alpha$. Then the family $\{T(K)\}_{K\in\mathcal{P}_\alpha}$ gives a topology $\mathcal{T}_\alpha$ of a topologically left linear ring on $\mathrm{End}(A)$. Obviously, if A will be an elementary Abelian group then the topologies $\mathcal{T}_\alpha$ forms a well ordered set.

A topology which plays an important role in the study of endomorphism rings, will be presented in what follows, together with a few results. This topology is called *the finite topology.* For a finite subset X of A, the X-neighborhood of $\alpha \in \mathrm{End}(A)$ is defined as follows:

$U_X(\alpha) = \{\eta \in \mathrm{End}(A) | \eta x = \alpha x \text{ for every } x \in X\}$

In this definition $U_X(\alpha) = \cap_{x \in X} U_x(\alpha)$ and $U_X(\alpha) = \alpha + U_X(0)$. Or more convenient we can define the neighborhoods of 0 as follows:

$U_x = \{\eta \in End(A) | \eta x = 0\}$ for every $x \in A$.

This is a Hausdorff topology, and U_x are ideals in $\mathrm{End}(A)$.

Theorem 1.3.23 ([**27**], Theorem 107.1)**.** *The endomorphism ring* $\mathrm{End}(A)$ *of an Abelian group A is complete in the finite topology.*

Remark 1.3.24. *The ring* $\mathrm{End}(A)$ *is discrete in the following cases:*

i) A is finitely generated;

ii) A is a torsion free group of finite rank.

Recall that if A is an Abelian group, p a prime number, then the *p-component* A_p of A is the subgroup

$\{x \in A : \quad \text{there exists} \quad k \in \mathbb{N}, p^k x = 0\}.$

Theorem 1.3.25. *[*[**27**]*, Theorem 107.4] The endomorphism ring* $\mathrm{End}(A)$*, of a group A, is compact in the finite topology, if and only if A is a torsion group whose p-components are finite direct sums of cocyclic groups.*

The finite topology on the endomorphism ring of an Abelian group has a generalization in the case of locally compact Abelian groups in the following way. It is considered *the compact-open topology.* As a base in 0 for the ring $\mathrm{End}_c(A)$ of continuous endomorphisms of A, we take the subsets of the form

$$T(K, U) = \{\alpha \in \mathrm{End}_c(A) : \alpha(K) \subseteq U\}$$

where K is a compact subset of A and U a neighborhood of 0.

Let ${}_RM$ be a locally compact left R-module and $\mathrm{End}_c({}_RM)$ furnished with the compact-open topology (R a locally compact ring).

Theorem 1.3.26. *[[**56**], Theorem 19.2] $S = \mathrm{End}_c({}_RM)$ is complete.*

We note that Theorem 1.3.26 generalizes Theorem 1.3.25.

$(R, \mathcal{T})$ is a right topological ring where $(R(+), \mathcal{T})$ is a topological group and for every $a \in R$, $R \overset{Ra}{\to} R$, $r \mapsto ra$, is a continuous function. The following theorem gives us the opportunity of studying endomorphism rings of Abelian groups, with the help of topological methods.

Theorem 1.3.27 ([**56**], Theorem 4.21)**.** *Let M be a left R-module and $\mathfrak{T}$ a compact topology on M. We assume that for every $r_0 \in R$ the mapping $M \to M$, $m \mapsto r_0 m$ is continuous. Then the ring $S = \mathrm{End}({}_RM)$ of all endomorphisms, not necessarily continuous on ${}_RM$ with the finite topology, is a topologically compact right ring.*

Corollary 1.3.28 ([**56**], Corollary 4.22)**.** *If M is an Abelian group which admits a compact group topology, then $\mathrm{End}(M)$ admits a compact right ring topology.*

Corollary 1.3.29 ([**56**], Corollary 4.23)**.** *Let p be a prime number and M an elementary Abelian group. If there exists a cardinal number $\mathfrak{m}$ such that$|M| = 2^{\mathfrak{m}}$, then the ring $\mathrm{End}(M)$ admits a compact right ring topology.*

Remark 1.3.30. *If A, B are two topologically Abelian groups, then $\mathrm{End}(A \oplus B)$ contains an isomorphic copy of $\mathrm{End}(A)$.*

CHAPTER 2

Boundedness of topological endomorphism rings

In the theory of vector spaces it is well-known the notion of a bounded subset.

This notion was extended for topological rings(see, for instance, [**33**] p. 44).

We note the importance that the bounded set has for the case of topological fields.

A subset S of a topological ring os called *right bounded* provided for every neighborhood U of 0 there exists a neighborhood V of 0 such that $V.S \subseteq U$([**32**], [**59**]). The notion of a left bounded subset of a topological ring is given in an analogous way. A subset is bounded if it is both left and right bounded.

The property of a subset to be bounded is not absolute in the following sense: If S is a bounded subset in a topological ring R and $R \subset R'$, then it is not necessary that S to be bounded in R'.

As an example let $\mathbb{R}$ be the topological ring of real numbers. Then $S = \mathbb{Z}$ is a discrete subring, hence, bounded, but S is not bounded in $\mathbb{R}$.

Left bounded rings (respectively, right bounded, bounded) admits some interesting characterizations:

i) A topological ring R is left bounded(respectively, right bounded, bounded) $\Leftrightarrow$ has a fundamental system of neighborhoods of 0 consisting of left ideals(respectively, right, bilateral) of the multiplicative sub-semigroup $R(\cdot)$.

ii) In a locally bounded ring, $R_0R = RR_0 = 0$ and the quotient ring R/R_0 has a fundamental system in 0 consisting of compact open ideals, where R_0 is the connected component of 0.

Important results about topological rings were obtained in [**33**], [**58**], [**60**].

Bounded subsets are important in those classes of topological rings with a big family of bounded subsets. We mention here locally compact rings and linearly left rings.

Recall [**5**] the following example: If A is an elementary infinite group, then $\mathrm{End}(A)$ is not locally bounded.

Because $\mathrm{End}(\mathbb{Z})$ is discrete, it is bounded. Hence the class of Abelian groups for which the endomorphism rings are compact is a proper subclass of the class of Abelian groups whose endomorphism rings are bounded.

If A is a "self-small" countable group then according to [**10**], Corollary 2.3 $\mathrm{End}(A)$ is discrete, hence bounded. For $A = Z(p^\infty), p \in \mathbb{P}$ the ring $\mathrm{End}(A)$ is bounded but according to [**10**], Propoition 3.1 the group A is not "self-small".

1. Preliminaries

Proposition 2.1.31. *If an Abelian group A is decomposed in a direct sum $A = \oplus_{i\in I} A_i$ of fully invariant subgroups, then* $\mathrm{End}(A) \cong_{top} \prod_{i\in I} \mathrm{End}(A_i)$.

Corollary 2.1.32. *Let A be a torsion Abelian group, $A = \oplus_{p\in\mathbb{P}} A_p$ its decomposition in p-primary components. Then* $\mathrm{End}(A) \cong_{top} \prod_{p\in\mathbb{P}} \mathrm{End}(A_p)$.

Remark 2.1.33. *The first part of Proposition 2.1.31 is stated in* [**27**]*, Chap. XV, § 106, Exercise 4, (a) and (b).*

Lemma 2.1.34. *(*[**56**]*, Proposition 19.21) Let $M = N \oplus N'$ be an Abelian group. Denote by pr_N the projection of M on N. Then the mapping: $\varphi : \mathrm{End}(N) \to \mathrm{End}(M)$, $\varphi(\alpha)(m) = \alpha(pr_N(m)), \alpha \in \mathrm{End}(N), m \in M$, is a topological isomorphism on its image.*

We will use the following criterion of compactness of subsets of the endomorphism ring $\mathrm{End}(A)$ of an Abelian group A :

Theorem 2.1.35. ([**56**]*, Theorem 19.4) A subset $X \subseteq \mathrm{End}(A)$ has a compact closure if and only if Xa is finite for each $a \in A$.*

2. Bounded endomorphism rings

Remark 2.2.36. *Let $R = (R, \mathfrak{T})$ be a topological ring and S its subring. If S is a bounded subset of R, then $(S, \mathfrak{T}\restriction_S)$ is a bounded topological ring.*

Recall that a topological ring is called *locally bounded* provided it has a bounded neighborhood of 0.

Lemma 2.2.37. *If* $0 \neq A = \bigoplus_{\mathfrak{m}} B$ *where* $\mathfrak{m}$ *is an infinite cardinal number, then* $\mathrm{End}(A)$ *is not locally bounded.*

Corollary 2.2.38. *Let A be a divisible Abelian torsion-free group. Then the following conditions are equivalent:*

i) $\mathrm{End}(A)$ *is discrete;*

ii) $\mathrm{End}(A)$ *is bounded;*

iii) $\mathrm{End}(A)$ *is locally bounded;*

iv) $A \cong \mathbb{Q}^n, n \in \omega$.

Remark 2.2.39. *Let p be a prime number,* $A = \mathbb{Q} \times Z(p^\infty)$. *Then the ring* $\mathrm{End}(A)$ *is unbounded.*

Theorem 2.2.40. *Let A be a divisible Abelian group. Then* $\mathrm{End}(A)$ *is bounded* $\Leftrightarrow$ $A \cong \mathbb{Q}^n$ *where* $n \in \omega$ *or* $A \cong \oplus_{p \in \mathbb{P}} A_p$ *where each* A_p *is a direct sum of a finite number of copies of* $Z(p^\infty)$.

Lemma 2.2.41. *Let* $n \in \omega$ *and* $Y \subseteq \mathbb{P}$, $A = \mathbb{Q}^n \oplus (\oplus_{p \in Y} Z(p^\infty)), n \in \omega$. *Then the ring* $\mathrm{End}(A)$ *is locally compact.*

Theorem 2.2.42. *Let A be a divisible Abelian group. Then the following conditions are equivalent:*

i) $\mathrm{End}(A)$ *is locally bounded;*

ii) $A \cong \mathbb{Q}^n \oplus (\oplus_{p \in \mathbb{P}} A_p)$ *where* $n \in \omega$ *and each* A_p *is a direct sum of a finite number of copies of the group* $Z(p^\infty)$;

iii) $\mathrm{End}(A)$ *is locally compact.*

Corollary 2.2.43. *Let A be a divisible Abelian group. Then the following conditions are equivalent:*

i) $\mathrm{End}(A)$ *is locally bounded and unbounded;*

ii) $A \cong \mathbb{Q}^n \oplus (\oplus_{p \in X} A_p)$ *where* $n \geq 1, \emptyset \neq X \subseteq \mathbb{P}$ *and each* A_p *is a direct sum of a finite number of copies of the group* $Z(p^\infty)$;

iii) $\mathrm{End}(A)$ *is locally compact and unbounded.*

Proposition 2.2.44. *Let A be a reduced p-group with the property that for each finite subset $F \subseteq A$ there exists a direct summand B of A such that $B \neq A$ and $F \subseteq B$. Then* $\mathrm{End}(A)$ *is unbounded.*

Corollary 2.2.45. *If A is an infinite reduced separable p-group then* $\mathrm{End}(A)$ *is unbounded.*

Corollary 2.2.46. *If A is an infinite bounded p-group then* $\mathrm{End}(A)$ *is unbounded.*

Corollary 2.2.47. *If A is an elementary group or a free group of finite rank, then* $\mathrm{End}(A)$ *is unbounded.*

Theorem 2.2.48. *For a reduced p-group $A, |A| \leq \aleph_0$ the following conditions are equivalent:*

i) $\mathrm{End}(A)$ *is compact;*

ii) $\mathrm{End}(A)$ *is bounded;*

iii) A is finite.

Theorem 2.2.49. *Let A be a countable torsion group. Then the following conditions are equivalent:*

i) $\mathrm{End}(A)$ *is compact;*

ii) $\mathrm{End}(A)$ *is bounded;*

iii) $A \cong \oplus_{p\in\mathbb{P}} A_p$ where each A_p is a finite direct sum of cocyclic groups.

Theorem 2.2.50. *If A is a divisible Abelian torsion group then a subset X of* $\mathrm{End}(A)$ *is bounded $\Leftrightarrow \overline{X}$ is compact.*

Proposition 2.2.51. *Let $p \in \mathbb{P}, A$ an Abelian p-group and B a basic subgroup of A. If* $\mathrm{End}(A)$ *is bounded, then $|B| \leq \aleph_0$.*

Corollary 2.2.52. *If A is an Abelian reduced p-group for which* $\mathrm{End}(A)$ *is bounded, then $|A| \leq 2^{\aleph_0}$.*

Remark 2.2.53. *The following question related to the subject of this paper remains open: Is it true that boundedness of the ring* $\mathrm{End}(A)$ *of an Abelian reduced p-group*

A implies that A is finite ? Theorem 2.2.48 and Corollary 2.2.52 show that if there exists an infinite reduced p-group A for which End(A) *is bounded, then* $\aleph_0 < |A| \leq 2^{\aleph_0}$.

An important class of ring topologies on End(A) is that of admissible topologies (see Chapter IV, section 7). In the following theorem we characterize bounded subsets in $(\mathrm{End}(A), \mathcal{T}_\lambda)$ (see the definition of $\mathcal{T}_\lambda$ topologies from section 7). The proof of this theorem was inspired from the proof of theorem 8 presented in paper [**22**].

Theorem 2.2.54. *Let A be an elementary group or a free group and λ an infinite cardinal number. Then the subset $S \subset$ End(A) is bounded in $(\mathrm{End}(A), \mathcal{T}_\lambda)$ if and only if for every subset $A_0 \subset A$, $|A_0| < \lambda$, $|S(A_0)| < \lambda$.*

3. Bounded and almost bounded topologies on endomorphism rings

In [**5**] it has been proved that the endomorphism ring of an infinite bounded p-group, endowed with the finite topology, is unbounded. We will prove that the ring of endomorphisms of an infinite bounded Abelian group does not admit non-discrete bounded topologies. Nevertheless, we prove that the ring of endomorphisms of $\oplus_\tau Z(p^n)$, where $p \in \mathbb{P}, n \in \mathbb{N}$ and τ is an infinite cardinal number, admits a non-discrete right linear topology.

Theorem 2.3.55. *If A is an infinite Abelian bounded group then its endomorphism ring admits a non-discrete right bounded topology.*

Recall that a topological ring $(R, \mathcal{T})$ is called *almost bounded* if there exists a coarser than $\mathcal{T}$ bounded topology.

Remark 2.3.56. *A ring R does not admit a non-discrete almost bounded ring topology if and only if it does not admit a non-discrete bounded ring topology.*

Theorem 2.3.57. *Let $A = \oplus_\tau Z(p^n)$, where n is a natural positive number, $p \in \mathbb{P}$ and τ a cardinal number. Then* End(A) *does not admit any non-discrete almost bounded ring topology.*

Remark 2.3.58. *The endomorphism ring of the Abelian group $A = \oplus_\tau Z(p^n)$ admits both non-discrete left bounded topology and non-discrete right bounded topology but does not admit a non-discrete bounded topology.*

Theorem 2.3.59. *Let D be a division ring and V a D-right vector space of dimension ω. Let $R = \mathrm{End}(V)/I$, where I contains all the endomorphisms of finite rank. Then R does not admit any non-discrete almost bounded ring topologies.*

CHAPTER 3

Regular endomorphism rings of topological Abelian groups

Regular in the sense of von Neumann rings form an important subclass of the class of associative rings. Recall that a ring R is called *regular* in the sense of von Neumann if for every $a \in R$ there exists $b \in R$ such that $aba = a$. We study in this paper the following questions:

(i) What are those topological Abelian groups in which every endomorphic image is a direct summand?

(ii) What are those locally compact Abelian groups for which the ring $\mathrm{End}_c(A)$ is regular?

Rangaswamy studied analogous questions for abstract Abelian groups [**46**]. We note that the Problem (i) has no complete answer in the class of abstract Abelian groups. We give a characterization of arbitrary topological Abelian groups whose rings $\mathrm{End}_c(A)$ are m-regular. A complete characterization of torsion (in abstract sense) LCA groups with regular $\mathrm{End}_c(A)$ is given (Theorem 3.1.72). We give examples of:

i) a LCA group $(A, \mathcal{T})$ for which $\mathrm{End}(A)$ is not regular but $\mathrm{End}_c(A)$ is regular;

ii) a LCA group $(A, \mathcal{T})$ for which $\mathrm{End}(A)$ is regular but $\mathrm{End}_c(A)$ is not.

We give a nontrivial example of a linearly compact Abelian group whose ring of continuous endomorphisms is regular. We indicate a natural ring topology for $End_c(A)$ for any linearly compact Abelian group which is analogous to the compact-open topology.

1. Regular rings of continuous endomorphisms of LCA groups

Lemma 3.1.60. *Let A be a topological Abelian not necessary Hausdorff group and α be an idempotent of* $\mathrm{End}_c(A)$. *Then* $\operatorname{im}\alpha$ *and* $\ker\alpha$ *are direct summands of A.*

Recall that an element a of a ring is called *m-regular* if there exists a positive integer m such that a^m is regular. A ring is *π-regular* if each of its elements is m-regular. A ring is called *m-regular* if all its elements are m-regular for a fixed m. The following theorem gives necessary and sufficient conditions under which an endomorphism α is m-regular.

Theorem 3.1.61. *Let A be a topological Abelian group and $\alpha \in \mathrm{End}_c(A)$. Then α is a m-regular element if and only if* $\mathrm{im}\,\alpha^{\mathrm{m}}$ *and* $\ker\alpha^m$ *are topological direct summands of A.*

Theorem 3.1.62. *The ring* $\mathrm{End}_c(A)$ *of all continuous endomorphisms of a topological Abelian group A is π-regular iff for every $\alpha \in \mathrm{End}_c(A)$ there exists a positive integer m such that* $\mathrm{im}\,\alpha^{\mathrm{m}}$ *and* $\ker\alpha^m$ *are topological direct summands of A.*

Corollary 3.1.63. *The ring* $\mathrm{End}_c(A)$ *of all continuous endomorphisms of a topological Abelian group A is regular if and only if the image and the kernel of every endomorphism are direct summands of the group.*

Lemma 3.1.64. *Let A be a topological Abelian group and* $\mathrm{End}_c(A)$ *is regular. If $p \in \mathbb{P}, x \in A$, and $p^2x = 0$ then $px = 0$.*

Corollary 3.1.65. *If A is a topological Abelian group,* $\mathrm{End}_c(A)$ *is regular and A is a p-group then $px = 0$ for every $x \in A$.*

Theorem 3.1.66. *Let A be a torsion free LCA-group. If* $\mathrm{End}_c(A)$ *is regular then* $\mathrm{End}_c(A)(+)$ *is divisible.*

Corollary 3.1.67. *If* $\mathrm{End}_c(A)(+)$ *is divisible then A is a divisible group.*

Example 3.1.68. *Let $p \in \mathbb{P}$ and $A = \prod_{i\in\omega}(R_i : S_i)$ where $R_i = \mathbb{Q}_p, S_i = \mathbb{Z}_p$. Then* $\mathrm{End}_c(A)$ *is not regular.*

Recall that a subgroup B of a topological group A is called *fully invariant* provided $\alpha B \subseteq B$ for every continuous endomorphism α of A.

Lemma 3.1.69. *If the locally compact Abelian group A is a direct sum $A = A_1 \oplus A_2$ of fully invariant subgroups A_1 and A_2 then* $\mathrm{End}_c(A) \cong_{top} \mathrm{End}_c(A_1) \times \mathrm{End}_c(A_2)$.

Rangaswamy [**46**] has proved that the ring $\mathrm{End}(A)$ where $A = \mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$ is not regular. We claim that if A is considered as a topological group with the product topology where $\mathbb{R}$ is furnished with the usual topology and each $\mathbb{Z}(p)$ with the discrete topology, then $\mathrm{End}_c(A)$ is regular.

We will give now an example of a group whose ring of all endomorphisms is regular, but the ring of all continuous endomorphisms is not regular.

Proposition 3.1.70. *Let* $A = (\mathbb{Z}/p\mathbb{Z})^{\mathfrak{m}} \oplus (\bigoplus_{\mathfrak{n}} \mathbb{Z}/p\mathbb{Z})$ *where* $p \in \mathbb{P}$ *and* $\mathfrak{m}, \mathfrak{n}$ *are infinite cardinal numbers. Then the ring* $\mathrm{End}_c(A)$ *is not regular.*

Remark 3.1.71. *The ring* $\mathrm{End}_c(A)$ *is prime.*

Theorem 3.1.72. *Let A be a torsion LCA group. Then the following conditions are equivalent:*

i) $\mathrm{End}_c(A)$ *is regular;*

ii) There exists $n \in \mathbb{N}$ *such that the subgroup* $\oplus_{i\geq n+1} A_{p_i}$ *is discrete and the group* A_{p_i} *is a compact infinite group satisfying the identity* $px = 0$.

2. Regular rings of quasi-injective discrete modules over compact rings

The following concept for discrete topological modules is analogous to the concept of a quasi-injective module in the theory of modules. A left discrete topological R-module M is called *quasi-injective* provided every homomorphism $f : N \to M$, where N is a submodule of M, has an extension to an endomorphism of M.

Theorem 3.2.73. *Let* I_R *be a quasi-injective* R*-module,* $S = \mathrm{End}(\mathrm{I_R})$ *and* $N = \{\alpha \in S : \alpha\, annihilates\, a\, large\, submodule\, of\, I_R\}$.

Then

(i) *S/N is regular in the sense of von Neumann;*

(ii) *N is the Jacobson radical of S;*

(iii) *idempotents modulo N can be lifted in S.*

Corollary 3.2.74. *Let R be a compact ring and P_R a projective right module in the category of profinite modules over R, and* $S = \mathrm{End}_c(A)$ *the ring of continuous endomorphisms furnished with the compact open topology. Then:*

i) $S/J(S)$ is regular;

ii) idempotents from $S/J(S)$ can be lifted modulo $J(S)$.

Definition 3.2.75. *Let M_R be a right discrete R-module.*

Two elements m, m' have the same order if the modules mR and $m'R$ are isomorphic.

Theorem 3.2.76. *Let I_R be a quasi-injective module. Then S is compact if and only if for every $m \in I_R$ the set of elements m' having the same order with m is finite.*

3. Linearly compact Abelian group with regular ring of endomorphisms

The following fact is well-known; we remind it for convenience of the reader:

If A is a linearly compact group, B and C are two closed subgroups, then the decomposition $A = B \oplus C$ is topological.

Consider the mapping $B \times C \to A, (b, c) \mapsto b + c.$ This mapping is a continuous isomorphism since $B \times C$ is linearly compact and it is open, hence a topological isomorphism.

Given any cardinal number α and a prime p, A will designate the group ${\mathbb{Q}_p}^\alpha$ where $\mathbb{Q}_p$ is the additive group of the locally compact field of p-adic numbers.

Lemma 3.3.77. *Every subspace $\mathbb{Q}_p a, a \in A$ is a topological direct summand.*

Lemma 3.3.78. *If B is a closed vector subspace of A then B is a topological direct summand.*

Lemma 3.3.79. *If $\alpha \in \mathrm{End_c(A)}$, then α is an endomorphism of a $\mathbb{Q}_p$-vector space.*

Lemma 3.3.80. *If $\alpha \in \mathrm{End_c(A)}$, then $im\alpha$ and $ker\alpha$ are closed $\mathbb{Q}_p$-subspaces of A.*

Theorem 3.3.81. $\mathrm{End_c(A)}$ *is a regular ring.*

Theorem 3.3.82. *Let A be a locally linearly compact Abelian group. Then $End_c(A)$ with the topology given by the family $\{T(K, V)\}$, where K runs all linearly compact subgroups, and V runs all open subgroups, is a topological ring.*

Open question: Classify all closed left(two-sided) ideals of $End_c(A)$.

CHAPTER 4

Endomorphism rings with special neighborhoods of zero

There exists a connection between local and global properties of a topological ring. For instance, a connected topological ring is commutative if and only if it has a commutative neighborhood of zero. Interrelations between local and global properties of a topological ring are weaker in the case of totally disconnected rings.

In this Chapter we study the connections between local and global properties of topological rings of endomorphisms. We show that although the rings of endomorphisms of Abelian groups are totally disconnected, there is a close relation between their local and global properties.

A fundamental notion in the theory of topological rings is the notion of a Q-inel (see, for instance,[**32**]). The importance of this concept lies in the following property of Q-rings: in a Q-ring every maximal left (right) modular ideal is closed [**56**]. In the case of noncommutative rings this notion splits formally in three different notions: Q_r-ring, Q_l-ring and Q-ring. It is not known (2010; see, [**32**]) if these notions are equivalent. We prove that for topological rings of endomorphisms these notions coincide (actually we prove a more general result). We study various conditions imposed on neighborhoods of zero of topological rings of endomorphisms.

We will prove, for instance, that $\mathrm{End}(A)$ is a Q-ring (i.e., has a neighborhood of 0 consisting of quasi-regular elements) if and only if its Jacobson is open. Evidently, this property is not characterizing the Abelian groups for which the Jacobson radical of endomorphism rings is open. Motivated by this result, we wil indicate some sufficient conditions under which the Jacobson radical of a endomorphism ring of an Abelian group is open. We note that it is slightly possible to find a classification of Abelian groups A for which $J(\mathrm{End}(A))$ is open. It is not known the classification of Abelian groups for which $\mathrm{End}(A)$ is discrete.

1. Q-rings of endomorphisms

The following notions are fundamental in the theory of topological rings (see, for instance, [**32**]): A topological ring $(R, \mathcal{T})$ is a Q_l-ring(Q_r-ring; Q-ring) if and only if 1 has a neighborhood consisting of left invertible (right invertible; invertible) elements.

Recall that a topological ring $(R, \mathcal{T})$ with 1 is called a *Gelfand ring* ([**12**], Exercise 11, pg. 110) provided the set of quasi-regular elements is open and $(R, \mathcal{T})$ is a ring with continuous inverse.

Theorem 4.1.83. *Let R be a left bounded topological ring. Then the following statements are equivalent:*

(i) *R is a Q_l-ring;*

(ii) *R is a Q_r-ring;*

(iii) *R is a Q-ring;*

(iv) *R is a Gelfand ring;*

(v) *$J(R)$ is open.*

Corollary 4.1.84. *Let A be an Abelian group. The following statements are equivalent:*

(i) $\mathrm{End}(A)$ *is a Q_l-ring;*

(ii) $\mathrm{End}(A)$ *is a Q_r-ring;*

(iii) $\mathrm{End}(A)$ *is a Q-ring;*

(iv) $\mathrm{End}(A)$ *is a Gelfand ring;*

(v) $J(\mathrm{End}(A))$ *is open.*

Example 4.1.85. *Let $\mathbb{Q}$ be the field of rationals. Then the set $\mathfrak{B}$ of all nonzero subgroups of the additive group of $\mathbb{Q}$ defines on $\mathbb{Q}$ a ring topology. The ring $(\mathbb{Q}, \mathcal{T})$ is a Q-ring, but the inversion is not continuous.*

2. Special 0-neighborhoods of rings of endomorphisms

Let us begin with the case of discrete rings.

Theorem 4.2.86. *Let $A = \oplus_{i\in I} A_i$ where $A_i \neq 0$ and I is infinite. Then the ring* $\mathrm{End}(A)$ *is non-discrete.*

The next Theorem shows that the class of non-discrete topological rings of endomorphisms is large.

Theorem 4.2.87. *The ring* $\mathrm{End}(A)$ *for a torsion Abelian group A is discrete if and only if A is finite.*

Theorem 4.2.88. *Let A be a separable p-group. Then the following conditions are equivalent:*

(i) $\mathrm{End}(A)$ *has a neighborhood of zero without non-zero topological nilpotent elements;*

(ii) $\mathrm{End}(A)$ *has a neighborhood of zero without non-zero nilpotent elements;*

(iii) *A is finite.*

Corollary 4.2.89. *If A is an Abelian separable p-group and* $\mathrm{End}(A)$ *has a 0-neighborhood without zero-divisors, then A is finite.*

Theorem 4.2.90. *If A is an Abelian separable group and* $\mathrm{End}(A)$ *has a commutative 0-neighborhood, then A is finite.*

Theorem 4.2.91. *If A is a torsion Abelian group and* $\mathrm{End}(A)$ *has a nilpotent 0-neighborhood, then A is a finite direct sum of p-groups.*

3. Sufficient conditions under which $\mathrm{End}(A)$ is a Q-ring

Theorem 4.1.83 is unsatisfactory since does not give any information about groups for which the ring of endomorphisms is a Q-ring. We will indicate in this section some assertions about the structure of Abelian groups A for which $\mathrm{End}(A)$ is a Q-ring.

Lemma 4.3.92. *Let p be a prime number and $n \in \mathbb{N}$. Then the Jacobson radical of* $\mathrm{End}(A)$*, where $A = \oplus_\omega Z(p^n)$, is not open.*

Theorem 4.3.93. *Let A be a reduced p-group and $J(\mathrm{End}(A))$ is open. If $B = \bigoplus_{n\in\mathbb{N}} B_{p^n}$ is a basic subgroup and $B_{p^n} = \oplus_{m_n} Z(p^n)$, then each cardinal number m_n is finite.*

Corollary 4.3.94. *If A is a reduced p-group, and $J(\mathrm{End}(A))$ is open, then $|A| \leq 2^{\omega}$.*

Theorem 4.3.95. *If A is an Abelian torsion divisible group then $J(\mathrm{End}(A))$ is open if and only if A is a finite direct sum of groups of type p^{∞}.*

Theorem 4.3.96. *If A is a reduced separable p-group then $J(\mathrm{End}(A))$ is open if and only if A is finite.*

Theorem 4.3.97. *Let A be a torsion group. Then the following conditions are equivalent:*

(i) $\mathrm{End}(A)$ *has no topologically nilpotent ideals;*

(ii) *A is a direct sum of elementary groups.*

Corollary 4.3.98. *If A is a torsion group and* $\mathrm{End}(A)$ *has no topologically nilpotent ideal then* $\mathrm{End}(A)$ *is a semi-simple left linearly compact ring.*

Corollary 4.3.99. *If A is a torsion group and* $\mathrm{End}(A)$ *has no topologically nilpotent ideal then the center of* $\mathrm{End}(A)$ *is discrete if and only if A has a finite exponent.*

4. Character and pseudo-character in endomorphism rings

Two fundamental notions in general topology are the character and the pseudo-character of a point in a topological space. The notion of pseudo-character of a point in a topological space was introduced by Aleksandrov and Urysohn in the classic paper "Memoire sur les espaces topologiques compacts" [**8**]. In This section we will study the following questions:

1) How can we obtain the character of $\mathrm{End}(A)$ through the invariants of the group A?

2) How can we obtain the pseudo-character of $\mathrm{End}(A)$ through the invariants of the group A?

3) Under which conditions the pseudo-character of the topological ring $(\mathrm{End}(A), \mathcal{T})$ coincides with the character?

Lemma 4.4.100. *Let $A = \oplus_{\alpha<\tau} A_{\tau}$ be a decomposition of an Abelian group A in a direct sum of nonzero subgroups, where τ is an infinite cardinal number. If $\chi(\mathrm{End}(A)) \leq \beta$, where β is a cardinal number, then $\tau \leq \beta$.*

Corollary 4.4.101. *Let A be an Abelian group which is a direct sum of nonzero groups of cardinality $\leq \omega$ and τ an infinite cardinal number. Then $\chi(\mathrm{End}(A)) \leq \tau$ if and only if $|A| \leq \tau$.*

Corollary 4.4.102. *If D is a non-zero divisible group, then $\psi(\mathrm{End}(D)) = \chi(\mathrm{End}(D))$ $= |D|$.*

Proposition 4.4.103. *If A is a nonzero group of cardinality $\leq \omega$ and τ an infinite cardinal number, then $\psi(\mathrm{End}(\oplus_\tau A)) = \chi(\mathrm{End}(\oplus_\tau A)) = \tau$.*

Corollary 4.4.104. *If A is a torsion group, τ a cardinal number and $\chi(\mathrm{End}(A)) \leq \tau$, then $|A| \leq 2^\tau$.*

Theorem 4.4.105. *Let A be the Kulikov's group. Then $\chi(\mathrm{End}(A)) = 2^\omega$ and $\psi(\mathrm{End}(A)) = \omega$.*

Lemma 4.4.106. *Let A be a reduced p-group and τ an infinite cardinal number. Then $\psi(\mathrm{End}(A)) \leq \tau$ if and only if the cardinality of every basic subgroup of A is $\leq \tau$.*

Theorem 4.4.107. *Let $A = \oplus_{p\in\mathbb{P}} A_p$ be an infinite torsion group, where A_p is a p-primary subgroup of A. Fix for every $p \in \mathbb{P}$ a decomposition $A_p = S_p \oplus D_p$, where D_p is the maximal divisible subgroup and S_p is a reduced group and let τ be an infinite cardinal number. Then $\psi(\mathrm{End}(A)) \leq \tau$ if and only if $|D_p| \leq \tau$ and for every basic subgroup B_p of S_p we have $|B_p| \leq \tau$.*

Corollary 4.4.108. *Let A be an infinite torsion group. Then $\psi(\mathrm{End}(A)) = \omega$ if and only if $A = (\oplus_{p\in\mathbb{P}} Z_p) \oplus D$ where D is a divisible subgroup with $|D| = \omega$ or $D = 0$ and every basic subgroup of $Z_p, p \in \mathbb{P}$ is finite or countable.*

Remark 4.4.109. *If $A = \oplus_{i\in I} A_i$ where $0 < |A_i| \leq \omega$ and I is infinite, then $\psi(\mathrm{End}(A)) = \chi(\mathrm{End}(A)) = |I|$.*

Theorem 4.4.110. *Let $A = \mathbb{Z}^{\mathbb{N}}$ be the Baer-Specker group. Thus $\chi(\mathrm{End}(A)) = 2^\omega$ and $\psi(\mathrm{End}(A)) = \omega$.*

5. A class of prime rings of continuous endomorphisms

In this section we will consider a "standard" ring of continuous endomorphisms. We will show that this ring is prime and we will construct an example connected with the notion of a topological ring of quotients. Usually the ring of quotients is a prime ring.

Theorem 4.5.111. *Let $p \in \mathbb{P}$ be a prime number and A a LCA group with $pa = 0$ for every $a \in A$. Then $\mathrm{End}_c(A)$ is a prime ring.*

Remark 4.5.112. *Let A be a topologically prime ring, in the abstract sense. If B is dense in A, then B is prime.*

We will present a general construction in the theory of topological rings (associativity is not assumed). Even though, the construction is not complicated, with its help we can obtain important examples of rings and modules. For instance, the notion of injective module in the category of topological modules can be obtained in the way describe further on:

Let R be an arbitrary ring and S a subring. We assume that we have on S a ring topology $\mathcal{T}$. Consider the subset $R' = \{x \in R :$ for every neighborhood V of 0_S there exists U a neighborhood of 0_S such that $xU \subset V$ and $Ux \subset V\}$. We will focus on modules case: Let R and S be two right topologically modules on a topological ring E. Then $R' = \{x \in R :$ for every neighborhood V of 0_S there exists U a neighborhood of 0_E such that $xU \subset V$.

Let R be an associative ring and $Q(R)$ Utumi's ring of quotients associated with R. If R is a topological ring then the analogue for $Q(R)$ can be defined in various forms. We mention the way proposed by Johnson:

We extend the ring R in $Q(R)$ by the above mentioned way. The ring that we obtained will be denoted by $Q'(R)$. We will illustrate this notion through the following example:

Let p be a prime number and $A = Z(p)^{\omega} \oplus (\bigoplus_{\omega} Z(p))$. Consider the ring $\mathrm{End}_c(A)$ with the compact-open topology. This ring can be embedded in the ring $\mathrm{End}(A)$ where A is considered as a discrete group. It is known that the ring $\mathrm{End}(A)$ coincides with its ring of quotients (these rings are called Kasch rings). From the structure of

A it results that the family $\{T(K, V)$: where K and V are compact open subgroups $\}$ forms a fundamental system of neighborhoods of the topological ring $\mathrm{End}_c(A)$. We will prove that $Q'(\mathrm{End}_c(A)) \neq \mathrm{End}_c(A)$.

Denote the generator of $Z(p)_i$ from the first summand by x_i and from the second by x'_i. It is obvious that the family $\{(x_i)_{i\in\omega}\}$ is linear independent on $\mathbb{F}_p$. Consider $\alpha \in \mathrm{End}(A)$ defined by the relation $\alpha(x_i) = (x_i, x'_i)$ where $i \in \omega$. We will prove that there is no neighborhood U of $0_{\mathrm{End}_c(A)}$ such that $U\alpha \subset T(K, K)$ where $K = Z(p)^\omega$. We can consider that U has the form $T(K_1, K_2)$ where $K_1 = \prod_{i=n}^{\infty} Z(p)_i$ and $K_2 = K \oplus (\bigoplus_{i=1}^{m} Z(p)_i)$, where $m \geq n$. We define $\beta \in \mathrm{End}(A)$ by the relation $\beta(K_2) = 0$ and $\beta \restriction_{\bigoplus_{j=m+1}^{\infty} Z(p)_j} = id_{\bigoplus_{j=m+1}^{\infty} Z(p)_j}$. Then $\beta\alpha(x_{m+1}) = \beta(x_{m+1} + x'_{m+1})$

$= x'_{m+1} \neq 0$, a contradiction.

6. Topological endomorphism rings with minimal topologies

The notion of a minimal topological space was introduced by A. S. Parhomenko in 1939 when he proved that compact Hausdorff spaces are minimal. Recall that a Hausdorff topological space $(X, \mathcal{T})$ is said to be *minimal* provided there is no Hausdorff topology on X coarser than $\mathcal{T}$. We study the following question: Let M_R be a right unitary module over a ring R. Under which conditions on M_R the ring $\mathrm{End}(M_R)$ furnished with the finite topology is a minimal topological ring? A positive answer is given for free modules over finite rings.

Recall [**13**] that a subring H of a topological ring R is called *essential* provided $I \cap H \neq 0$ for each closed ideal $I \neq 0$.

The following Theorem was proved for compact groups by R.M. Stephenson [**55**] in 1971 and was generalized for algebras by Banaschewski [**13**] in 1974. For reader's convenience, we will give the proof for the case of topological rings:

Theorem 4.6.113. *Let $(R, \mathcal{T})$ be a topological Hausdorff ring and H a dense subring. Then $(H, \mathcal{T}|H)$ is minimal $\Leftrightarrow$ $(R, \mathcal{T})$ is minimal and H is essential in R.*

The next Theorem indicates a class of minimal topological rings.

Theorem 4.6.114. *Let M_R be a free R-module over a finite ring with 1 and* $N := \{q \in End(M_R) : |imq| < \infty\}$. *If P is a subring of* $End(M_R)$, $N \subseteq P \subseteq End(M_R)$ *and* $\mathcal{T}_0$ *is the finite topology on* $End(M_R)$, *then* $(P, \mathcal{T}_0|P)$ *is minimal.*

7. Admissible topologies on endomorphism rings

Let A be an Abelian group and $\mathrm{End}(A)$ its ring of endomorphisms. Then ${}_{EndA}A$ is a discrete left topological module where $(\mathrm{End}(A), \mathcal{T}_0)$ is a topological ring and $\mathcal{T}_0$ is the finite topology. A ring topology $\mathcal{T}$ on $\mathrm{End}(A)$ is called *admissible* provided ${}_{EndA}A$ is a topological module where A is furnished with the discrete topology. Clearly, a ring topology $\mathcal{T}$ is admissible if $\mathcal{T} \geq \mathcal{T}_0$.

We indicate in this section a class of admissible ring topologies $\mathcal{T}_\lambda$, over the endomorphism ring of an infinite Abelian group. We also give an example of a non-discrete Hausdorff ring topology which is not comparable with the finite one, hence not admissible. We construct an example of an admissible ring topology, different from the topologies $\mathcal{T}_\lambda$.

We introduce the notion of an admissible group topology on $\mathrm{End}(A)$ and indicate some constructions of such topologies.

Consider an infinite Abelian group A and denote $|A| = \tau$. Fix an infinite cardinal number $\lambda \leq \tau$. We take as a fundamental system of neighborhoods of 0 for a ring topology $\mathcal{T}_\lambda$ on $\mathrm{End}(A)$ the family of left ideals $\{T(K)\}$ where $|K| < \lambda$.

We obtain in this way a well-ordered set of admissible topologies.

Theorem 4.7.115. *For every infinite* $\lambda \leq \tau$ *the topological ring* $(\mathrm{End}(A), \mathcal{T}_\lambda)$ *is complete.*

Bellow A is an elementary group and I is the set of all $\alpha \in \mathrm{End}(A)$ with finite $im\alpha$. The family $\mathfrak{B} = \{Ann_r(U)\}$ where U runs all the finite subsets of I, gives on $\mathrm{End}(A)$ a nondiscrete Hausdorff ring topology.

If $K \subset_f A$, then since $\mathrm{End}(A)$ is a regular von Neumann ring, $\mathrm{End}(A)K = \mathrm{End}(A)\alpha$ for some $\alpha \in \mathrm{End}(A)$ (see von Neumann's Lemma, [**39**], p. 68). Obviously, if $K \subset I$, then $\alpha \in I$. Therefore, if $K \subset I$, then $Ann_r(K) = Ann_r(\alpha)$ for some $\alpha \in I$.

Theorem 4.7.116. *The topology given by the family $\mathfrak{B}$ is not comparable with the finite topology, hence it is not admissible.*

Remark 4.7.117. *Let $\{\mathcal{U}_i\}_{i\in I}$ be a family of admissible topologies. Then $\mathcal{T} = \vee_{i\in I}\mathcal{U}_i$ is an admissible topology.*

Remark 4.7.118. *There exists a non-discrete maximal Hausdorff admissible topology.*

Recall [**44**] that a functorial topology on the category A of all Abelian groups is a functor T on A to the category of topological Abelian groups such that TA is the group A with a topology T_A and every homomorphism is continuous.

In what follows we will give an example of an admissible topology different from topologies $\mathcal{T}_\lambda$.

Let A be the free countable group and p a prime number. The family $\{p^n A\}_{n\in\omega}$ gives a nondiscrete functorial group topology $\mathcal{U}_p$ on A. Let $\mathcal{K}$ be the family of all nonempty compact subsets of $(A,\mathcal{U}_p)$. Denote by $\mathcal{V}$ the group topology given by a filter basis $\{T(K)\}_{K\in\mathcal{K}}$.

We note that $(A,\mathcal{V})$ is a complete group. We claim that $\mathcal{V}$ is a ring topology. Indeed, let $\alpha \in \mathrm{End}(A)$ and $K \in \mathcal{K}$. Since every every endomorphism of A is continuous with respect to the topology $\mathcal{V}$, $\alpha K \in \mathcal{K}$ and $T(\alpha K)\alpha \subset T(K)$. We have proved that $\mathcal{V}$ is a ring topology and thereby it is an admissible topology. We have to prove that $\mathcal{V}$ is different from the finite topology $\mathcal{T}_0$. Let $\{x_n\}_{n\in\omega}$ be a family of free generators. Thus the set $K = \{0\} \cup \{2^n x_n : n \in \omega\}$ is compact. We claim that $T(K)$ is not open with respect to the finite topology. Assume on the contrary that there exists a finite subset N of A such that $T(N) \subset T(K)$. We can assume without loss in generality that $N = \{x_0, \dots, x_n\}$, where $n \in \omega$.

Set $\alpha \in \mathrm{End}(A), \alpha(x_i) = 0$ for $i = 0, \dots, n$ and $\alpha(x_{n+1}) = x_{n+1}$. Thus $\alpha \in T(N)$ and since $\alpha(2^{n+1}x_{n+1}) = 2^{n+1}x_{n+1} \neq 0, \alpha \notin T(K)$, a contradiction.

A group topology $\mathcal{T}$ on $\mathrm{End}(A)$ is called admissible if $\mathcal{T}_0 \leq \mathcal{T}$, where $\mathcal{T}_0$ is the finite topology.

Recall that an ultrafilter $\mathfrak{F}$ on a topological group X is called *fixed ultrafilter* provided there exists $x \in X$ such that $\mathfrak{F} = \{A \subset X : x \in A\}$.

Let A be an Abelian group and $\mathfrak{F}$ a non-fixed ultrafilter on A. The family $\mathfrak{B} = \{T(A \setminus Y)\}_{Y \in \mathfrak{F}}$ where $T(A \setminus Y) = \{\alpha \in \mathrm{End}(A) : \alpha(A \setminus Y) = 0\}$ forms a fundamental system of neighborhoods of 0 of $\mathrm{End}(A)$.

Proposition 4.7.119. *The topology for which $\mathfrak{B}$ is a fundamental system of neighborhoods of 0, is an admissible group topology on* $\mathrm{End}(A)$*.*

References

[1] H. F. Abrudan: *On Regular Endomorphism Rings of Topological Abelian Groups*, accepted for publication in "Czechoslovak Mathematical Journal".

[2] H. F. Abrudan: *Topological endomorphism rings with minimal topologies*, Automation, Computers, Applied Mathematics **19**(2010), 33-36.

[3] H. F. Abrudan: *Bounded topologies on endomorphism rings*, accepted for publication in "Creative Mathematics".

[4] H. F. Abrudan: *Admissible topologies on endomorphism rings*, accepted for publication in "Carpathian Journal of Mathematics".

[5] H. F. Abrudan, M. Ursul: *Boundedness of topological endomorphism rings of torsion Abelian groups*, Contr. to Gen. Algebra **17** (2006), 1-8.

[6] H. F. Abrudan, M. Ursul: *Endomorphism rings with special neighborhoods of zero*, sent for publication in Algebra Colloquium.

[7] P. S. Aleksandrov: *Uber die Struktur der bikompakten topologischen Raume*, Math. Ann., **92** (1924), 267-274.

[8] P. S. Aleksandrov, P. Urysohn: *Memoire sur les espaces topologiques compacts*, Koninkl. Nederl. Akad. Wetensch., Amsterdam,(1929).

[9] A. Z. Anan'in: *On representability of a finite local ring*, J. of Algebra, **228** (2000), 417-427.

[10] D. M. Arnold, C. E. Murley: *Abelian groups, A, such that HOM(A,-) preserves direct sums of copies of A*, Pacific J. Math. **56** (1975), 7-20.

[11] L. Ausenhofer: *A duality property of an uncountable product of* $\mathbb{Z}$, Mathematische Zeitschrift. **257** (2007), 231-237.

[12] R. Baer: *The decomposition of abelian groups into direct summands*, Quart. J. Math. Oxford **6** (1935), 222-232.

[13] B. Banaschewski: *Minimal topological algebras* Math. Ann., **211** (1974), 107-114.

[14] G. Bergman: *Some examples in PI-ring theory*, Isr. J. Math. **18** (1974), 257.

[15] N. Bourbaki: *Obscaia topologia. Topologiceskie gruppy. Cisla i sviazannye s nimi gruppy i prostranstva*, Izdatelstvo "Nauka", Moskva, (1969).

[16] S. Brenner, M. C. R. Butler: *Endomorphism rings of vector spaces and torsion free abelian groups*, J. London Math. Soc. **40** (1965), 183-187.

[17] B. Brown, N. H. McCoy: *The Maximal Regular Ideal of a Ring*, Proc. Am. Math. Soc. **1** (1950), 165-171.

[18] A. V. Budanov: *On the Jacobson Radical of the Endomorphism Ring of a Homogeneous Separable Group*, Mat. Zametki **87** (2010), 133-136.

[19] A. L. S. Corner: *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math Soc. **13** (1963), 687-710.

[20] A. L. S. Corner: *Endomorphism rings of torsion-free abelian groups*, Proc. Internat. Conf. Theory Groups, (New York, London, Paris), (1967), 59-69.

[21] A. L. S. Corner: *Endomorphism algebras of large modules with distinguished submodules*, J. Algebra **11** (1969), 155-185.

[22] A. I. Costash, M. Ursul: *Bounded Subsets in Some Topological Rings*, Buletinul A.S. a Republicii Moldova (1995), 36-42.

[23] P. C. Eklof: *Set theoretic methods in homological algebra and Abelian groups*, Les Presses de L'Universite de Montreal, Montreal, (1980).

[24] R. Engelking: *General topology*, Heldermann Verlag, Berlin, (1989).

[25] C. Faith: *Rings and Things and a Fine Array of Twentieth Century Associative Algebra*, Mathematical Surveys and Monographs, Volume 65 American Mathematical Society (1999).

[26] L. Fuchs: *Infinite Abelian groups*, Vol. I, Academic Press, New York, (1970).

[27] L. Fuchs: *Infinite Abelian groups*, Vol. II, Academic Press, New York, (1973).

[28] L. Fuchs, K. M. Rangaswamy: *On generalized regular rings*, Math. Zeitschr. **107** (1968), 71-81.

[29] S. Glaz, W. Wickless: *Regular and principal projective endomorphism rings of mixed Abelian Groups*, Communications in Algebra **22** (1994), 1161-1176.

[30] K. R. Goodearl: *von Neumann Regular Rings*, 2nd ed., Krieger Publishing Company, Malabar, (1991).

[31] M. Hall: *The theory of groups*, New York,(1959).

[32] I. Kaplansky: *Topological rings*, Amer. J. Math. **69**(1947), 153-183.

[33] I. Kaplansky: *Selected papers and other writings*, Springer Verlag, (1995).

[34] Z. M. Kishkina: *Endomorphisms of p-primitive torsion-free abelian groups [Russian]*, Izv. Akad. Nauk SSSR **9**(1945), 201-232.

[35] M. Krol: *The automorphisrn groups and endomorphism rings of torsion-free abelian groups of rank two*, Dissertationes Math **55**(1967), 153-183.

[36] A. G. Kurosh: *Lectii po obscei algebre*, Nauka, Moscow, (1973).

[37] A. G. Kuros: *Teoria Grupurilor*, Editura Tehnica, Bucuresti, (1959).

[38] T. Lam: *A First Course in Noncommutative Rings*, Second Edition, Springer Verlag, (2001).

[39] J. Lambek: *Lectures on Rings and Modules*, Chelsea Publishing Company, New York, (1986).

[40] W. Liebert: *Charakterisierung der Endomorphismenringe endlicher abelscher Gruppen*, Arch. Math. **18** (1967), 128-135.

[41] W. Liebert: *Charakterisierung der Endomorphismenringe beschrankter abelscher Gruppen*, Math. Ann **174** (1967), 217-232.

[42] W. Liebert: *Endomorphism rings of abelian p-groups*, Studies on Abelian Groups, Paris, (1968),239-258.

[43] H. Leptin: *Linear kompakte moduln und ringe*, Math. Z. **63** (1955), 241-267.

[44] A. Mader: *Basic Concepts on Functorial Topologies*, Abelian Group Theory - Proceedings of the Oberwolfach Conference, January 12-17,1981 Springer-Verlag, Berlin Heidelberg New York (1981), 251-271.

[45] A. S. Parhomenko: *Uber eineindeutige stetige Abbildungen*, Math. Sb., **5(47)**(1939), 197-210.

[46] K. M. Rangaswamy: *Abelian groups with endomorphic images of special types*, J. of Algebra **6** (1969), 271-280.

[47] K. M. Rangaswamy: *Representing Baer rings as endomorphism rings*, Math. Ann. **190** (1970), 167-176.

[48] K. M. Rangaswamy: *Regular and Baer rings*, Proc. Am. Math. Soc. **42** (1974), 354-358.

[49] Ph. Schultz,: *On a paper of Szele and Szendrei on groups with commutative endomorphism rings*, Acta Math. Acad. Sci. Hung., **24** (1973), 59-63.

[50] K. Shoda: *Uber die Automorphismen einer endlichen abeleschen Gruppe*, Math. Ann. **100** (1928), 674-686.

[51] R. M. Stephenson Jr.: *Minimal topological groups* em Math. Ann. **192** (1971), 193-195.

[52] T. Szele: *Gruppentheoretische Beziehungen der Primkorper*, Mat. Aineiden Aikakauskirja, **13** (1949), 80-85.

[53] T. Szele: *Uber die Abelschen Gruppen mit nullteilerfreiem Endomorphismenring*, Publ. Math. (Debrecen), **1** (1949), 89-91.

[54] T. Szele, J. Szendrei: *On abelian groups with commutative endomorphism ring*, Acta Math. Acad. Sci. Hung., **2** (1951), 309-324.

[55] M. Stroppel: *Locally Compact Groups*, Version of 24.IV.1998.

[56] M. Ursul: *Topological Rings Satisfying Compactness Conditions*, Mathematics and its Applications, Vol. 549, Kluwer Academic Publishers, Dordrecht-Boston-London, (2002).

[57] M. Ursul: *Unele proprietăţi ale inelului de endomorfisme*, Bul. Acad. de Ştiinţe a Rep. Moldova, 2(**12**) (1993), 17-21.

[58] M. Ursul: *Boundedness in locally compact rings*, Topol. and its Appl., (**55**) (1994), 17-21.

[59] S. Warner: *Topological rings*, North-Holland Mathematics Studies, Vol. 178, Elsevier, Amsterdam/London/New York/Tokyo,(1993).

[60] E. Weiss: *Boundedness in topological rings*, Pacific J. Math., (**6**) (1956), 149-158.

[61] D. Zelinsky: *Linearly compact modules and rings*, Amer. J. Math. **75** (1953), 79-90.