MINISTERUL EDUCAŢIEI, CERCETĂRII ŞI TINERETULUI
UNIVERSITATEA DE NORD DIN BAIA MARE
FACULTATEA DE ŞTIINŢE
DEPARTAMENTUL DE MATEMATICĂ ŞI INFORMATICĂ

MIRCEA D. FARCAŞ

Approximation by linear operators of functions of one or several variables

Abstract of PHD Thesis

Scientific advisor: C.S. I Dr. ION PĂVĂLOIU

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Contents

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Introduction

For a function $f(x)$ defined on a closed interval $[0,1]$, the expression

$$(B_m f)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right),$$

$m \in \mathbb{N}$, is called the Bernstein polynomial of order m of the function $f(x)$. $(B_m f)(x)$ is a polynomial in x of degree $\leq m$. The polynomials $B_m f$ were introduced by S.N. Bernstein in the paper [40]. Bernstein polynomials are linear with respect to the function $f(x)$ and with the help of them we shall prove the famous theorem of Weierstrass.

H. Bohman in paper [44] and P.P. Korovkin in paper [69] discovered a test criterion for the convergence of a sequence of linear and positive operators to the identity operator in the space $C[a,b]$. Let $e_j : [a,b] \to \mathbb{R}$ be the test functions defined by $e_j(x) = x^j$, $j \in \{0,1,2\}$ and let $L_m : C([a,b]) \to C([a,b])$, $m \in \mathbb{N}$ be a sequence of linear positive operators. If $\lim\limits_{m\to\infty} L_m e_j = e_j$, $j \in \{0,1,2\}$, uniformly on $[a,b]$, then for any function $f \in C([a,b])$, we have $\lim\limits_{m\to\infty} L_m f = f$, uniformly on $[a,b]$.

O. Shisha and B. Mond in paper [110] obtained some estimations for the errors of approximation.

Chapter 1 contains some basic results (approximation theorems, modulus of continuity, evaluation of errors theorems) and some new results about the approximation of one-variable functions: moments and central moments of Bernstein and Schurer operators, recursive formulas, the study of approximation properties for a class of linear positive operators obtained by choosing the nodes, Voronovskaja type theorems.

Chapter 2 contains some known results about the approximation of bivariate functions (Korovkin type and Shisha-Mond type theorems, B-continuous and B-differentiable functions, total and mixed modulus of continuity) and some new results about mean-valued theorems for B-differentiable functions (section 2.2), Bernstein bivariate operators (section 2.3), Schurer bivariate operators (section 2.4), Stancu and Schurer-Stancu bivariate operators (section 2.5), Kantorovich and Durrmeyer bivariate operators (section 2.6), approximation of functions of several variables (section 2.8).

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1. Approximation of univariate functions

This chapter presents some notions and results about approximation of univariate functions. In papers [52] and [94] we get some results about Bernstein operators.

Lemma 1. *For $m \in \mathbb{N}$ and $q \in \mathbb{N}_0$, we have*

$$(B_m e_q)(x) = \frac{1}{m^q} \sum_{k=0}^{q} a_{q,k}(x) m^k, \tag{1}$$

where

$$a_{q,k}(x) = \sum_{j=k}^{q} S(q,j) s(j,k) x^j, \tag{2}$$

for $k \in \{0, 1, \ldots, q\}$.

Theorem 1. *We have the formula*

$$(B_m e_{q+1})(x) = x(B_m e_q)(x) + \frac{x(1-x)}{m}(B_m e_q)'(x) \tag{3}$$

for $m \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $x \in [0,1]$.

Theorem 2. *The central moments of Bernstein polynomials admit the representation*

$$(B_m(\cdot - x)^q)(x) = \frac{1}{m^q} \sum_{k=0}^{q} b_{q,k}(x) m^k \tag{4}$$

where

$$b_{q,k}(x) = \sum_{j=0}^{k} (-1)^j \binom{q}{j} x^j a_{q-j,k-j}(x), \tag{5}$$

$k \in \{0, 1, \ldots, q\}$.

Theorem 3. *We have $\lim\limits_{m\to\infty} (B_m(* - x)^q)(x) = 0$, for $q \geq 1$, $\lim\limits_{m\to\infty} m(B_m(* - x)^q)(x) = 0$, for $q \geq 3$ and $\lim\limits_{m\to\infty} m^2(B_m(* - x)^q)(x) = 0$, for $q \geq 5$.*

In paper [53] we obtain some results about Schurer polynomials.

Lemma 2. *We have the formula*

$$\tilde{B}_{m,p}e_q = \frac{1}{m^q}\sum_{k=0}^{q}\tilde{a}_{q,k}(x)m^k, \tag{6}$$

where

$$\tilde{a}_{q,k}(x) = \sum_{\nu=k}^{q}\binom{\nu}{k}a_{q,\nu}(x)p^{\nu-k}, \tag{7}$$

$k \in \{0,1,\dots,q\}$ *and* $a_{q,\nu}(x)$ *are given by* (2).

Theorem 4. *The central moments of Bernstein-Schurer operators associated to the test functions admit the representation*

$$(\tilde{B}_{m,p}(\cdot - x)^q)(x) = \frac{1}{m^q}\sum_{k=0}^{q}\tilde{b}_{q,k}(x)m^k, \tag{8}$$

where

$$\tilde{b}_{q,k}(x) = \sum_{\nu=0}^{k}(-1)^\nu\binom{q}{\nu}\tilde{a}_{q-\nu,k-\nu}(x)x^\nu, \tag{9}$$

$k \in \{0,1,\dots,q\}$.

Theorem 5. *We have* $\lim_{m\to\infty}(\tilde{B}_{m,p}(\cdot - x)^q)(x) = 0$, *for* $q \geq 1$, $\lim_{m\to\infty} m(\tilde{B}_{m,p}(\cdot - x)^q)(x) = 0$, *for* $q \geq 3$ *and* $\lim_{m\to\infty} m^2(\tilde{B}_{m,p}(\cdot - x)^q)(x) = 0$, *for* $q \geq 5$.

In papers [54], [95], [96] and [97] we studied a class of linear operators with changed nodes.

We define the operators $A_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$, by

$$(A_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x)f(x_{m,k}), \tag{10}$$

where the numbers $x_{m,k}$ verify the relations:

$$x_{m,k} \in [0,1], \tag{11}$$

for any $m \in \mathbb{N}$ and $k \in \{0,1,\dots,m\}$ and

$$\lim_{m\to\infty}\alpha_m = 0, \tag{12}$$

where

$$\alpha_m = \max_{k\in\{0,1,\dots,m\}}\left|x_{m,k} - \frac{k}{m}\right|.$$

Further, we define the operators $\tilde{A}_m : C([0, 1 + p]) \to C([0, 1])$, $m \in \mathbb{N}$, by

$$(\tilde{A}_m f)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k} f(x_{m,k}) \tag{13}$$

where the numbers $x_{m,k}$ verify the relations:

$$x_{m,k} \in [0, 1 + p], \tag{14}$$

for any $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m + p\}$ and

$$\lim_{m\to\infty} \beta_m = 0, \tag{15}$$

where

$$\beta_m = \max_{k\in\{0,1,\ldots,m+p\}} \left| x_{m,k} - \frac{k}{m} \right|$$

Theorem 6. *If $f \in C([0, 1])$ then for any $x \in [0, 1]$ and $m \in \mathbb{N}$ we have that*

$$|(A_m f)(x) - f(x)| \leq 2\omega_f(\delta_m) \tag{16}$$

where $\delta_m = \sqrt{4\alpha_m + \frac{1}{4m}}$.

Theorem 7. *If $f \in C([0, 1 + p])$ then for any $x \in [0, 1]$ and $m \in \mathbb{N}$, $m > p^2 - p$ we have that*

$$|\tilde{A}_m f)(x) - f(x)| \leq 2\omega_f(\delta_{m,p}) \tag{17}$$

where

$$\delta_{m,p} = \sqrt{2(2 + p)\beta_m + \frac{(m + p)^2}{4m^2(m - p^2 + p)}}.$$

Let I, J be real intervals with $I \cap J \neq \emptyset$ and $p_m = m$ for any $m \in \mathbb{N}$ (the finite case) or $p_m = \infty$ for any $m \in \mathbb{N}$ (the infinite case). For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$. We suppose that for any compact $K \subset I \cap J$ there exists the sequence $(u_m(K))_{m\geq 1}$, depending on K so that

$$\lim_{m\to\infty} u_m(K) = 0 \tag{18}$$

uniformly on K and

$$\left| \sum_{k=0}^{p_m} \varphi_{m,k}(x) - 1 \right| \leq u_m(K) \tag{19}$$

for any $x \in K$, any $m \in \mathbb{N}$ and we note $u(K) = \sup\{|u(x)| : x \in K\}$.

We consider the operators $(L_m)_{m\geq 1}$ defined by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k}) \tag{20}$$

for any $f \in E(w)$, $x \in J$ and $m \in \mathbb{N}$, with the property that $\lim_{m\to\infty} L_m f = f$, uniformly on any compact $K \subset I \cap J$.

For $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, consider the nodes $y_{m,k} \in I$ such that

$$\alpha_m = \sup_{k\in\{0,1,\ldots,p_m\}\cap\mathbb{N}_0} |x_{m,k} - y_{m,k}| < \infty \tag{21}$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \alpha_m = 0. \tag{22}$$

For $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, denote $\alpha_{m,k} = x_{m,k} - y_{m,k}$.

Definition 1. *For $m \in \mathbb{N}$ define the operator $K_m : E(I) \to F(J)$ by*

$$(K_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(y_{m,k}), \tag{23}$$

for any $x \in I$.

Theorem 8. *For any function $f \in E(I) \cap C(I)$, we have*

$$\lim_{m\to\infty} (K_m f)(x) = f(x) \tag{24}$$

uniformly on any compact $K \subset I \cap J$.

Theorem 9. *If $f \in E(I \cap J) \cap C(I \cap J)$, then for any $x \in K = [a, b] \subset I \cap J$ şi orice $m \in \mathbb{N}$, we have*

$$\begin{aligned} |(K_m f)(x) - f(x)| &\leq |f(x)||(L_m e_0)(x) - 1| + \\ &+ ((L_m e_0)(x) + 1)\omega(f; \delta_{m,x}) \leq M u_m(K) + (2 + u_m(K))\omega(f; \delta_m), \end{aligned} \tag{25}$$

where
$\delta_{m,x} = \sqrt{(L_m e_0)(x)[(L_m \psi_x^2)(x) + 2\alpha_m (L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x)]}$,
$\delta_m = \sqrt{(1 + u_m(K))[w_m(K) + 2\alpha_m(b + v_m(K) + (\alpha_m^2 + 2b\alpha_m)(1 + u_m(K))]}$ *and* $M = \sup\{|f(x)| : x \in K\}$.

We can obtain convergence and approximation theorems for the new operators by particularizing of the sequence $y_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$.

Furthermore, we consider a weight function $w : I \to (0, \infty)$ with the property that there exists a constant $L > 0$ so that $L \leq w(x)$, for any $x \in I$ and the set

$$E_w(I) = \{f : I \to \mathbb{R}|\ wf \text{ bounded on } I\}. \tag{26}$$

In the following, let s be fixed natural number, s even. For any $x \in I \cap J$ we suppose that $\psi_x^i \in E(w)$, where $i \in \{0, 1, \ldots, s+2\}$. For $m \in \mathbb{N}$ and $i \in \{0, 1, \ldots, s+2\}$ define

$$T_{m,i}(x) = m^i \sum_{k=0}^{p_m} (x_{m,k} - x)^i \varphi_{m,k}(x) \tag{27}$$

for any $x \in I \cap J$.

Theorem 10. *If $f \in E_w(I)$ is a s times differentiable at $x \in I \cap J$ (if $s = 0$ consider f continuous at x) and we suppose that there exists $\lambda_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ so that $\frac{(T_{s+2}L_m)(x)}{m^{\lambda_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, then for any γ so that*

$$\gamma < s + 2 - \lambda_{s+2}, \tag{28}$$

we have

$$\lim_{m\to\infty} m^\gamma \left[(L_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} (T_i L_m)(x) f^{(i)}(x) \right] = 0. \tag{29}$$

If $f \in E_w(I)$ is a s times differentiable on I and for the compact $K \subset I \cap J$ there exists $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and for any $x \in K$ we have

$$\frac{(T_{s+2}L_m)(x)}{m^{\lambda_{s+2}}} \leq k_{s+2}(K), \tag{30}$$

then the convergence above is uniform on K.

Theorem 11. *If f is continuous on I and $K \subset I \cap J$ is a compact, then we have*

$$\left| (L_m f)(x) - \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x) \right| \leq (k_0(K) + k_2(K)) \omega \left(f; \frac{1}{\sqrt{m^{2-\lambda_2}}} \right) \tag{31}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Theorem 12. *If $f \in E_w(I)$ is continuous at $x \in I \cap J$, then*

$$\lim_{m\to\infty} (K_m f)(x) = f(x). \tag{32}$$

If the function f is continuous on I and $K \subset I \cap J$ is a compact, then the convergence above is uniform on K and we have

$$\left| (K_m f)(x) - \left(\sum_{k=0}^{p_m} \varphi_{m,k}(x) \right) f(x) \right| \leq (k_0'(K) + k_2'(K)) \omega \left(f; \frac{1}{\sqrt{m^{2-\lambda_2}}} \right) \tag{33}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Theorem 13. *If $f \in E_w(I)$ is twice differentiable at $x \in I \cap J$, with $f^{(2)}$ continuous at x and $\dfrac{(T_4 L_m)(x)}{m^{\lambda_4}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, then*

$$\lim_{m\to\infty} m^{2-\lambda_2}\Bigg[(K_m f)(x) - (T_0 L_m)(x) f(x) - \frac{1}{m}(T_1 L_m)(x) f^{(1)}(x) - \\ - \frac{1}{2m^2}(T_2 L_m)(x) f^{(2)}(x)\Bigg] = 0. \tag{34}$$

2. Approximation of bivariate functions

This chapter presents some results about the approximation of continuous, B-continuous and B-differentiable bivariate functions. For B-differentiable functions we obtain some mean-value theorems of Pompeiu, Boggio and Ivan-type, in paper [98].

Theorem 14. *Let $f : [a, b] \times [a', b'] \to \mathbb{R}$ be a B-differentiable function on $[a, b] \times [a', b']$ and $d \notin [a, b]$, $d' \notin [a', b']$. Then there exists a point $(\xi, \eta) \in (a, b) \times (a', b')$ such that*

$$\Delta \frac{1}{(\cdot - d)(* - d')}\,[(a, a'), (b, b')] D_B \frac{f(\cdot\,, *)}{(\cdot - d)(* - d')}\,(\xi, \eta) = \\ = \Delta \frac{f(\cdot\,, *)}{(\cdot - d)(* - d')}\,[(a, a'), (b, b')] D_B \frac{1}{(\cdot - d)(* - d')}\,(\xi, \eta). \tag{35}$$

If in addition f admits the derivatives f'_x, f'_y, f''_{xy} on $[a, b] \times [a', b']$ and the derivative f''_{xy} is continuous on $(a, b) \times (a', b')$ then

$$\frac{aa'f(b, b') - a'bf(a, b') - ab'f(b, a') + bb'f(a, a')}{(a - b)(a' - b')} - \xi\eta f''_{xy}(\xi, \eta) + \\ + \xi f'x(\xi, \eta) + \eta f'y(\xi, \eta) - f(\xi, \eta) = (dd' - \xi d' - \eta d) f''_{xy}(\xi, \eta) + \\ + d f'_x(\xi, \eta) + d' f'y(\xi, \eta) + \\ - \frac{(dd' - ad' - a'd) f(b, b') - (dd' - bd' - a'd) f(a, b')}{(a - b)(a' - b')} + \\ + \frac{(dd' - ad' - b'd) f(b, a') - (dd' - b'd - bd') f(a, a')}{(a - b)(a' - b')}. \tag{36}$$

Theorem 15. *Let $f,g:[a,b]\times[a',b']\to\mathbb{R}$ be two functions B-differentiable on $[a,b]\times[a',b']$. If $g(x,y)\neq 0$ for any $(x,y)\in[a,b]\times[a',b']$, then there exists $(\xi,\eta)\in(a,b)\times(a',b')$ such that*

$$\begin{aligned}(37)\qquad &\Delta\frac{f(\cdot\,,*)}{g(\cdot\,,*)}\,[(a,a'),(b,b')]D_B\frac{*}{g(\cdot\,,*)}\,(\xi,\eta)=\\ &=\Delta\frac{*}{g(\cdot\,,*)}\,[(a,a'),(b,b')]D_B\frac{f(\cdot\,,*)}{g(\cdot\,,*)}\,(\xi,\eta).\end{aligned}$$

If in addition f, g admit the derivatives f'_x, g'_x, f'_y, g'_y, f''_{xy}, g''_{xy} on $[a,b]\times[a',b']$ and the derivatives f''_{xy}, g''_{xy} are continuous on $(a,b)\times(a',b')$ then

$$\begin{aligned}(38)\qquad &\left[\frac{f(a,a')}{g(a,a')}-\frac{f(b,a')}{g(b,a')}-\frac{f(a,b')}{g(a,b')}+\frac{f(b,b')}{g(b,b')}\right]\big[2\eta g'_x(\xi,\eta)g'_y(\xi,\eta)-\\ &-g(\xi,\eta)g'_x(\xi,\eta)-\eta g(\xi,\eta)g''_{xy}(\xi,\eta)\big]=\\ &=\left[\frac{a'}{g(a,a')}-\frac{a'}{g(b,a')}-\frac{b'}{g(a,b')}+\frac{b'}{g(b,b')}\right]\big[2f(\xi,\eta)g'_x(\xi,\eta)g'_y(\xi,\eta)-\\ &-g(\xi,\eta)f'_x(\xi,\eta)g'_y(\xi,\eta)-g(\xi,\eta)f'_y(\xi,\eta)g'_x(\xi,\eta)-\\ &-f(\xi,\eta)g(\xi,\eta)g''_{xy}(\xi,\eta)+g^2(\xi,\eta)f''_{xy}(\xi,\eta)\big].\end{aligned}$$

Theorem 16. *Let $f:[a,b]\times[a',b']\to\mathbb{R}$ be a B-differentiable function, $d\in\mathbb{R}$ such that $f(x,y)\neq d$ for any $(x,y)\in[a,b]\times[a',b']$. Then there exists $(\xi,\eta)\in(a,b)\times(a',b')$ such that*

$$\begin{aligned}(39)\qquad &\Delta\frac{\cdot\,*}{f(\cdot\,,*)-d}\,[(a,a'),(b,b')]D_B\frac{*}{f(\cdot\,,*)-d}\,(\xi,\eta)=\\ &=\Delta\frac{*}{f(\cdot\,,*)-d}\,[(a,a'),(b,b')]D_B\frac{\cdot\,*}{f(\cdot\,,*)-d}\,(\xi,\eta).\end{aligned}$$

If in addition f admits the derivatives f'_x, f'_y, f''_{xy} on $[a,b]\times[a',b']$, f''_{xy} continuous on $(a,b)\times(a',b')$ then

$$\begin{aligned}(40)\qquad &\left[\frac{aa'}{f(a,a')-d}-\frac{ba'}{f(b,a')-d}-\frac{ab'}{f(a,b')-d}+\frac{bb'}{f(b,b')-d}\right]\cdot\\ &\cdot\big[2\eta f'_x(\xi,\eta)f'_y(\xi,\eta)-(f(\xi,\eta)-d)f'_x(\xi,\eta)-\\ &-\eta(f(\xi,\eta)-d)f''_{xy}(\xi,\eta)\big]=\\ &=\left[\frac{a'}{f(a,a')-d}-\frac{a'}{f(b,a')-d}-\frac{b'}{f(a,b')-d}+\frac{b'}{f(b,b')-d}\right]\cdot\\ &\cdot\big[f^2(\xi,\eta)-\eta f(\xi,\eta)f'_y(\xi,\eta)-\xi f(\xi,\eta)f'_x(\xi,\eta)-\\ &-\xi\eta f(\xi,\eta)f''_{xy}(\xi,\eta)+2\xi\eta f'_x(\xi,\eta)f'_y(\xi,\eta)\big].\end{aligned}$$

For the Bernstein bivariate operators defined on the triangle Δ_2, we gave a recursive formula in paper [52] and a formula for the moments $B_m e_{pq}$ in paper [55].

Theorem 17. *If $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$, then*

$$(B_m e_{pq})(x,y) = \frac{1}{m^{p+q}} \sum_{i=0}^{p} \sum_{j=0}^{q} m^{[i+j]} S(p,i) S(q,j) x^i y^j, \tag{41}$$

for any $x, y \in \Delta_2$.

Theorem 18. *If $m \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ şi $(x,y) \in \Delta_2$, then*

$$(B_m e_{p+1q})(x,y) = \frac{x(1-x)}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) + x (B_m e_{pq})(x,y) - \tag{42}$$
$$- \frac{xy}{m} \frac{\partial}{\partial y} (B_m e_{pq})(x,y),$$

$$(B_m e_{pq+1})(x,y) = \frac{y(1-y)}{m} \frac{\partial}{\partial y} (B_m e_{pq})(x,y) + y (B_m e_{pq})(x,y) - \tag{43}$$
$$- \frac{xy}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y).$$

In papers [99] şi [100] we gave the approximation theorems for continuous, respectively B-continuous and B-differentiable bivariate functions on Δ_2 with Bernstein operators.

Theorem 19. *If $f \in C(\Delta_2)$, then for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, we have*

$$|(B_m f)(x,y) - f(x,y)| \le \left(1 + \delta_1^{-1} \frac{1}{2\sqrt{m}}\right) \cdot \tag{44}$$
$$\cdot \left(1 + \delta_2^{-1} \frac{1}{2\sqrt{m}}\right) \omega_{total}(f; \delta_1, \delta_2),$$

for any δ_1, $\delta_2 > 0$ and

$$|(B_m f)(x,y) - f(x,y)| \le 4 \omega_{total} \left(f; \frac{1}{2\sqrt{m}}, \frac{1}{2\sqrt{m}}\right). \tag{45}$$

Theorem 20. *If $f \in C_b(\Delta_2)$, then for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, we have*

$$|(UB_m f)(x,y) - f(x,y)| \le \left(1 + \delta_1^{-1} \frac{1}{2\sqrt{m}} + \delta_2^{-1} \frac{1}{2\sqrt{m}} + \right. \tag{46}$$
$$\left. + \delta_1^{-1} \delta_2^{-1} \frac{1}{2m}\right) \omega_{mixed}(f; \delta_1, \delta_2),$$

for any δ_1, $\delta_2 > 0$ and

$$|(UB_m f)(x,y) - f(x,y)| \le \frac{5}{2} \omega_{mixed} \left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right). \tag{47}$$

Theorem 21. *If $f \in D_b(\Delta_2)$, with $D_B f \in B(\Delta_2)$, then for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, $m \geq 2$, we have*

$$(48) \qquad |(UB_m f)(x,y) - f(x,y)| \leq \frac{3}{2m}\|D_B f\|_\infty + \left(\frac{1}{2m} + \delta_1^{-1}\frac{1}{2m\sqrt{m}} + \right.$$

$$\left. + \delta_2^{-1}\frac{1}{2m\sqrt{m}} + \delta_1^{-1}\delta_2^{-1}\frac{1}{4m^2}\right)\omega_{mixed}(D_B f; \delta_1, \delta_2),$$

for any δ_1, $\delta_2 > 0$ and

$$(49) \qquad |(UB_m f)(x,y) - f(x,y)| \leq \frac{3}{2m}\|D_B f\|_\infty +$$

$$+ \frac{7}{4m}\omega_{mixed}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right).$$

In paper [56] we studied the approximation on Δ_2 by bivariate Schurer operators

$$(50) \qquad (\tilde{B}_{m,p} f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m+p}} p_{m+p,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right),$$

The aprroximation on Δ_2 of the bivariate functions by Stancu and Schurer-Stancu operators was studied in papers [57] şi [58].

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be real parameters which verify the relations $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. For $m \in \mathbb{N}$, the operator $S_m^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} : C([0,1] \times [0,1]) \to C(\Delta_2)$, defined for any function $f \in C([0,1] \times [0,1])$ by

$$(51) \qquad (S_m^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{m+\beta_2}\right),$$

for any $(x,y) \in \Delta_2$, is a bivariate operator of Stancu type.

Let $p \in \mathbb{N}_0$ be given and $\alpha_1, \beta_1, \alpha_2, \beta_2$ real parameters such that $0 \leq \alpha_1 \leq \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. For $m \in \mathbb{N}$, the operator $\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$, defined for any function $f \in C([0,1+p] \times [0,1+p])$ by

$$(52) \qquad (\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m+p}} p_{m+p,k,j}(x,y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{m+\beta_2}\right),$$

for any $(x,y) \in \Delta_2$, is a bivariate operator of Schurer-Stancu type.

In papers [101] and [102], we studied the approximation properties for Kantorovich and Durrmeyer bivariate operators on Δ_2 and we proved some approximation theorems.

For $m \in \mathbb{N}$, consider the operator $\mathcal{K}_m : L_1([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ defined for any function $f \in L_1([0,1] \times [0,1])$ by

$$(\mathcal{K}_m f)(x,y) = (m+1)^2 \sum_{\substack{k,j=0 \\ k+j \le m}} p_{m,k,j}(x,y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} f(s,t)\, ds\, dt \tag{53}$$

for any $(x,y) \in \Delta_2$, which is a Kantorovich type operator.

For $m \in \mathbb{N}$, the operator $\mathcal{M}_m : L_1(\Delta_2) \to \mathcal{F}(\Delta_2)$, defined for any function $f \in L_1(\Delta_2)$ by

$$(\mathcal{M}_m f)(x,y) = (m+1)(m+2) \sum_{\substack{k,j=0 \\ k+j \le m}} p_{m,k,j}(x,y) \iint_{(\Delta_2)} p_{m,k,j}(s,t) f(s,t)\, ds\, dt \tag{54}$$

for $(x,y) \in \Delta_2$, is a Durrmeyer type operator.

In paper [59] we studied the approximation properties for some bivariate opertators obtained by changing the nodes:

$$(K_{m,p} f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \le m+p}} \varphi_{m+p,k,j}(x,y) f(u_{m,k}, v_{m,j}). \tag{55}$$

In the paper [60] we studied the approximation properties for the Bernstein operators of three variables on the tetrahedron Δ_3 for continuous and B-continuous functions.

In paper [61] we gave a formula for Bernstein multivariate fundamental polynomials:

Theorem 22. *For $m, n \in \mathbb{N}$ şi $\mathbf{x} \in \Delta_n$, we have the formula*

$$\sum_{\substack{\mathbf{i} \\ 0 \le |\mathbf{i}| \le m}} \prod_{j=1}^{n} \left(\frac{i_j}{m} - x_j \right) p_{m,\mathbf{i}}(\mathbf{x}) = \frac{x_1 x_2 \dots x_n}{m^n} \sum_{i=0}^{n} a_i m^i \tag{56}$$

where

$$a_i = \sum_{j=0}^{i} (-1)^j \binom{n}{j} s(n-j, i-j), \tag{57}$$

$i \in \{0, 1, \dots, n\}$.

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