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SUMAR — CONTENTS — SOMMAIRE

Recenzii — Book Reviews — Comptes rendus

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ERROR ESTIMATES IN THE APPROXIMATION OF THE FIXED POINTS FOR A CLASS OF φ-CONTRACTIONS

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REZUMAT. — Estimări ale erorii în aproximarea punctelor fixe pentru o clasă de φ-contracţii. Lucrarea prezintă o clasă de φ-contracţii, cu φ funcţie de comparaţie care verifică condiţia de convergenţă (c), pentru care estimarea erorii de aproximare a punctului fix prin metoda aproximaţiilor succesive este dată de aceeaşi formulă ca şi cea din teorema de contracţie a lui Banach ([3]).

1. **Introduction.** The paper shows that, for the class of φ-contractions with φ a comparison function which satisfies a convenient convergence condition (c): There exist the numbers k_0 and α, $0 < \alpha < 1$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} a_k$ such that

$$\varphi^{k+1}(r) \leqslant \alpha[\varphi^k(r) + a_k], \text{ for each } k \geqslant k_0 \text{ and } r \in \mathbf{R}_+,$$

the estimation of aproximation error of the fixed point by means of the succesive approximations method is given by the same formulas as in the Banach's contractions theorem ([3]).

To this end, we make use of a generalization of the ratio (or D'Alembert's) test for the series of positive terms, established in [1].

Let (X, d) be a complete metric space, $f: X \to X$ a mapping and $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ a monotone increasing comparison function, such as: $\varphi: \mathbb{R}^+ \to$

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)), \text{ for all } x, y \in X. \quad (1) \tag{1}$$

We construct the sequence of succesive approximations, $(x_n)_{n\in N}, x_n = f(x_{n-1})$ $n \geqslant 1$ and $x_0 \in X$, and we obtain from (1), using the montonicity of φ, that, (see [3], p. 80). $x_n = f(x_{n-1})$

$$d(x_n, x_{n+p}) \leqslant \sum_{k=n}^{n+p-1} \varphi^k(d(x_0, x_1)) \text{ for each } p \geqslant 1, \quad n \in \mathbf{N}^*. \quad (2) \tag{2}$$

If the series of positive terms

$$\sum_{k=1}^{\infty} \varphi^k(r), \quad (3) \tag{3}$$

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converges, for every $r \in \mathbf{R}_+$, then the sequence $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence, hence $(x_n)_{n\in\mathbf{N}}$ is convergent for all $x_0 \in X$.

The main purpose of this paper is to prove that, if φ fulfils the condition (c), then the series (3) converges for all $r \in \mathbf{R}^+$ and, consequently, we have the estimation (5).

The sequence $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence even when φ satisfies more weaker conditions (see [4], Theorem 3.3.1) which are, generally, insufficent to assure the convergence of the series (3).

However, in this case, in order to evaluate the approximation error, we need some additional hypotheses (see [4], Remark 3.3.1).

2. **A necessary and sufficient test for the convergence of the series of decreasing positive terms.** In [1] has been given the following generalization of the ratio test.

THEOREM 1. *Let* $\sum_{n=1}^{\infty} u_n$ *be an infinite series of positive terms. If there exists convergent series of nonnegative terms* $\sum_{n=1}^{\infty} v_n$ *and two numbers* k, n_0, *such as*

$$\frac{u_{n+1}}{u_n + v_n} \leqslant k < 1, \; for \; n \geqslant n_0 \tag{4}$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

Remarks. 2. Recall that a series is of positive (nonnegative) terms if all its terms are strictly positive (respectively positive, and an infinity of them may be equal to zero).

3. The Theorem 1 applies in some typical situations when the ratio test fails (see [1]). Obviously, the ratio test is obtained from Theorem 1, for $v_n = 0$, $n \in \mathbf{N}$.

In what follows we give a short new proof of Theorem 1:

From (4) we obtain, by an elementary calculation,

$$u_n \leqslant u_1 k^{n-1} + (v_1 k^{n-1} + v_2 k^{n-2} + \ldots + v_{n-1} k).$$

In view of Martens's theorem ([2]), to prove Theorem 1, it suffices to observe that Σk^n and Σv_n are both absolutely convergent.

For the series of decreasing positive terms, we can prove the converse of theorem 1. Thus, we obtain:

THEOREM 2. *A series* $\sum_{n=1}^{\infty} u_n$ *of decreasing positive terms converges if and only if there exists a convergent series of nonnegative terms* $\sum_{n=1}^{\infty} v_n$, *and two numbers* k, n_0, *such as the condition* (4) *is satisfied.*

Proof. The sufficiency follows from Theorem 1. To prove the necessity is enough to take $v_n = a u_n$, with $a > 0$.

Remark 4. Throughout this paper we shall consider series of decreas positive terms, because, for any comparison function, $\varphi(t) < t$ implies $\varphi^{k+1}(t)$ $\leqslant \varphi^{k}(t)$.

3. **The error evaluation.** For the definitions and basic properties conc ning comparison functions we refer to [4].

DEFINITION 1. ([4]) A monotone increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ wh satisfies the condition

(i) $(\varphi^n(r))_{n\in\mathbf{N}}$ converges to 0, as $n \to \infty$, for any $r \in \mathbf{R}_+$, is called *compar function.* comparison

DEFINITION 2. ([4]) Let (X, d) be a metric space. A mapping $f : X \to$ is called φ-*contraction* if and only if there exists a comparison function φ so th (1) is fulfilled. $f: X \to X$

DEFINITION 3. A monotone increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ which s tisfies the condition (c) is called *(c)-comparison function.*

Every (c)-comparison function is a comparison function.

Proof. Applying Theorem 1, we deduce that series (3) converges for a $r \in \mathbf{R}^+$, hence $\varphi^k(t)$ tends to zero as k tends to infinity, which proves lemm

Remark 3. It is not quite obvious that every φ-contraction is a continuo mapping. The following property of a comparison function.

$$\varphi(t) < t, \text{ for each } t < 0$$

(see [4], lemma 3.1.3) suggests a way to remove any doubts. The main res of this paper is given by

THEOREM 3. *Let (X, d) be a complete metric space and $f : X \to X$ a φ-contra tion with φ (c)-comparison function.* a φ-contract

Then

1) $F_f = \{x^*\}$;

2) *The sequence $(x_n)_{n\in\mathbf{N}}$, $x_n = f(x_{n-1})$, $n \geqslant 1$, $x_0 \in X$ converges to x^*, for an $x_0 \in X$;* for any

3) *We have*

$$d(x_n, x^*) \leqslant s(d(x_0, x_1)) - S_{n-1}(d(x_0, x_1)), \quad (5) \qquad (5)$$

where $s(r)$, $S_{n-1}(r)$ denote the sum, respectively the partial sum of rank $n - 1$ of th series (3); of the

4) *If, in addition, φ is subadditive and there exists a mapping $g : X \to X$ and $\eta > 0$ such as* $g: X \to X$

$$d(f(x), g(x)) \leqslant \eta, \text{ for any } x \in X,$$

then

$$d(y_n, x^*) \leqslant \eta + s(\eta) + s(d(x\ , x_1)) - S_{n-1}(d(x_0, x_1)), \quad (6) \qquad (6)$$

where $y_n = g\,(x\ )$.

Proof. Since φ is (c)-comparison function, the convergence of $(x_n)_{n\in\mathbf{N}}$ follow immediately from (2). Let x^* be its limit.

Then, the continuity of f, leads to $x^* = f(x^*)$ and x^* is the unique fixed point of f. Thus 1) and 2) are proved.

Also, (5) follows from (2), latting $p \to \infty$.

Finally, we have

$$d(y_n, x^*) \leqslant d(y_n, x_n) + d(x_n, x^*)$$

and

$$d(y_n, x_n) \leqslant \eta + \varphi(d(y_{n-1}, x_{n-1})) \leqslant \ldots \leqslant \eta + \varphi(\eta) + \ldots + \varphi^n(\eta).$$

Since $S_n(\eta) \leqslant s(\eta)$, the preceding inequalities together with (5), give the required estimation (6).

Remarks. 5. If $\varphi(t) = at$, $0 < a < 1$, then, from Theorem 3 we obtain Theorem 3.2.1 [2];

6. Theorem 2 shows that the estimation (5) holds if and only if φ is a (c)-comparison function.

7. Finally, let us observe that $s(r) - S_{n-1}(r)$ is the remainder of rank n of the series (3). By (5) we deduce that $(x_n)_{n\in\mathbf{N}}$ converges to x^* no more quickly than the series (3) to s.

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