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Existence and Approximation of Fixed Points of Enriched Contractions and Enriched φ-Contractions

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Abstract: We obtain existence and uniqueness fixed point theorems as well as approximation results for some classes of mappings defined by symmetric contractive type conditions in a convex metric space in the sense of Takahashi. By using a new approach, i.e., the technique of enrichment of contractive type mappings, we obtain general results which extend the well known Banach contraction mapping principle from metric spaces as well as other corresponding results for enriched mappings defined on Banach spaces. To indicate the relevance of our new results, we present some important particular cases and future directions of research.

Keywords: metric space; uniform structure; enriched contraction; enriched φ-contraction; fixed point; iterative method

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1. Introduction

Fixed point theory offers important tools for nonlinear analysis in the study of the existence and approximation of the solutions of nonlinear functional equations (differential equations, integral equations, integro-differential equations, etc.) (see the comprehensive monograph [1]). The sought solution of such a functional equation is expressed as the fixed point of a suitable operator, i.e., as the solution of the fixed point problem,

$$x = Tx, \tag{1}$$

where T is defined on a space X endowed with a certain structure, while the problem (1) is solved by applying an appropriate fixed point theorem.

The most useful metrical fixed point theorem in nonlinear analysis is without doubt the famous Banach contraction mapping principle, which is based on a *symmetric* contraction condition

$$d(Tx, Ty) \leq c \cdot d(x, y),\ x, y \in X \quad (0 \leq c < 1). \tag{2}$$

Most of the contractive conditions which generalize Banach's contraction condition (2) are symmetric (see the important monograph [2]), but there exist important fixed point theorems (e.g., the ones corresponding to *almost contractions*, see [3]), which are not symmetric but still ensure the existence and approximation of fixed points.

On the other hand, for many of the fixed point theorems established in metric spaces, we need some additional geometric properties of the space X, related to convexity in the usual sense for subsets of Euclidian space, and expressed by the fact that, for any two distinct points x and y in X, there exists a third point z in X *lying between* x and y.

In 1970, Takahashi [4] introduced a notion of *convexity structure* in a metric space with the aim of studying the fixed point problem for nonexpansive mappings in such

spaces. A convex metric space offers the minimal tools for constructing various fixed point iterative methods for approximating fixed points of nonlinear operators, such as Krasnoselskij, Mann and Ishikawa fixed point iterative schemes, which require the linearity and convexity of the ambient topological space.

This is the main reason, after the pioneering work by Takahashi [4], several authors studied fixed point problems in the setting of a Takahashi convex metric space, e.g., Machado [5], Talman [6], Itoh [7], Naimpally, Singh and Whitfield [8,9], Ding [10], Ciric [11], Shimizu and Takahashi [12], Huang [13], Popa [14], Beg [15], Chang, Kim and Jin [16], Sharma and Deshpande [17], Tian [18], Beg and Abbas [19,20], Beg, Abbas and Kim [21], Aoyama, Eshita and Takahashi [22], Shimizu [23], Abbas [24], Agarwal, O'Regan and Sahu [25], Xue, Lv and Rhoades [26], Phuengrattana and Suantai [27,28], Khan and Abbas [29], and Siriyan and Kangtunyakarn [30], among others.

On the other hand, in some recent papers [31–45], the authors used the technique of enrichment of contractive type mappings to generalize, in the setting of a Banach space, well known and important classes of *symmetric* contractive type mappings from the metric fixed point theory, e.g. Banach contractions [46], Kannan contractions [47], Chatterjea contractive mappings [48], nonexpansive and Lipschitzian mappings, etc.

For example, in [43], the following concept is introduced. Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an *enriched contraction* if there exist $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that the following symmetric contraction condition

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x-y\|, \text{ for all } x, y \in X \tag{3}$$

holds.

Obviously, the class of enriched contractions includes the usual Banach contractions (2), obtained from (3) for $b = 0$, but also some nonexpansive and Lipschitzian mappings, being a genuine extension of the class of Banach contractions. It was proven by Berinde and Păcurar [43] that any enriched contraction in a Banach space has a unique fixed point, x^*, which can be approximated by means of the Krasnoselskij iterative scheme.

Similar results for enriched Kannan contractive mappings and enriched Chatterjea contractive type mappings are obtained in [44,45], respectively.

An enriched Kannan mapping is defined similarly (see [44]). Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an *enriched Kannan mapping* if there exist $a \in [0, 1/2)$ and $k \in [0, \infty)$ such that

$$\|k(x-y) + Tx - Ty\| \leq a[\|x - Tx\| + \|y - Ty\|], \text{ for all } x, y \in X. \tag{4}$$

It is easily seen that the usual Kannan contractions [47,49] are obtained from (4) for $k = 0$.

Other similar results addressed the following classes of mappings: strictly pseudocontractive mappings [31], nonexpansive mappings in Hilbert spaces [32] and nonexpansive mappings in Banach spaces [33].

Note that, due to the particular form of the contractive conditions (3) and (4) and the corresponding ones in [31–33], which involve explicitly the linearity and convexity of the space, all the results presented in [31–45] are established in the case of a Banach (or Hilbert) space.

As the basic fixed point theorems established in literature for Picard–Banach contractions [50], Kannan mappings [47,49], Chatterjea mappings [48] etc. are stated in the setting of a complete metric space, the main aim of the present paper is to extend some of the above-mentioned results for enriched contractions [43] to the more general case of a convex metric space in the sense of Takahashi.

To this end, we need some concepts and basic results related to the theory of Takahashi convex metric spaces, mainly taken from the works in [4,25], which are presented in the next section.

2. Preliminaries: Convex Metric Spaces

There have been a few attempts to introduce the structure of convexity outside linear spaces. The one introduced in 1970 by Takahashi [4] turned out to be very useful in fixed point theory.

Definition 1. *[4] Let (X, d) be a metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0, 1]$,*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for any } u \in X. \tag{5}$$

A metric space (X, d) endowed with a convex structure W is called a Takahashi convex metric space and is usually denoted by (X, d, W).

Obviously, any linear normed space and each of its convex subsets are convex metric spaces, with the natural convex structure

$$W(x, y; \lambda) = \lambda x + (1 - \lambda)y, x, y \in X; \lambda \in [0, 1]. \tag{6}$$

but the reverse is not valid: there are various examples of convex metric spaces which cannot be embedded in any Banach space (see [4,29] (Example 1.2, Example 1.3), [25,51]).

The next lemmas present some fundamental properties of a convex metric space in the sense of Definition 1 (see [4,25] for more details).

Lemma 1. *Let (X, d, W) be a convex metric space. For all $x, y \in X$ and any $\lambda \in [0, 1]$,*

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y). \tag{7}$$

Proof. By the triangle inequality and (5), we get

$$d(x, y) \leq d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) \leq \lambda d(x, x) +$$

$$(1 - \lambda)d(x, y) + \lambda d(x, y) + (1 - \lambda)d(y, y) = d(x, y).$$

□

Lemma 2. *Let (X, d, W) be a convex metric space. For all $x, y \in X$ and any $\lambda \in [0, 1]$, we have*

$$d(x, W(x, y; \lambda)) = (1 - \lambda)d(x, y) \text{ and } d(W(x, y; \lambda), y) = \lambda d(x, y).$$

Proof. By (5), we get

$$d(x, W(x, y; \lambda)) \leq (1 - \lambda)d(x, y)$$

and

$$d(W(x, y; \lambda), y) \leq \lambda d(x, y).$$

If we had strict inequality in either of the above two inequalities, then, by Lemma 1, we would reach the contradiction

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) < d(x, y).$$

□

Lemma 3. *Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following:*

(i) $W(x, x; \lambda) = x$; $W(x, y; 0) = y$ and $W(x, y; 1) = x$; and

(ii) $|\lambda_1 - \lambda_2| d(x, y) \leq d(W(x, y; \lambda_1), W(x, y; \lambda_2))$.

Let (X, d, W) be a convex metric space and $T : X \to X$ be a self mapping. Denote by $Fix\,(T)$ the set of all fixed points of T, that is,

$$Fix\,(T) = \{x \in X : Tx = x\}.$$

The next lemma is a partial extension of a result given in Corollary to Theorem 5 in [52] from the setting of Banach spaces to that of convex metric spaces.

Lemma 4. *Let (X, d, W) be a convex metric space and $T : X \to X$ be a self mapping. Define the mapping $T_\lambda : X \to X$ by*

$$T_\lambda x = W(x, Tx; \lambda), x \in X. \tag{8}$$

Then, for any $\lambda \in [0, 1)$,

$$Fix\,(T) = Fix\,(T_\lambda). \tag{9}$$

Proof. For $\lambda = 0$, $T_\lambda = T$ and the assertion is trivial. Assume $\lambda \in (0, 1)$ and let $a \in Fix\,(T)$. This means $a = Ta$ and therefore

$$d(a, T_\lambda a) = d(a, W(a, Ta; \lambda)) \leq \lambda d(a, a) + (1 - \lambda)d(a, Ta) = 0,$$

i.e., $a \in Fix\,(T_\lambda)$.

Conversely, assume $a \in Fix\,(T_\lambda)$. This means that $d(a, T_\lambda a) = 0$, which implies

$$d(a, W(a, Ta; \lambda)) = 0.$$

By Lemma 2,

$$d(a, W(a, Ta; \lambda)) = (1 - \lambda)d(a, Ta),$$

so it follows that

$$(1 - \lambda) \cdot d(a, Ta) = 0,$$

which, in view of the fact that $1 - \lambda \neq 0$, implies $d(a, Ta) = 0$. □

3. Enriched Contractions in Convex Metric Spaces

Definition 2. *Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $c \in [0, 1)$ and $\lambda \in [0, 1)$ such that*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq cd(x, y), \text{ for all } x, y \in X. \tag{10}$$

To specify the parameters c and λ involved in (10), we also call T a (λ, c)-enriched contraction.

It is easily seen that, in view of Lemma 3, a $(0, c)$-enriched contraction is a usual Banach contraction. The next result is a significant extension of the main fixed point theorem in [43] (Theorem 2.4) from the case of a Banach space setting to that of an arbitrary complete convex metric space.

Theorem 1. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a (λ, c)-enriched contraction. Then,*

(i) $Fix\,(T) = \{p\}$, for some $p \in X$.

(ii) The sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process

$$x_{n+1} = W(x_n, Tx_n; \lambda),\ n \geq 0, \tag{11}$$

converges to p, for any $x_0 \in X$.

(iii) *The following estimate holds*

$$d(x_{n+i-1}, p) \leq \frac{c^i}{1-c} \cdot d(x_n, x_{n-1}) \quad n = 1, 2, \ldots; \; i = 1, 2, \ldots \tag{12}$$

Proof. By the enriched contractive condition (10), we have that the mapping $T_\lambda : X \to X$ defined by (8) satisfies

$$d(T_\lambda x, T_\lambda y) \leq c \cdot d(x, y), \text{ for all } x, y \in X, \tag{13}$$

that is, T_λ is a c-contraction. We note that the Picard iteration associated to T_λ is actually the Krasnoselskij iterative process $\{x_n\}_{n=0}^{\infty}$ associated to T and defined by (11), i.e.,

$$x_{n+1} = T_\lambda x_n, \; n \geq 0. \tag{14}$$

Now, we take $x = x_n$ and $y = x_{n-1}$ in (13) to get

$$d(x_{n+1}, x_n) \leq c \cdot d(x_n, x_{n-1}), \; n \geq 1, \tag{15}$$

which inductively implies

$$d(x_{n+1}, x_n) \leq c^n \cdot d(x_1, x_0), \; n \geq 1. \tag{16}$$

As $c \in [0, 1)$, from (16), we deduce that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0, \tag{17}$$

i.e., $\{x_n\}_{n=0}^{\infty}$ is *asymptotically regular*. By triangle inequality and (13), for any n and any $k > 0$, we have

$$d(x_{n+k+1}, x_{n+1}) \leq c \cdot d(x_{n+k}, x_n)$$
$$\leq c[d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+1}) + d(x_{n+1}, x_n)]$$

which yields

$$d(x_{n+k+1}, x_{n+1}) \leq \frac{c}{1-c}[d(x_{n+k}, x_{n+k+1}) + d(x_{n+1}, x_n)].$$

Thus, by (17), it follows that $\lim_{n \to \infty} d(x_{n+k+1}, x_{n+1}) = 0$, uniformly with respect to k, which shows that $\{x_n\}_{n=0}^{\infty}$ is Cauchy (note that the conclusion can also be drawn directly from the estimate (19)). Hence, $\{x_n\}_{n=0}^{\infty}$ is convergent and let us denote

$$p = \lim_{n \to \infty} x_n. \tag{18}$$

By letting $n \to \infty$ in (14) and using the continuity of T_λ (which follows by (13)), we immediately obtain

$$p = T_\lambda p,$$

i.e., $p \in Fix\,(T_\lambda)$.

Next, we prove that p is the unique fixed point of T_λ. Assume that $q \neq p$ is another fixed point of T_λ. Then, by (13),

$$0 < d(p, q) = d(T_\lambda p, T_\lambda q) \leq c \cdot d(p, q) < d(p, q),$$

a contradiction. Hence, $Fix\,(T_\lambda) = \{p\}$ and, therefore, in view of Lemma 4, $p \in Fix\,(T)$, which proves (i).

Conclusion (ii) follows by (18).

To prove (iii), first by (16) and (15), one obtains routinely the following estimates

$$d(x_{n+m}, x_n) \leq c^n \cdot \frac{1-c^m}{1-c} \cdot d(x_1, x_0),\ n \geq 1,\ m \geq 1. \tag{19}$$

$$d(x_{n+m}, x_n) \leq \frac{c}{1-c} \cdot d(x_n, x_{n-1}),\ n \geq 1,\ m \geq 1. \tag{20}$$

By letting $m \to \infty$ in (19) and (20), we get

$$d(x_n, p) \leq \frac{c^n}{1-c} \cdot d(x_1, x_0),\ n \geq 1 \tag{21}$$

and

$$d(x_n, p) \leq \frac{c}{1-c} \cdot d(x_n, x_{n-1}),\ n \geq 1, \tag{22}$$

respectively. Now, one can merge (21) and (22) to get the unifying error estimate (12). □

Remark 1. *If T is a $(0, c)$-contraction, then, by Theorem 1, we obtain the contraction mapping principle in the setting of a metric space. Note that, in this case, the Krasnoselskijj type iterative process (11) reduces to the Picard iteration. Note that we can give a shorter proof to Theorem 1, but here we intended to illustrate the important role of asymptotic regularity. This is a very important concept in the metric fixed point theory and was formally introduced and used by Browder and Petryshyn [53]. For some interesting very recent developments on the role of asymptotic regularity in fixed point theory, see the works of Górnicki [54] and Berinde and Rus [55]. By Theorem 1, we obtain in particular the main result in [43], given by the next corollary.*

Corollary 1. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (b, θ)-enriched contraction, that is, a mapping for which there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x-y\|, \text{ for all } x, y \in X.$$

Then,
(i) $Fix\,(T) = \{p\}$, for some $p \in X$.
(ii) There exists $\lambda \in (0, 1]$ such that the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{23}$$

converges to p, for any $x_0 \in X$.
(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \leq \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\| \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, \tag{24}$$

where $c = \dfrac{\theta}{b+1}$.

Proof. On the Banach space X, we consider the natural convexity W defined by (6) with $\lambda = \dfrac{b}{b+1}$ and apply Theorem 1. □

The local variant of Banach contraction mapping principle (see, e.g., [56]), which involves an open ball B in a complete metric space (X, d) and a nonself contraction map of B into X that has the essential property that it does not displace the center of the ball too far, is important in concrete applications. The analog of this result in the case of enriched contractions in convex metric spaces is given by the following theorem.

Theorem 2. *Let (X, d, W) be a complete convex metric space, $x_0 \in X$, $r > 0$, $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ and let $T : B \to X$ be a (λ, c)-enriched contraction. If*

$$d(Tx_0, x_0) < \frac{1-c}{1-\lambda} \cdot r,$$

then T has a fixed point.

Proof. We can choose $\varepsilon < r$ such that

$$d(Tx_0, x_0) \leq \frac{1-c}{1-\lambda} \cdot \varepsilon < \frac{1-c}{1-\lambda} \cdot r. \tag{25}$$

Since T is a (λ, c)-enriched contraction, there exists $c \in [0, 1)$ such that

$$d(T_\lambda x, T_\lambda y) \leq c \cdot d(x, y), \text{ for all } x, y \in B,$$

for any $\lambda \in (0, 1)$, where we denote as before $T_\lambda x := W(x, Tx; \lambda)$.

By Lemma 2, we have that

$$d(T_\lambda x_0, x_0) = d(W(x_0, Tx_0; \lambda), x_0) = (1-\lambda) \cdot d(x_0, Tx_0),$$

and therefore (25) implies that $d(T_\lambda x_0, x_0) \leq (1-c)\varepsilon$.

We now prove that the closed ball

$$\overline{B}_\varepsilon := \{x \in X : d(x, x_0) \leq \varepsilon\}$$

is invariant with respect to T_λ. Indeed, for any $x \in \overline{B}_\varepsilon$, we have

$$d(T_\lambda x, x_0) \leq d(T_\lambda x, T_\lambda x_0) + d(T_\lambda x_0, x_0) \leq cd(x, x_0) + (1-c)\varepsilon \leq \varepsilon.$$

Since $\overline{B}_\varepsilon$ is complete, the conclusion follows by Theorem 1.

□

Remark 2. *If X is a Banach space, then by Theorem 2 we obtain the local variant of the enriched contraction mapping principle established in [43].*

Corollary 2. *Let $(X, \|\cdot\|)$ be a Banach space, $r > 0$, $B = B(x_0, r) := \{x \in X : \|x - x_0\| < r\}$ and let $T : B \to X$ be a (b, θ)-enriched contraction, that is, a mapping for which there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x - y\|, \text{ for all } x, y \in X.$$

If $\|Tx_0 - x_0\| < (b+1-\theta)r$, then T has a fixed point.

Proof. We consider on X the natural convexity W defined by (6) with $\lambda = \dfrac{b}{b+1}$ and apply Theorem 2. □

There exist mappings which are not (enriched) contractions in the sense of Definition 2, but a certain iterate of them is an (enriched) contraction (see Example 2 in [43]).

In such cases, we cannot apply Theorem 1 and therefore the following result could be useful in applications.

Theorem 3. *Let (X, d, W) be a complete convex metric space and let $U : X \to X$ be a mapping with the property that there exists a positive integer N such that U^N is a (λ, c)-enriched contraction. Then,*

(i) $Fix(U) = \{p\}$, for some $p \in X$.

(ii) The sequence $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process*

$$x_{n+1} = W(x_n, U^N x_n; \lambda)\ n \geq 0,$$

converges to p*, for any* $x_0 \in X$*.*

Proof. We apply Theorem 1 (i) for the mapping $T = U^N$ and obtain that $Fix\,(U^N) = \{p\}$. We also have

$$U^N(U(p)) = U^{N+1}(p) = U(U^N(p)) = U(p),$$

which shows that $U(p)$ is a fixed point of U^N. However, U^N has a unique fixed point, p, hence $U(p) = p$ and so $p \in Fix\,(U)$.

The remaining part of the proof follows by Theorem 1. □

4. Enriched φ-Contractions in Convex Metric Spaces

There are various generalizations of the contraction mapping principle in arbitrary complete metric spaces which are based on considering instead of the original Banach contraction condition

$$d(Tx, Ty) \leq c \cdot d(x, y), x, y \in X,\ c \in [0, 1),$$

a weaker contractive condition of the form

$$d(Tx, Ty) \leq \varphi(d(x, y)), x, y \in X \tag{26}$$

or of the form

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y), x, y \in X, \tag{27}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to [0, 1)$ are functions possessing some suitable properties.

Thus, Rakotch [57] considered mappings satisfying (27) with ψ nonincreasing and $\psi(t) < 1$ for $t > 0$; Browder [58] considered mappings satisfying (26) with φ right continuous, nondecreasing and such that $\varphi(t) < t$ for $t > 0$; Boyd and Wong [59] weakened the right continuity of φ to the right upper continuity; and Matkowski [60,61] extended Boyd and Wong's results (see [2] for more details and a comprehensive list of references).

In the following, we consider $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ to be a *comparison function* (see [3]), if it satisfies the following conditions:

(i_φ) φ is nondecreasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$.

(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$.

Note that any comparison function also satisfies:

(iii_φ) $\varphi(t) < t$, for $t > 0$.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *φ-contraction* if it satisfies (26) with φ a comparison function. The first main result of this section is the following fixed point theorem which extends Theorem 1, Theorem 5.2 in [56] and Theorem 2.7 in [3], the last ones from the class of φ-contractions to the more general class of enriched φ-contractions.

Theorem 4. *Let* (X, d, W) *be a complete convex metric space and let* $T : X \to X$ *be an enriched* φ*-contraction, i.e., a mapping for which there exists a comparison function* φ *such that, for some* $\lambda \in [0, 1)$*,*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq \varphi(d(x, y)),\ \text{for all } x, y \in X. \tag{28}$$

Then,

(i) $Fix\,(T) = \{p\}$*, for some* $p \in X$*.*

(ii) The sequence $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process*

$$x_{n+1} = W(x_n, Tx_n; \lambda),\ n \geq 0, \tag{29}$$

converges to p, for any $x_0 \in X$.

Proof. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$, $x_n = T_\lambda x_{n-1} = T_\lambda^n x_0$, $n = 1, 2, \ldots$, be the Picard iteration associated to $T_\lambda(x) := W(x, Tx; \lambda)$. Then, by (28), we obtain

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1))$$

which, by (ii$_\varphi$), implies $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, that is,

$$d(T_\lambda^n x_0, T_\lambda^{n+1} x_0) \to 0, \text{ as } n \to \infty, \tag{30}$$

which expresses the fact that T_λ is asymptotically regular.

We now prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}_{k=0}^{\infty}$, $\{x_{m_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon, \text{ for all } k \geq 0. \tag{31}$$

Moreover, for each k, one can choose n_k to be the smallest integer satisfying the above conditions. Using (30), it follows (see the proof of Theorem 5 for details)

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon \text{ and } \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Now, by (28), we have

$$d(x_{n_k}, x_{m_k}) = d(T_\lambda x_{n_k-1}, T_\lambda x_{m_k-1}) \leq \varphi(d(x_{n_k-1}, x_{m_k-1})), \text{ for all } k \geq 0.$$

Letting $k \to \infty$ in the previous inequality and using the continuity of T_λ, one gets

$$\varepsilon \leq \varphi(\varepsilon),$$

a contradiction with (iii$_\varphi$), since $\varepsilon > 0$.

Thus, $\{T_\lambda^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence. As (X, d) is a complete metric space, $\{T_\lambda^n x_0\}_{n\in\mathbb{N}}$ is convergent.

Let $p = \lim_{n\to\infty} T_\lambda^n x_0$. Hence,

$$p = T_\lambda \left(\lim_{n\to\infty} T_\lambda x_{n-1}\right) = T_\lambda p,$$

which shows that $p \in Fix(T_\lambda)$.

Assume there exists $q \in Fix(T_\lambda)$, $q \neq p$. Then, $d(p, q) > 0$ and the condition of φ-contractiveness implies

$$0 < d(p, q) = d(T_\lambda p, T_\lambda q) \leq \varphi(d(p, q)) < d(p, q),$$

which is a contradiction. To finish the proof, we apply Lemma 4. □

By combining Theorems 3 and 4, we get the following result.

Corollary 3. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a mapping with the property that there exists a positive integer N such that T^N is an enriched φ-contraction. Then,*

(i) $Fix(T) = \{p\}$, for some $p \in X$.

(ii) There exists $\mu \in [0, 1)$ such that the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = W(x_n, T^N x_n; \mu)\ n \geq 0,$$

converges to p, for any $x_0 \in X$.

The next result unifies and extends several important fixed point results for the generalized contractions considered in this section. It is based on considering an auxiliary function $\alpha : \mathbb{R}_+ \to [0,1)$ possessing the following property:

(g) If $\{t_n\} \subset \mathbb{R}_+$ is nonincreasing and $\alpha(t_n) \to 1$ as $n \to \infty$, then $t_n \to 0$ as $n \to \infty$.

Denote by $\mathcal{A}$ the set of all functions α satisfying (g). For example, if $\alpha(t) = \exp(-t)$, for $t \geq 0$, then $\alpha \in \mathcal{A}$.

Theorem 5. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be an enriched α-contraction, i.e., a mapping for which there exists a function $\alpha \in \mathcal{A}$ such that, for some $\lambda \in [0,1)$,*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq \alpha(d(x,y)) \cdot d(x,y), \text{ for all } x, y \in X. \tag{32}$$

Then,

(i) $Fix\,(T) = \{p\}$, for some $p \in X$.

(ii) The sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process

$$x_{n+1} = W(x_n, Tx_n; \lambda),\ n \geq 0, \tag{33}$$

converges to p, for any $x_0 \in X$.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=0}^{\infty}$,

$$x_n = T_\lambda x_{n-1} = T_\lambda^n x_0,\ \ n = 1, 2, \ldots,$$

be the Picard iteration associated to $T_\lambda(x) := W(x, Tx; \lambda)$. If there exists some $n \geq 0$ such that $x_{n+1} = x_n$, then $Fix\,(T) = \{x_n\}$ and the proof is done.

Thus, assume in the following that $x_{n+1} \neq x_n$, for all $n \geq 0$. Then, by (32), we obtain

$$d(x_{n+1}, x_{n+2}) \leq \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \tag{34}$$

which implies that the sequence of nonnegative real numbers $\{d(x_n, x_{n+1})\}$ is decreasing, hence convergent. Denote

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0.$$

Suppose $r > 0$. Then, all terms of the sequence are positive and, thus, by (34), we obtain

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \alpha(d(x_n, x_{n+1})) < 1.$$

By letting $n \to \infty$ in the previous inequalities, we get

$$\lim_{n\to\infty} \alpha(d(x_n, x_{n+1})) = 1$$

and, since $\alpha \in \mathcal{A}$, this yields

$$d(x_{n+1}, x_{n+2}) = d(T_\lambda^n x_0, T_\lambda^{n+1} x_0) \to 0, \text{ as } n \to \infty, \tag{35}$$

which expresses the fact that T_λ is asymptotically regular.

We now prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_n\}_{n=0}^{\infty}$ is not a Cauchy sequence.

Then, there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}_{k=0}^{\infty}$, $\{x_{m_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon. \tag{36}$$

Corresponding to the given m_ks, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k > k$ and satisfying (36). Then, we have

$$d(x_{n_k - 1}, x_{m_k}) < \varepsilon. \tag{37}$$

By (36) and (37), we have

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}).$$

that is,

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}). \tag{38}$$

By letting $k \to \infty$ and using (35), we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{39}$$

Now, by using the triangle inequality, we get

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \tag{40}$$

and

$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}). \tag{41}$$

Letting $k \to \infty$ in (40) and (41) and using (35) and (39), we have

$$\varepsilon \leq 0 + \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) + 0$$

and

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq 0 + \varepsilon + 0,$$

which yields

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

By (34), we have

$$d(x_{n_k}, x_{m_k}) \leq \alpha(d(x_{n_k-1}, x_{m_k-1}))d(x_{n_k-1}, x_{m_k-1}) < d(x_{n_k-1}, x_{m_k-1})$$

and, by letting $k \to \infty$ in the previous inequality, we get

$$\varepsilon \leq \lim_{k\to\infty} \alpha(d(x_{n_k-1}, x_{m_k-1}) \cdot \varepsilon \leq \varepsilon,$$

which implies

$$\lim_{k\to\infty} \alpha(d(x_{n_k-1}, x_{m_k-1}) = 1.$$

Since $\alpha \in \mathcal{A}$, we obtain

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = 0.$$

This result and (39) yields $\varepsilon = 0$, which is a contradiction.

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and, since X is complete, $\{x_n\}_{n=0}^{\infty}$ is convergent. Denote

$$\lim_{n\to\infty} x_n = p. \tag{42}$$

Then,

$$d(p, T_\lambda p) \leq d(p, T_\lambda x_n) + d(T_\lambda x_n, T_\lambda p) \leq$$
$$d(p, x_{n+1}) + \alpha(d(x_n, p)) \cdot d(x_n, p).$$

Letting $n \to \infty$ in the previous inequality, we get $d(p, T_\lambda p) = 0$, that is, p is a fixed point of T_λ.

To prove the uniqueness, we suppose that there exists $q \in Fix(T_\lambda)$, $q \neq p$. Then, $d(p, q) > 0$ and, by (32), we have

$$d(p, q) = d(T_\lambda p, T_\lambda q) \leq \alpha(d(p, q))d(p, q) < d(p, q),$$

a contradiction.

Now, (i) follows by Lemma 4, while (ii) follows by (42).

□

Remark 3. *Theorem 5 is a very general result. Apart from Theorems 1 and 4 in this paper, we also mention the following important classical particular cases of it.*

(1) If $\lambda = 0$, then by Theorem 5 we obtain the Geraghty fixed point theorem (see [62,63]).

(2) If $\lambda = 0$ and $\alpha(t)$ is monotone decreasing, then by Theorem 5 we obtain the pioneering fixed point result of Rakotch ([57], p. 463).

(3) If $\lambda = 0$ and $\alpha(t) = \frac{\varphi(t)}{t}$, with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ right continuous, nondecreasing and such that $\varphi(t) < t$ for $t > 0$, then by Theorem 5 we obtain Browder's fixed point theorem ([58], p. 27).

(4) If $\lambda = 0$ and $\alpha(t) = \frac{\varphi(t)}{t}$, with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ right upper continuous, nondecreasing and such that $\varphi(t) < t$ for $t > 0$, then by Theorem 5 we obtain Boyd and Wong's fixed point theorem ([59], p. 331).

(5) If $\lambda = 0$ and $\alpha(t) = \frac{\varphi(t)}{t}$, with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function, then by Theorem 5 we obtain Matkowski's fixed point theorem ([61]) (see also [64]).

5. Conclusions and Future Work

(1) In this paper, we introduce and study, in the setting of a Takahashi convex metric space, the existence and approximation of fixed points for the class of enriched contractions, an important class of mappings known to include the Picard–Banach contractions and some nonexpansive and Lipschitzian mappings.

(2) We show that any enriched contraction in a complete convex metric space has a unique fixed point that can be approximated by means of a Kransnoselskij type iterative process expressed by means of the mapping that defines the convexity structure of the convex metric space. In particular, from the fixed point results established in this paper, we obtain the classical contraction mapping principle in the setting of a metric space as well as the main fixed point results established in [43] for enriched contractions in Banach spaces.

(3) We obtain a local fixed point result (Theorem 2) as well as an asymptotic fixed point result (Theorem 3) for enriched contractions in convex metric spaces. These results significantly extend the corresponding ones in [43] from Banach spaces to complete convex metric spaces.

(4) Finally, we prove very general fixed point theorems for enriched φ-contractions (Theorem 4, Corollary 3 and Theorem 5) which generalize and extend various important related results existing in literature for φ-contractions, due to Rakotch [57], Boyd and Wong [59], Browder [58], Geraghty [62,63], and Matkowski [60,61], among others (see also [2,3]).

(5) There exist other important results regarding the solution of the fixed point problem in convex metric spaces (see [11,12,14,17,19–25,27–30]) or in Banach spaces, metric spaces and generalized metric spaces (see [2,26,31–43,47–49,56,65,66]) that could be developed by means of the approach considered in the present paper.

(6) A similar technique to the one used in this paper can be utilized in the case of hyperbolic spaces, taking as starting points, for example, the works in [67–70] and also some related results established in usual metric spaces (see [34–42,71–78]).

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